DEVELOPING PROSPECTIVE TEACHERS' PROFESSIONAL NOTICING OF STUDENTS' PROPORTIONAL REASONING THROUGH VIDEO CASE ENHANCED LEARNING ENVIRONMENT

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF SOCIAL SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ÖZNUR SOYAK

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
THE DEPARTMENT OF ELEMENTARY EDUCATION

JULY 2023

Approval of the thesis:

DEVELOPING PROSPECTIVE TEACHERS' PROFESSIONAL NOTICING OF STUDENTS' PROPORTIONAL REASONING THROUGH VIDEO CASE ENHANCED LEARNING ENVIRONMENT

submitted by **ÖZNUR SOYAK** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in **Elementary Education**, **the Graduate School of Social Sciences of Middle East Technical University** by,

Prof. Dr. Sadettin KİRAZCI
Dean
Graduate School of Social Sciences _____

Prof. Dr. Semra SUNGUR
Head of Department
Department of Mathematics and Science Education _____

Prof. Dr. Mine IŞIKSAL BOSTAN
Supervisor
Department of Mathematics and Science Education _____

Prof. Dr. Erdinç ÇAKIROĞLU
Co-Supervisor
Middle East Technical University
Department of Mathematics and Science Education _____

Examining Committee Members:

Prof. Dr. Sinan OLKUN (Head of the Examining Committee)
Ankara University
Department of Elementary Education _____

Prof. Dr. Mine IŞIKSAL BOSTAN (Supervisor)
Middle East Technical University
Department of Mathematics and Science Education _____

Prof. Dr. Didem AKYÜZ
Middle East Technical University
Department of Mathematics and Science Education _____

Assoc. Prof. Dr. Fadime ULUSOY
Kastamonu University
Department of Mathematics and Science Education _____

Assist. Prof. Dr. Işıl İŞLER BAYKAL
Middle East Technical University
Department of Mathematics and Science Education _____

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: Öznur SOYAK

Signature:

ABSTRACT

DEVELOPING PROSPECTIVE TEACHERS' PROFESSIONAL NOTICING OF STUDENTS' PROPORTIONAL REASONING THROUGH VIDEO CASE ENHANCED LEARNING ENVIRONMENT

SOYAK, Öznur
Ph.D., The Department of Elementary Education
Supervisor: Prof. Dr. Mine IŞIKSAL BOSTAN
Co-supervisor: Prof. Dr. Erdinç ÇAKIROĞLU

July 2023, 320 pages

The primary purpose of this study was to investigate the development of prospective teachers' professional noticing of students' mathematical thinking in the domain of proportional reasoning as they participated in a classroom teaching experiment designed in a video clip-enhanced learning environment. Data were collected from six prospective teachers during the 2018-2029 spring semester within an elective course. Multiple data sources were utilized to collect data, including clinical pre-post interviews, reflection papers, initial and revised lesson plans, teaching experiment sessions, group discussions, and field notes. The teaching experiment sessions were mainly based on analyzing and discussing the micro-case video clips, including the key understandings of proportional reasoning in Carpenter's (1999) four-level developmental trajectory.

Jacobs et al.'s professional noticing of children's mathematical thinking framework was used to analyze the data. Data analysis revealed that prospective teachers'

professional noticing of students' thinking developed through video case-based learning environment. Micro-case video clips and video-based group discussions led to a more detailed exploration of student thinking. In this vein, this study revealed that prospective teachers' professional noticing of students' thinking in the domain of proportional reasoning improved after teacher experiment sessions which required analyzing and discussing students' thinking in the video clips. On the other hand, it was revealed that this trend to develop prospective teachers' professional noticing of student thinking showed a variation in video case-based analyzes of key understandings of proportional reasoning. In this vein, the prospective teachers proposed a different type of interpretation and response in relation to the students' understanding in each video clip.

In this sense, this study has theoretical and practical contributions to researchers and mathematics educators in terms of promoting prospective teachers' professional noticing of students' thinking in the particular domain of proportional reasoning through video case-enhanced learning environment.

Keywords: Teacher Noticing, Micro-Case Video Clips, Students' mathematical Thinking, Proportional Reasoning, Prospective Teacher Education

ÖZ

VİDEO VAKALARIYLA ZENGİNLEŞTİRİLMİŞ ÖĞRENME ORTAMI İLE ÖĞRETMEN ADAYLARININ ÖĞRENCİLERİN ORANTISAL AKIL YÜRÜTMELERİNİ FARK ETME BECERİLERİNİN GELİŞTİRİLMESİ

SOYAK, Öznur
Doktora, İlköğretim Bölümü
Tez Yöneticisi: Prof. Dr. Mine IŞIKSAL BOSTAN
Ortak Tez Yöneticisi: Prof. Dr. Erdinç ÇAKIROĞLU

Temmuz 2023, 320 sayfa

Bu çalışmanın amacı, video vakalarıyla zenginleştirilmiş bir öğrenme ortamında tasarlanmış bir sınıf öğretimi deneyine katılan öğretmen adaylarının, orantısal muhakeme bağlamında öğrencilerin matematiksel düşünmelerine ilişkin profesyonel fark etme becerilerindeki gelişimlerini incelemektir. Veriler, 2018-2019 bahar dönemi boyunca seçmeli bir ders kapsamında altı öğretmen adayından toplanmıştır. Veri toplama sürecinde klinik ön ve son görüşmeler, yansıtıcı düşünce raporları, ders planları ve revize edilmiş ders planları, öğretimsel deney oturumları, grup tartışmaları ve araştırmacı alan notları gibi çoklu veri kaynaklarından faydalanılmıştır. Öğretimsel deney oturumları Carpenter ve arkadaşlarının dört seviyeli gelişimsel yörüngesindeki orantısal akıl yürütmenin anahtar kavramlarını içeren mikro vaka video kliplerinin analiz edilmesine ve tartışılmasına dayanmaktadır.

Öğretmen adaylarının öğrencilerin matematiksel düşünme şekillerini fark etme becerilerinin incelenmesinde Jacobs ve arkadaşları tarafından tanımlanmış mevcut bir

teorik çerçeveden faydalanılmıştır. Bulgular, öğretmen adaylarının öğrencilerin düşüncelerini fark etme becerilerinin video vakalarıyla zenginleştirilmiş öğrenme ortamı aracılığı ile geliştiğini ortaya koymuştur. Mikro (vaka) video klipleri ve video tabanlı grup tartışmaları, öğrenci düşüncesinin daha ayrıntılı bir şekilde keşfedilmesini sağlamıştır. Bu bağlamda video durumlarındaki öğrencilerin düşünüşlerini analiz etmeyi ve tartışmayı gerektiren öğretmen deney oturumları sonrasında, öğretmen adaylarının öğrenci düşünmelerine ilişkin farkındalıklarının geliştiği görülmüştür. Öte yandan, öğretmen adaylarının öğrenci düşüncelerini profesyonel olarak fark etmelerini geliştirmeye yönelik bu eğilimin, orantısal akıl yürütmenin temel anlayışlarının video vakaya dayalı analizlerinde bir çeşitlilik gösterdiği ortaya çıkmıştır. Bu bağlamda, öğretmen adayları video kliplerdeki öğrencilerin düşüncelerine yönelik farklı türde bir yorumlama ve öğretimsel strateji önermişlerdir.

Bu anlamda, öğretmen adaylarının öğrencilerin orantısal muhakeme alanındaki düşüncelerini profesyonel olarak fark etmelerini teşvik etme açısından araştırmacılara ve matematik eğitimcilerine bu çalışma, video vakalarıyla zenginleştirilmiş öğrenme ortamı aracılığı ile teorik ve pratik katkılar sunmuştur.

Anahtar Kelimeler: Öğretmenin Fark etmesi, Mikro durum video clipleri, Öğrencilerin Matematiksel Düşünmeleri, Orantısal Akıl Yürütme, Öğretmen Adayı Eğitimi

To My Family

ACKNOWLEDGMENTS

Firstly, I would like to express my sincere gratitude to my super supervisor Prof. Dr. Mine IŞIKSAL BOSTAN for the continuous support of my Ph.D study and for her patience, motivation, and immense knowledge. Her guidance helped me in all the time of research and writing of this dissertation. I cannot thank you enough for the long hours spent together on developing the ideas in this dissertation over the years. I could not have imagined having a better advisor and mentor for my Ph.D study.

I am truly indebted to my co-advisor Assoc. Prof. Dr. Prof. Dr. Erdinç ÇAKIROĞLU who always kept this study in perspective. I can't express enough gratitude for your support and mentorship.

Besides my advisor, I would like to thank the rest of my thesis committee:
Prof. Dr. Sinan OLKUN, Prof. Dr. Didem AKYÜZ, Assoc. Prof. Dr. Fadime ULUSOY and Assist. Prof. Dr. Işıl İŞLER BAYKAL for for their feedback, contributions, and suggestions for my study.

I also thank the prospective middle school teachers who agreed to participate in this study. Without you, this dissertation could not be done.

I would like to thank my close friends, Emine, Merve, Ayşenur, Dilek, Rukiye, and Fetiye provided me with unwavering moral and academic support. I am fortunate to have people like you who are always willing to offer insightful suggestions and comments. In short, thank you to all who embellish my life.

Following that, I would like to thank my close friends Tuğçe, Beyza, Nurbanu, Emine Ç, Nazlı and Gözde. They have always backed me in this endeavor. I wish you all the best in your future lives. I would also like to thank my dear friend for sharing with me valuable suggestions and her point of view.

I really appreciate and thank to my family for their endless support and encouragement through this challenging process. Furthermore, I would like to thank Dr. Çağrı KAYA, who has always been there to support and encourage me in all facets of life.

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LIST OF ABBREVIATIONS

MCVs	Micro-case video clips
NCTM	National Council of Teachers of Mathematics
MoNE	Ministry of National Education
PSTs	Prospective Teachers
CSU	Case-specific-undetailed
CSD	Case-specific-detailed

CHAPTER 1

INTRODUCTION

Mathematics educators have a growing interest in noticing, which has been extensively studied in the past decade (Jacobs & Spangler, 2017). Since research has shown that noticing is one of the essential skills teachers must possess (e.g., Nickerson et al., 2017; Philipp, 2014). Researchers described noticing as how people focus on and make sense of complex situations (Amador, 2022; Jacobs et al., 2010). Although the ability to notice has been defined from several viewpoints (Jacobs et al., 2010; Mason, 2002, 2011; Sherin et al., 2011; van Es, 2008), they all emphasize the significance of finding and interpreting the essential parts of teaching and learning settings in order to make instructional decisions.

The development of noticing skills in teacher education programs has become significant; however, it is a complex process (Amador, 2020; Ivars et al., 2018; Star & Strickland, 2008; van Es & Sherin, 2002, 2008). As teaching is a multidimensional activity, teachers might not be able to see and respond to every classroom event. Previous research reveals that teachers require assistance in learning to attend to and interpret students' mathematical thinking and significant classroom events. The process of noticing helps teachers break down and evaluate teaching practices to learn students' mathematical thinking. In other words, noticing is a practice that needs to be developed to promote student thinking (Jacobs et al., 2010; Mason, 2002; McDuffie et al., 2013; Sherin & Han, 2004; Star & Strickland, 2008; van Es & Sherin, 2002, 2008). Noticing skills could be developed by shifting from concentrating on teachers' actions and classroom climate to students' thinking and from evaluative to interpretive statements based on evidence (van Es, 2011). Furthermore, as teachers learn how to respond to and expand upon student thinking, student thinking becomes a significant resource for teacher learning (Kazemi & Franke, 2004; Wilson & Berne, 1999).

Teachers at all levels, from novices to expert teachers, have pre-existing lenses through which they view their professional interactions with students and colleagues. These lenses shape how teachers interpret and respond to various aspects like lesson plans and student thinking, as well as their professional development and reflection (Mason, 2011). It is not sufficient to tell teachers to notice something; a development process must be fostered. While supporting the development of noticing skills as a part of the teaching profession, "It is quite common for people to become aware of the detail that they had previously overlooked, because they begin to look, not simply to see what they have become habituated to see" (Mason, 2002, p. 37). Prospective and in-service teachers require opportunities to concentrate on more specific, detailed components of practice and student thinking to cultivate their noticing skills. These opportunities and experiences can be utilized to better comprehend the process of learning to notice (Amador, 2020). However, prospective teachers (PSTs) face significant difficulties in learning noticing skills (Peterson & Leatham, 2009; Stockero, 2008). It has been claimed that prospective teachers have limited knowledge of students' thinking due to their limited experience with students in a classroom environment.. Prior studies suggest that classroom experience improves the ability to notice, which, unfortunately, prospective teachers do not have. Thus, prospective teachers should be s upported to learn how to notice student thinking and develop their noticing skills (Stockero & Rupnow, 2017). Even though prospective teachers do not have classroom experience, they can notice, and noticing skilss can be cultivated (Males, 2017). In this regard, this study is essential in developing prospective teachers' noticing of students' mathematical thinking in the context of proportional reasoning through a video case enhanced learning environment.

Recent research has demonstrated that in order to investigate and develop prospective teachers' and in-service teachers' noticing skills, researchers have utilized different tools such as students' written works (i.e., Fernandez et al., 2013; Jacobs et al., 2010) or different video types such as the video of an unknown teacher's classroom or one's own classroom or one-on-one student interview videos (i.e., Jacobs et al. 2010; McDuffie et al., 2013; Santagata & Guario, 2010; Schack et al., 2013; Star & Strickland, 2008; van Es & Sherin, 2002). Studies on how prospective teachers learn to see things have revealed that specifically different video types influence how

prospective teachers notice things (Superfine et al., 2017). The videos used in teacher education serve various purposes and emphasize various aspects of learning and teaching while hiding others. The findings indicate that watching educational videos shifted noticing abilities from just delivering the events based on chronological order to attending to and interpreting the significant mathematical details of the process (Jacobs et al., 2010; Kazemi et al., 2011; van Es, 2011). However, prospective teachers tend to focus on unrelated details in videos, specifically in classroom videos. Although classroom videos encapsulate the events in classrooms, prospective teachers tend to focus on teachers' or students' behavior or classroom management rather than students' mathematical reasoning in the videos (Krupa et al., 2017; van Es et al., 2020). Even if they watch their teaching practices, it is difficult for prospective teachers to detect relevant events in classroom videos (Star & Strickland, 2008; van den Bogert et al., 2014) because they show tendency to focus on a range of issues including classroom climate, teachers' pedagogies, student learning and classroom management. On the other hand, studies further indicate that videos which depict only one student's problem-solving process help prospective teachers to focus more on the student's mathematical thinking (Jacobs et al., 2011; Stockero et al., 2017a; Ulusoy & Cakiroglu, 2020; van Es & Sherin, 2006). From this point, the current study aims to support prospective teachers-learners' development of noticing of student thinking in the domain of proportional reasoning through one-on-one student video clips. .

Recent studies have emphasized the importance of teacher noticing in specific mathematical domains (e.g., Carpenter et al., 1988, 1989; Dreher & Kuntze, 2015; Jacobs et al., 2010). Therefore, proportional reasoning was chosen to support the development of prospective teachers' professional noticing skills. Proportional reasoning plays a significant role in children's mathematical development since it is a foundational subject in school mathematics, including elementary and secondary school mathematics (Lamon, 2007). NCTM's Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, Commission on Standards for School Mathematics, 1989)stated that, "proportional reasoning is of such great importance that it merits whatever time and effort must be expended to assure its careful development. Students need to see many problem situations that can be modeled and solved through proportional reasoning" (p. 82).

Studies and classroom experience have demonstrated that students apply memorized methods, such as cross-product algorithms (Karplus et al., 1983; Lesh et al., 1988). National and international mathematics textbooks begin with the definitions of ratio and proportion, which require students to find the value of ratios and solve the missing value problem, these practices only demand computational skills. In parallel with the content of textbooks, students are taught the use of to use cross-product algorithms without having a meaningful understanding of the relationship between given quantities and developing intuitive knowledge (Lamon, 1994). However, studies show that solving problems using cross product algorithm is not an indicator of proportional reasoning (Cramer & Post, 1993a, 1993b; Lamon, 1995; Lesh et al., 1988). Therefore, prospective teachers must learn students' thinking while solving various proportional word problems before being in classrooms.

Prospective Teachers' Professional Noticing of Students' Thinking in the Context of Proportional Reasoning

Early research suggested a strong need to improve prospective teachers' noticing skills (Sherin & van Es, 2005). Previous studies showed that although they do not have classroom experience, they could develop this ability through professional development (Males, 2017). In this vein, prior research has provided promising results in terms of prospective teachers' noticing skills (e.g., Walkoe, 2014). As Mason stated, the awareness of mathematical situations is critical because teachers may identify critical mathematical ideas that students present and make instructional decisions based on the students' thinking and understanding (Mason, 2002). Researchers are increasingly emphasizing the importance of student thinking in making instructional decisions. Therefore, prospective teachers need support in learning to notice students' mathematical thinking (e.g., Sherin et al., 2011) because teacher educators cannot presume that prospective teachers know how to attend to students' strategies and interpret and decide how to respond to students' understandings (Stockero, 2008). Hence, it has become essential to teach prospective teachers how to notice students' mathematical thinking to improve student success (Goldsmith & Seago, 2011; Jacobs et al., 2010; Sherin & van Es, 2009; Star & Strickland, 2007; van Es, 2011). Researchers explored teachers' noticing of students' understanding, mostly by

watching classroom videos and analyzing students' written work in any domain (Colestock & Sherin, 2009). Sherin et al. (2017) criticized this implementation by saying that a teacher without expertise in noticing classroom climate would also not have expertise in noticing student thinking. Thus, this study contributes to the literature by elaborating on what prospective teachers attend to, how they interpret students' understandings, and and how they decide how to respond to students' understandings in the domain of proportional reasoning, in which much of the complexity of classrooms is eliminated using micro-case video clips depicting only a single student engaging in the problem-solving process.

Recently, researchers have focused on investigating professional teacher noticing in various domains such as pattern generalization thinking (e.g., Callejo & Zapatera, 2017; LaRochelle et al., 2019; Lee & Lee, 2021), algebra (e.g., Walkoe, 2014), derivative (e.g., Sánchez-Matamoros et al., 2014), proportional reasoning (e.g., Fernandez et al., 2012; Amador et al., 2022), and geometry (e.g., Ulusoy & Cakiroglu, 2021; Wilkie, 2022). These studies examine the significance of elementary teachers' noticing of students' mathematical reasoning in particular domains (e.g., Garet et al., 2001; Goldsmith & Seago, 2011; Kennedy, 1998). From this point of view, to understand teachers' noticing of students' mathematical thinking and how this might be used to enhance student learning, it is necessary to examine teachers' noticing in a specific mathematical domain (Walkoe, 2014).

In the present study, I propose a specific content emphasis, which is proportional reasoning. This process requires individuals to recognize, examine, analyze, and propose evidence 'in support of assertions about proportional relationships' (Lamon, 2007, p. 647). Accordingly, both students and teachers struggle with it (Lamon,2007). As Amador et al. (2022) state, "Teachers need specific support to advance their own proportional reasoning content knowledge and to make instructional decisions that are responsive to how students reason about proportions" (p. 164). Since teachers are exposed to various stimuli in classrooms, it is challenging for them to pay close attention to students' thoughts on each topic. While this is difficult for in-service teachers, it will be more difficult for prospective teachers with little experience. To put it differently, prospective teachers cannot disregard the unrelated elements or instances

that arise during instruction that do not contribute to learning how to notice student thinking (Teuscher et al., 2011). In this vein, this study utilized micro-case video clips, which include the key understandings of proportional reasoning in the developmental trajectory of proportional reasoning, to support and improve the noticing skills of prospective teachers.

In video-based interventions, researchers mostly examine classroom videos that provide multiple data about a classroom environment, such as the teacher, the student, climate, pedagogy, management, and mathematical content (e.g., Sherin et al., 2007). Even though examining classroom videos assists prospective teachers in developing their noticing skills (Santagata et al., 2007), they also face several difficulties in noticing significant instances and details in classroom videos (Peterson & Leatham, 2009; Stockero et al., 2017b; Ulusoy & Cakiroglu, 2021). When this type of video is utilized in professional development programs, prospective teachers tend to focus on situations unrelated to students' mathematical understandings, such as pedagogy or management (Sangata et al., 2007; Star & Strickland, 2008). Besides, utilizing raw videos for teacher education is not typically productive without making specific edits (Seago et al., 2018; Ulusoy, 2020). On the other hand, there is a limited number of studies examining the development of prospective teachers' professional noticing of students' mathematical thinking using micro-case video clips, specifically in a particular domain (Fernández et al., 2011, 2013; Weiland et al., 2020; Star & Strickland, 2008; Fisher et al., 2017).

In light of the constraints of classroom videos, we propose that the production and the use of video clips, namely micro-cased video clips, can be a productive approach for developing prospective teachers' noticing abilities. In this regard, we utilized video clips so that teachers could slow down instructional interactions and focus on students' proportional reasoning as video format permits pausing and rewatching. The micro-cased video clips generally include only one student's task-based interviews, and each student in the video clip focuses on various proportional reasoning problems. In conclusion, the present study aimed to examine the development of professional noticing of students' mathematical thinking in the context of the key understandings

of proportional reasoning through video clip-enhanced learning environments. The study addresses the following research questions:

1.1. Research Questions

1. How do prospective teachers develop professional noticing of student's mathematical thinking expertise in the domain of proportional reasoning as they participate in a video clip-enhanced learning environment?

a) How do the prospective teachers attend to students' strategies in the micro-case video clips?

b) How do the prospective teachers interpret students' thinking in the micro-case video clips?

c) What next instructional actions do the prospective teachers propose in response to students' thinking in the micro-case video clips?

1.2. Significance of the Study

This study aimed to investigate the development of professional noticing of students' thinking in the context of key understandings of proportional reasoning through a video case-enhanced learning environment. The most significant aspects of this research are explained in the following paragraphs.

Teaching is a complicated process requiring teachers to focus on some actions and ignore others (Teuscher et al., 2017). As Sherin and Star (2011) highlighted, "a teacher is bombarded with a blooming, buzzing confusion of sensory data" (p. 69). In this vein, teachers must filter these data in order to make decisions that enhance student learning. This statement highlights the importance of *teacher noticing* because it is central to managing the "most salient features of teaching" (Doyle, 1977, p. 52). In recent years, mathematics educators have demonstrated interest in noticing constructs to understand how teachers make sense of classroom events, where attending and responding to every event is not possible (Jacobs et al., 2011). Some studies have only focused on

teachers' attention (Star et al., 2011) while others have addressed how teachers reason about what they observe in the classroom (Sherin, 2007; van Es, 2011). Jacobs et al. (2010) narrowed their focus and investigated a particular aspect of teacher noticing called children's mathematical thinking, namely, *professional noticing of children's mathematical thinking*. They conceptualized noticing expertise as three interrelated skills: attending to children's strategies, interpreting children's understandings, and deciding how to respond based on children's understandings (Jacobs et al., 2010, p. 172). Prospective teachers must learn to notice student thinking (NCTM, 2000). The professional noticing framework provides prospective teachers a window into the development of students' mathematical thinking and understanding (Stigler & Hiebert, 1999).

Furthermore, school experience, as a significant part of teacher education programs at universities, enables one to make this transition (Ishler & Kay, 1981). However, prospective fourth-year teachers may not succeed in making sense of "*what they observe in meaningful ways*" and in using it to make instructional decisions (Jacobs et al., 2011, p. 99). Star and colleagues (2011) believed that before preservice teachers make sense of classroom characteristics, they must first learn to pay attention to relevant and noteworthy details. As seen, prospective teachers need opportunities to focus on students' thinking to develop their noticing skills before they start teaching in real classrooms. Studies have shown that noticing skills can be learned (Jacobs et al., 2010; Star et al., 2011; Stockero, 2014), but they don’t develop naturally (Stockero, 2014; Star & Strickland, 2008). Therefore, in this study, video-based professional development is designed to help prospective teachers assess noticing (van Es & Sherin, 2005). I used micro-case video clips, including selected and edited cases related to a student's thinking process on proportional reasoning tasks. We believe that the production and the use of micro-case video clips can assist in developing prospective teachers' noticing of students' thinking.

Given the advances in new technologies, video is extensively used in teacher preparation and professional development programs (van Es et al., 2011). As an artifact of teaching, videos (i.e., classroom videos or individual student videos) have become a powerful and key resource for helping in-service and pre-service teachers

learn to notice (e.g., Sherin, 2004; van Es et al., 2014; Van Zoest & Stockero, 2008; van Es et al., 2019). Van Es, Tekkumru-Kisa and Seago (2020) highlight the impact of videos as follows: "The genre of video captures the specificity of the teaching and learning experience, allowing for fine-grained observation and sense-making" (p. 23). Several studies documented how videos in a collaborative context support noticing students' thinking (e.g., Jacobs et al., 2022; Larison et al.,2022; Stockero et al., 2017). For this reason, in this study, as a researcher, I asked prospective teachers to view video clips, and then I used specific prompts about the video clips to elicit their noticing skills through video-based intervention. Moreover, videos enable teachers to "zoom in" on practice in a less complicated learning environment when the purpose is not to act in the moment of teaching, but rather to understand the complexity of teaching (Goldsmith & Seago, 2011; Tunney & van Es, 2016). In this regard, classroom videos can encourage teachers to develop a shared language and shift their focus to students' thinking and particular teacher moves (van Es, 2011). Even though classroom videos provide multiple channels of information, they also provide multiple channels of distraction (Goldsmith & Seago, 2011, p. 184). Even though utilizing unedited classroom videos provides complex classroom context, it might be difficult for specifically prospective teachers to pay attention to the relevant moments in a lesson since prospective teachers who did not have teaching experience may primarily focus on the class environment or teacher pedagogy in classroom videos (Jacobs et al., 2011; Stockero, 2014, 2017; Stockero & Rupnow, 2017). Besides, some critical details of students' thinking may not appear in classroom videos (Stockero & van Zoest, 2013). This situation may cause students' thinking to go unnoticed (Peterson & Leatham, 2009). Due to the aforementioned limitations of classroom videos, this study proposed *micro-case video clips* for prospective teachers, which focused only on one student explaining his/her reasoning in the interview, eliminating the distractions of a classroom session and revealing the student's thinking in detail. Therefore, the complexity of students' mathematical thinking can be noticed by prospective teachers because the current study aims to develop prospective teachers' professional noticing of students' thinking. Undoubtedly, the most prominent contribution of the present study is to show the efficacy of utilizing micro-case video clips to promote prospective teachers' professional noticing in a video-case-based learning environment. Consequently, micro-case video clips would provide valuable contributions to teacher

educators by extending studies on prospective and practicing teachers' professional noticing of students' understandings.

There is widespread consensus in the research community that teachers' noticing is contingent on their expertise and knowledge (e.g., Star et al., 2011; van Es, 2010). However, what type of professional knowledge and expertise indicates noticing? Professional noticing is a domain-specific expertise (Jacobs & Empson, 2016); hence, teachers' efficacy in making sense of and deciding how to respond to students' thinking will vary based on the mathematical domain (LaRochelle, 2018). In this study, *proportional reasoning* was chosen as the domain. It is a milestone in students' mathematical development (Lesh et al., 1988). Concerning this, the NCTM's Curriculum and Evaluation Standards (1989) underlines the importance of proportionality: "It is of such great importance that it merits whatever time and effort must be expanded to assure its careful development. Students need to see many problem situations that can be modeled and solved through proportional reasoning" (NCTM, 1989, p. 82). On the other hand, the study of proportional reasoning is a challenging area of mathematics. For instance, students have faced difficulties in conceptualizing proportional relationships (Lesh et al., 1988). However, in order to reason proportionally, students must have a strong understanding of foundational concepts such as a sense of co-variation and invariant-covariant relationships (Hilton & Hilton, 2019). For instance, in Turkish mathematics textbooks, proportional reasoning begins with the definition of ratio and proportion. Students are asked to find the ungiven value of ratio forms and missing value problems. Students are taught cross-product algorithms without making sense of multiplicative relationships between the given quantities (Lamon, 1995). Therefore, they fail to distinguish between additive and multiplicative relationships (van Dooren et al., 2010). Moreover, the Turkish curriculum (MoNE, 2018) does not include all key understandings of proportional reasoning (i.e., additive and multiplicative reasoning). All these bring about a significant challenge since teachers must conceptualize proportional relationships to "employ explicit teaching strategies to promote students' proportional reasoning and to enhance the underlying foundational concept" (Hilton et al., 2016, p. 194).

On the other hand, numerous studies have demonstrated that students and pre-service teachers have similar understandings and difficulties in the context of ratio and proportion (Harel & Behr, 1995; Hilton & Hilton, 2019; Lamon, 2007; Sowder & Philipp, 1995). Studies indicated that some prospective teachers even had difficulties in distinguishing the additive and multiplicative situations (Sowder & Philipp, 1995). According to Amador et al. (2022), "Teachers need specific support to advance their own proportional reasoning content knowledge to make instructional decisions that are responsive to how student reason about proportions" (p.2). By addressing issues mentioned above, we utilized video clips containing various video recordings of a task-based interview of students related to key understandings of proportional reasoning (i.e., additive-multiplicative thinking, unitizing, norming, and multiplicative relationships) to direct prospective teachers' attention to mathematically noteworthy details of student thinking. These key understandings were introduced to students with five proportional word problems based on Carpenter and colleagues' developmental trajectory of proportional reasoning. In this sense, the current study shows promise for developing PSTs' ability to professionally notice student thinking through viewing and reflecting upon videos of clinical student interviews in the particular context of proportional reasoning. Prospective teachers can use trajectory-oriented noticing to focus on tailoring responses to individual students because they watched particular video clips, including students' interviews related to big ideas of proportional reasoning in sessions, and each video clip had various types of students' reasoning (e.g., incorrect or correct responses).

Research on teacher noticing focuses on teachers' professional learning, from preparation courses to professional development programs for prospective teachers (Sherin et al., 2011). In recent years, teacher educators have become increasingly aware of the importance of teacher noticing to promote student learning (Amador et al., 2020). Professional development programs are considered to be a promising approach to promote prospective teachers' noticing of students' thinking (Stein et al., 2008). Only gaining teaching experience in teaching practice courses at universities is insufficient for prospective teachers to develop their teaching practice before teaching in real classrooms (LaRochelle, 2018). When prospective teachers learn how to attend to students' strategies, interpret their understanding, and respond to students' thinking,

student thinking can promote teacher learning during professional development (Franke et al., 2001). Therefore, it is necessary to design a course as a professional development program in teacher education . Thereupon, the results of the current study would contribute to the literature by designing a video case enhanced learning environment to support the prospective teachers' noticing of students' thinking. With all of these taken into consideration, the study aimed to promote prospective teachers' professional noticing of students' thinking in the domain of proportional reasoning through micro-case videos.

1.3. Definition of the Important Terms

Professional Noticing

Researchers describe noticing in a variety of ways, but the common factor is individuals' ability to make sense of complex events (e.g., Goodwin, 1994; Mason, 2002; van Es & Sherin, 2009). Jacobs et al. (2010) described the professional noticing of children's mathematical thinking "as a set of interrelated skills; (a) *attending* to children's strategies, (b) *interpreting* children's understandings, and (c) *deciding* how to respond on the basis of children's understandings" (Jacobs et al., 2010, p. 172).

Attending was described as "the extent to which teachers attend to the mathematical details in children's strategies" (Jacobs et al., 2010, p.172). For this study, we characterized attending as the extent to which prospective teachers attended to the details of students' strategies in the micro case video clips. Jacobs et al. (2010) describe *interpreting* children's understandings as "the extent to which teacher's reasoning is consistent with both the details of the specific child's strategies and the research on children's mathematical development" (p. 172). In this study, we characterized interpreting as the extent to which prospective teachers interpreted the students' thinking in the micro case video clips. Finally, Jacobs et al. (2010) identified *deciding* as "the extent to which teachers use what they have learned about children's understandings" and "whether their reasoning is consistent with the research on children's mathematical development" (p. 173). In this study, we characterized *deciding how to respond* (we refer to from here on as "responding") as

intended responses of prospective teachers that are related to the students' understandings in the micro case video clips.

Micro-case video clips

In this thesis, "micro-case videos" refers to the purposefully created, designed, and edited video clips, including one student's problem-solving instead of a classroom environment (Ulusoy & Cakiroglu, 2021, p. 255). Therefore, this video type was designed for prospective teachers to focus on students' mathematical understanding. It included the following main characteristics: (a) the student's mathematical thinking process, (b) a selection of specifically created and edited mathematical events, (c) structured ratio and proportion-related tasks, and (d) isolated out-of-class learning environment in order to remove "the distractions of a classroom lesson to feature the complexity of student thinking" (van Es et al., 2020. p.37).

Proportional Reasoning

The term proportional reasoning refers to any reasoning that emphasizes the relationship between two ratios (Vergnuad, 1983). In this thesis, proportional reasoning refers to mathematical reasoning that demands "a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information" (Lesh et al., 1988, p. 93).

CHAPTER 2

LITERATURE REVIEW

The primary purpose of this study was to investigate the development of prospective teachers' professional noticing of students' mathematical understanding in the context of proportional reasoning in a video clip-enhanced learning environment. This chapter was categorized into three main sections. The first section reviewed the literature about the construct of noticing in a video clip enhanced learning environment and provided a description of the professional noticing conceptual framework. The second section includes the literature on the content domain of proportional reasoning. Existing studies highlight that professional noticing of students' mathematical understanding is a domain-specific expertise. Therefore, this section is a significant starting point for understanding prospective teachers' professional noticing of students' mathematical thinking in this study. In the third section, I summarize the literature related to prospective teachers' professional noticing of students' thinking in the domain of proportional reasoning.

A literature review summary is presented at the end of the chapter.

2.1. Noticing and Teacher Noticing

Noticing is seen as a natural component of human sense-making. We see and interpret the events in our daily lives based on our interpretations and objectives (Ball, 2017). Noticing entails a professional vision that involves observing and making sense of events (Goodwin, 1994). The mathematics classroom is a complex setting with simultaneous actions. For this reason, what teachers see must be reconstructed because although they each look at the same thing, their perceptions may vary (Gibson, 1979). Even if attending to all actions is impossible, teachers must decide on what to pay

attention to and how to proceed with that complexity. In this vein, given the growth in research in the field of professional noticing, this large and growing body of literature has investigated the definitions of noticing differently, and teacher noticing has been characterized in a variety of ways (i.e., Jacobs et al., 2010; Sherin et al., 2010; Schack et al., 2017). In this regard, definitions often include attention to and interpretation of students' thinking, consequent pedagogical decisions, and the relationship between those interpretations and broader teaching and learning principles (Jacobs, Lamb, & Philipp, 2010; van Es & Sherin, 2008). While van Es and Sherin (2008) describe noticing as the "ability to notice features of a practice that a particular social group values" (p. 244), Star and Strickland (2007) define noticing as attending to the significant moments in the classroom, referring to the first component of van Es and Sherin (2008). Jacobs et al., (2010) define noticing as a set of three interrelated skills: attending, interpreting, and responding. Researchers and teacher educators have increasingly recognized the importance of developing teachers' noticing skills (Jacobs et al., 2010; Mason, 2002; Sherin & van Es, 2009; Star & Strickland, 2007). Teachers' expertise and, therefore, their noticing improve through cognition and reflection on their teaching methods and experiences. According to Jacobs and Spangler (2017), noticing expertise is crucial for detecting and making sense of significant instructional aspects in the mathematics classroom as "the act of focusing attention on and making sense of situation features in a visually complex world" is a required skill for teachers (p. 771).

In the past two decades, many researchers have focused on fostering teachers' ability to notice students' thinking (Schack et al., 2017; Sherin et al., 2011; Stahnke et al., 2016). These efforts have been made to better understand teachers' noticing (Schack et al., 2017). The significance of noticing as a theoretical framework for comprehending teaching extends far beyond any simple definition of this notion. However, there have been substantial discrepancies in how noticing is produced and measured, particularly regarding researchers' methodological procedures to gather and analyze data on teacher noticing. For instance, some academics have developed frameworks for learning to notice (e.g., van Es, 2011). Others have eloquently framed noticing as a professional practice and differentiated between different levels of proficiency in noticing (e.g., Jacobs et al., 2010). Studies on noticing have focused on

teachers in a variety of fields, such as mathematics (e.g., Goldsmith & Seago, 2011; Jacobs et al., 2010; Jacobs et al., 2011; Mason, 2002; Sherin & van Es, 2009; Star & Strickland, 2008; van Es & Sherin, 2002) and science (e.g., Barnhart & van Es, 2015; Luna, 2018; Luna et al., 2018). The literature on noticing in mathematics education reveals that Mason (2002; 2011), van Es and Sherin (2002; 2006; 2008), Santagata, Zannoni, and Stigler (2007), and Jacobs et al. (2010) have produced the most cited or referred publications. The recent research into noticing has a connection to some of these works or all of them (e.g., Amador, 2016; Kaiser et al., 2021; Santagata & Guarino, 2011; Ulusoy & Cakiroglu, 2020).

Over the previous decade, teacher noticing and its variations (i.e., professional noticing of children's mathematical thinking) gained recognition as a construct of significant interest to educators in mathematical environments (Schack et al., 2017; Sherin et al., 2011). Many would argue that the origins of noticing may be traced back to studies on professional vision (Goodwin, 1994). Goodwin (1994) defined *professional vision* as the accuracy with which a practitioner in a particular domain articulates features of that profession. Teachers need to be sensitive to classroom characteristics in the teaching profession, which is very important to enhance students' development (Mason, 2002, 2011).

Noticing has been investigated by numerous researchers (Amador et., 2016; Jacobs et al., 2011; Sangata et al., 2007; Sherin & Star, 2011; Stockero, 2014; Ulusoy & Cakiroglu, 2021; van Es, 2011). While some researchers defined noticing solely as attending (e.g., Star et al., 2011), others described noticing as both attending to and making sense of the complex teaching practice (e.g., Goldsmith & Seago, 2011; van Es, 2011). For example, Jacobs and her colleagues (2010) discussed that noticing involved attending to students' strategies, interpreting students' mathematical understandings, and deciding how to respond based on students' understandings. Star and Strickland (2009) state that noticing involves "what catches their (teachers') attention and what they miss—when they view a classroom lesson" (p.111). This approach concerns what teachers attend to and do not attend to (Miller, 2011). van Es and Sherin (2021) proposed "shaping" as another dimension of noticing. They identified it as "teachers constructing interactions, in the midst of noticing, to gain

access to additional information that further supports their noticing" (p. 23). Similarly, Stockero et al. (2017) introduced "building" as a dimension of noticing. They identified it as teachers' practice of making students' thinking the focus of the class to better understand the essential mathematical ideas. Jacobs et al. (2010) adopted an even broader approach. They characterized professional noticing as including teachers' strategies for responding to classroom events in addition to attending to and interpreting these events. The intended response in the description of professional noticing shows that all three processes (i.e., attending, interpreting, and deciding how to respond) are conceptually intertwined (Jacobs et al., 2010; Amador et al., 2021). Definitions of noticing locate the phenomenon to be researched differently, and each has promoted the overall understanding of teacher noticing (e.g., Sherin & Star, 2011). Even though there is some difference in the interpretation of noticing dimensions, most researchers refer to noticing as being able to recognize and concentrate on significant events in the classroom or details of students' strategies (Erickson, 2011). Although studies have differences in the conceptualization of noticing, there is a consensus on the significance of noticing in mathematics education (Amador, 2016; Jacobs et al., 2010; van Es, 2011).

Another difference in how researchers conceptualize noticing is whether they narrow down their concentration on a specific component of noticing, such as student thinking (e.g., Jacobs et al., 2010), specific content domain (e.g., Schifter, 2018), or novice and expert noticing (Kaiser et al., 2022). These conceptual differences in noticing also have methodological implications (Sherin et al., 2011). For example, some research focuses on individual teachers' noticing, while others focus on groups of teachers (Jacobs et al., 2010). Besides, there is a difference among classroom artifacts in terms of evaluating teacher noticing, such as student' written work (e.g., Goldsmith & Seago, 2011), video cases (e.g., Stockero, 2008), and live classrooms (Sherin et al., 2011). Researchers who employ video vary in terms of video selection (video of an unknown teacher's classroom or one's own classroom or one-on-one student interviews) and the length of videos (short video clips, edited video clips, or whole class period). The studies also differ in collecting participants' noticing moments, examining teachers' discussions (Choy et al., 2011), and analyzing teachers' written responses to noticing prompts (Jacobs et al., 2010). One of the distinct and novel methods entailed placing

wearable cameras to capture moments that teachers define as interesting (Sherin et al., 2011). It is evident that these various definitions of noticing and methodological approaches serve as the source of the same goal: the development of the construct of teachers noticing (Jacobs et al., 2010; Leatham et al., 2015; Sherin & van Es, 2005; Star & Strickland, 2008). Previous studies revealed that prospective and in-service teachers had difficulties in sidentifying relevant important teaching and learning events without deliberate focus and training. Concerning in-service teachers, studies tended to concentrate more on their actions than students' actions (e.g., Colestock & Sherin, 2009; Santagata, 2009; Sherin & van Es, 2009; van Es & Sherin, 2008). Besides, it was revealed that in-service teachers routinely monitored students' activities more than prospective teachers and paid even attention. In contrast to in-service teachers, prospective teachers have significant difficulties in learning to notice (Peterson & Leatham, 2009; Stockero, 2008) and recognizing relevant events in classroom videos. Thus, teacher educators suggest that there is a strong need to prepare prospective teachers to utilize student thinking to guide and shape their future instructional actions to improve students' learning (Jacobs et al., 2010; Star & Strickland, 2008; van Es, 2011). At this point, some researchers offer frameworks for learning to notice students' thinking (e.g., Choy, 2015; Jacobs et al., 2010; van Es, 2011). The frameworks of teacher noticing and related studies which utilized related frameworks are expressed below.

A body of research stated that the ability to recognize and make interpretations of salient classroom events and to make instructional decisions based on those interpretations is among the most crucial elements of teaching expertise, specifically for prospective teachers (Borko et al., 2015; Jacobs et al., 2010; Kersting et al., 2010, 2012, 2021; van Es & Sherin, 2008). The most comprehensive research on noticing in mathematics education has been conducted by van Es and Sherin (e.g., Sherin, 2002, 2004, 2007, 2009; Sherin & van Es, 2003, 2005, 2006, 2008, 2009, 2021; van Es et al., 2014; van Es et al., 2022).

In order to examine the development of teachers' ability to notice, Sherin and van Es (2002) created a trajectory to present the levels of development for learning to notice. They focused on three important dimensions of noticing: (a) identifying what is

important or noteworthy about a classroom situation, (b) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent, and (c) using what one knows about the context to reason about classroom interactions (van Es, 2002, p.573). This trajectory has four levels focusing on three dimensions. Teachers made descriptive and general comments about the video at the first level.

At the second level, they started to make not only descriptive comments but also evaluative comments. While teachers focused on specific noteworthy events at the third level, their comments still included judgmental statements. The fourth level contained specific noteworthy statements with evidence to support the claims (van Es & Sherin, 2002). This study suggests that video analysis enables teachers to identify noteworthy events with specific evidence. Moreover, teachers who advanced to Levels 3 and 4 focused on student thinking and were more likely to explain how the discourse would be shaped based on the student thinking they noticed. While defining noticing, they underlined the significance of interpreting for classroom interactions. How teachers analyze what they notice becomes as significant as what they notice (Sherin, 2002, p.575). Based on these works, Sherin and van Es (2009) concentrated on video club meetings to understand what teachers learn and also how teachers learn in the video club context. Sherin (2007) describes the ability to notice as a professional vision that includes two primary components: selective attention and knowledge-based reasoning (Sherin, 2007; van Es & Sherin, 2009). Selective attention refers to "how the teacher decides where to pay attention at a given moment," and knowledge-based reasoning points to "the ways in which a teacher reasons about what is noticed based on his or her knowledge and understanding" (Sherin & van Es, 2009, p. 22).

Jacobs and her colleagues (2010) investigated a specialized type of noticing and narrowed their focus to the professional noticing of elementary school students' mathematical thinking. They conceptualized "a set of three interrelated skills: *attending* to children's strategies, *interpreting* children's understandings, and *deciding* how to respond on the basis of children's understanding" (Jacobs et al., 2010, p. 172). Philip et al. (2017) highlighted that these three practices are *"highly interrelated and often occur seemingly simultaneously"* (p. 114). *Attending* was

described as "the extent to which teachers attend to the mathematical details in children's strategies (Jacobs et al., 2010, p.172). Jacobs et al. (2010) describe *interpreting* children's understandings as "the extent to which teacher's reasoning is consistent with both the details of the specific child's strategies and the research on children's mathematical development" (Jacobs et al., 2010, p.172, p. 172). They identified *deciding* as the " the extent to which teachers use what they have learned about the children's understandings from the specific situation and whether their reasoning is consistent with the research on children's mathematical development" (Jacobs et al., 2010, p. 173). They found that practicing teachers provided evidence of attending to students' strategies and interpreting the students' mathematical understandings. A few practicing teachers showed limited evidence of deciding how to respond based on students' understanding, and only one participant showed robust evidence. On the other hand, novices (i.e., prospective teachers and initial participants as experienced practicing teachers) experienced difficulty in all components. The findings indicate that, although teaching experience seems to develop noticing skills, in-service teachers and novice teachers should be supported in professional development to acquire noticing expertise (Jacobs et al., 2011). In this study, we use Jacobs et al.'s three interrelated skills (attending to, interpreting, and deciding how to respond to student thinking – which we sometimes refer to from here on as "responding"). Further details about the *Professional Noticing* framework are explained in the following sections.

Lee and Choy examined prospective and in-service teachers to investigate the role of noticing in teachers' development from Lesson Study. They built on van Es's (2010) study to investigate what and how teachers notice while planning and reviewing the lessons. In addition, they utilized the Professional Noticing framework (Jacobs et al., 2010) to examine the extent to which prospective and pre-service teachers attend to, interpret, and decide how to respond during lesson study discussions. The Three-Points framework includes the following points: (1) thinking about how teachers emphasize relevant and critical mathematical points (the Key Point); (2) recognizing the difficulties students encounter while trying to understand vital mathematical points (the Difficult Point); and (3) providing a strategy for assisting students as they reach the lesson's objectives (the Critical Point) (Yang & Ricks, 2013, p.54). The study

revealed that during initial observations, both prospective and in-service teachers had difficulty concentrating on noteworthy mathematical details. Nonetheless, by the end of lesson study discussions, both groups of teachers began to attend to students' thinking. On the other hand, the findings show that neither prospective teachers nor in-service teachers attained a degree of noticing to decide how to respond to students' mathematical thinking. In this sense, this framework supported both groups of teachers' noticing skills; none of them reached the highest level of Professional noticing. These findings indicate the difficulty of assisting even experienced teachers in learning to respond to students' understanding (Philip et al., 2017).

Studies indicate that the decision-making process occurs after attention and interpretation, so teachers have to attend to noteworthy events of a classroom lesson and propose the reasoning for what they notice (Sangata, 2011). In this vein, there is a strong alignment between these two elements: what teachers notice in classroom settings shapes how teachers interpret and reason about what they notice or vice versa (Jacobs et al., 2011; Sherin & van Es, 2009). Also, the research analyzed what teachers see or miss (Star & Strickland, 2008; Star et al., 2010). This perspective leads to constructing what teachers see and what they decide not to attend or how they see complex situations in a particular way (Jacobs et al., 2010; Sherin et al., 2011). In other words, researchers discuss how and why teachers focus on specific situations rapidly in their classrooms and filter out other situations (Miller, 2011; Sherin & van Es, 2009) to develop a more extensive perspective of teacher noticing.

Studies to promote teaching and learning highlighted the requirement for students to be active in their learning to construct the meaning of mathematics (NCTM, 2000). This approach demands teachers to investigate the students's elicited mathematical thinking in order to assist student learning. It is not surprising that a body of research in recent years concentrates on teachers' noticing of students' mathematical thinking (e.g., Jacobs et al., 2010; Schifter, 2018; Sherin & van Es, 2009; Teuscher et al., 2017). It has been demonstrated that increased focus on student thinking provides more opportunities for student learning (Schifter, 2018; Carpenter et al., 1989; Jacobs et al., 2007; Franke et al., 2001). Even though there is an increasing awareness about noticing in educational environments, the construct of noticing children's mathematical

thinking is still missing. Mason (2002) states that "every act of teaching depends on noticing: noticing what children are doing, how they respond, evaluating what is being said or done against expectations and criteria, and considering what might be said or done next" (p. 7). By many accounts, noticing places students' mathematical thinking and reasoning at the center of instructional decision-making (Anthony et al., 2015; Jacobs et al., 2010; van Es & Sherin, 2008). To summarize, in recent times, noticing students' mathematical understanding is seen as a critical component of teachers' expertise (Bastian et al., 2021). In this vein, the present study aims to promote prospective teachers' professional noticing skills in the context of students' mathematical thinking.

Noticing expertise is not something teachers routinely possess, but research has consistently shown that it is a learnable practice (Jacobs & Spangler, 2017). Teachers do not develop noticing expertise through teaching experience alone. However, studies provide substantial evidence that with support, teachers and prospective teachers can improve their noticing skills (Fernandez et al., 2012; Goldsmith & Seago, 2011; Jacobs et al., 2010; Sangata & Guarino, 2011; Sherin & Han, 2004; Sherin & van Es, 2009; Star & Strickland, 2007; van Es, 2011; van Es & Sherin, 2008; Walkoe, 2014). van Es and Sherin (2008) pointed out that when teachers notice students' mathematical thinking outside the classroom through professional development, they increase their noticing skills similarly inside the classroom. Similarly, Mason (2011) discussed how people could respond to events they noticed in a given educational setting and how teachers' noticing alters over time as they become more experienced in the given context. The Goldsmith and Seago (2011) study asserts that teachers who gain expertise in students' mathematical understanding can retrieve the details of students' strategies, as they will purposefully pay attention to details in complex situations. Therefore, recent studies highlighted that professional development experiences are a significant component of noticing expertise in terms of providing deliberate linking of research to practice (Goldsmith & Seago, 2011; Jacobs et al., 2010; Jacobs et al., 2011; Sherin & Han, 2004; Sherin & van Es, 2009; Star & Strickland, 2007; van Es, 2011; van Es & Sherin, 2008).

Studies clearly showed that noticing can be learned (Roth McDuffie et al., 2014; Star & Strickland, 2008; van Es & Sherin, 2006; van Es et al., 2017; Walkoe, 2015) and teacher noticing can be improved through professional development related to noticing students' mathematical understanding (Amador, 2021; Goldsmith & Seago, 2011; Kaiser et al., 2021; Sherin & Han, 2004; Star & Strickland, 2008; van Es & Sherin, 2002). Concerning the professional noticing of students' mathematical thinking, much of the extant research focused on experts' and novices' development (e.g., Lee & Choy, 2017; Fernández et al., 2012; Goldsmith & Seago, 2011; Superfine et al., 2017; Stockero & Rupnow, 2017). Noticing levels of novice and expert teachers are not identical (Sherin et al., 2011), and what experts and novices notice about teaching practice differs significantly (Superfine et al., 2017). One of the differences between experts and novices is an awareness of what to react to and what to disregard (Miller, 2011). In the work of Goldsmith and Seago (2011), in-service teachers who attended a professional development program required "fewer prompts from the facilitators to explore the logic of students' thinking or to ground their conversations in specific details from the artifacts" (p. 177). On the other hand, novices (i.e., prospective teachers) could not attend to the details as deeply as experts (Jacobs et al., 2010) and had difficulty in determining which details of classroom practice need attention (van Es, 2011). They showed a tendency to focus on trivial features of the classroom (e.g., classroom management) (Star & Strickland, 2008) and did not attend to students' strategies (Crespo, 2000; Superfine et al., 2017). The reason for this difference might result from a lack of teaching experience (Fernández et al., 2012). Thus, researchers have proposed various approaches to specifically support prospective teachers' noticing. For instance, Amador and Weiland (2015) involved prospective teachers in a systematic lesson study process to promote their noticing of students' mathematical thinking.

Choy et al. (2017) presented the FOCUS framework to direct prospective teachers' attention to important mathematical details while investigating what and how teachers notice while planning, teaching, and reflecting on mathematics lessons (p. 463). These studies on noticing students' mathematical thinking indicate that prospective teachers cannot attend to the noteworthy details of students' strategies and have difficulties in interpreting their thinking in depth (Star & Strickland, 2008; Star et al., 2011).

Similarly, Jacobs et al. (2011) found that teachers who have more teaching experience (i.e., experienced practicing teachers) demonstrated better performance in attending to students' mathematical strategies, and they gained skills to divide students' strategies into meaningful parts (Bransford et al., 2000). All these studies indicate that prospective teachers struggle to notice students' thinking, but they can learn to notice in professional development programs (Jacobs et al., 2010; McDuffie et al., 2013; Schack et al., 2013). In this regard, the present study focused on the development of prospective teachers' noticing of students' mathematical understanding through a video case-enhanced learning environment.

A fundamental aspect of teaching expertise is noticing classroom events and students' thinking (Mason, 2002). Studies showed that classroom artifacts could promote the quality of teachers' noticing expertise (Goldsmith & Seago, 2011; Lee, 2018). Different artifacts effectively achieve various aims, such as developing professional noticing (Franke et al., 2001; Jacobs et al., 2010; Kersting et al., 2010; van Es, 2008). The existing literature demonstrates several promising methods for promoting teachers' noticing of students' mathematical thinking. Types of artifacts ranged from knowledge-based research (Carpenter et al., 1989) to students' written work (Crespo, 2000; Jacobs et al., 2010; LaRochelle, 2018) and from analyses of videos (Santagata & Guarino, 2011; Sherin & Han, 2004)to discussion of narratives about mathematical topics (Ivars & Fernandez, 2018). For instance, Ivors and Fernadez (2018) aimed to analyze the potential effects of narratives and expert feedback on pre-service teachers' noticing of students' thinking. They asserted that narratives were practical artifacts to lead teachers to focus on particular mathematical elements. Even though these methods help teachers reach information about students' thinking, some of the artifacts can be inconvenient for illustrating students' thinking and improving the instructional implementation (Choy, 2015).

Studies have demonstrated that specifically videos and written documents as common artifacts play a central role in professional teacher development (e.g., Gibson & Ross, 2016; Goldsmith & Seago, 2011; Jacobs et al., 2011; Gonzalez & Vargas, 2020). Students' written works and video clips are also tools for teachers' professional development (Ball, 1992; Nikula et al., 2006). Jacobs et al. (2011) asked participants

to examine students' written work or watch classroom videos to respond to the prompts about attending, interpreting, and deciding on how to respond (p.177). LaRochelle (2018) carried out research on how professional teacher noticing was developed among prospective teachers, in-service teachers, and emerging teacher leaders. In order to compare the professional noticing expertise among these three groups, written student artifacts involving six different proportional reasoning strategies and understandings, as pointed out in the literature, were used. Goldsmith and Seago (2011) investigated how various types of artifacts (e.g., written work and video clips) shape what teachers are likely to notice. Regarding previous studies, in this study, micro-case video clips were primarily used to support prospective teachers' noticing of students' thinking.

2.1.1. Using video clips as a context for professional noticing

Research findings show that although both artifacts provide opportunities to promote teachers' noticing of students' mathematical thinking, students' written works presented fewer clues about students' solution process (Teuscher et al., 2017). Video is commonly used as a primary artifact to develop teachers' noticing expertise (Sherin & Han, 2004; Sherin & van Es, 2005).

A growing body of research on noticing detailed the kinds of videos that were used, the purposes they served, and the type of educational activities integrated into research (Santagata & Guario, 2011). Classroom video is one of the sources of video-based research. Studies show that three types of classroom video-based activities are mainly utilized: classroom videos of unknown teachers, classroom videos of colleagues, and classroom videos of the teacher's own teaching (Sherin &Han, 2004). Classroom videos of unknown teachers are used to analyze classroom practices, and teachers who watched the videos apply what they noticed in these videos in their classrooms (Santagata & Guarino, 2011). Viewing videos of colleagues is another type of classroom video source. The main advantages of using classroom videos of colleagues are to see others, have a window into different practices (Zhang et al., 2011), and recognize that they encounter similar difficulties (Borko et al., 2008). As for videos of the teacher's own teaching, Zhang et al. (2011) stress that these videos are like "having

a mirror placed in my face" (p. 458). Analyzing their own classroom experience allows teachers to gradually recognize themselves and detect critical points in their teaching practices (Borko et al., 2008; Star & Strickland, 2008). All these types of videos provide various information about distraction sources (Goldsmith & Seago, 2011). Another promising activity to improve noticing expertise is watching individual students' videos (Santagata & Guarino, 2011; Jacobs et al., 2010; van Es & Sherin, 2006). These videos are designed with a specific purpose to help teachers concentrate solely on students' mathematical thinking without being exposed to many distractions (Sangata & Guarino, 2011; van Es & Sherin, 2006). In this regard, van Es and Sherin (2006) examined how specific types of videos promote the development of teachers' noticing as a part of professional development programs. They used two different videos for two groups of teachers. While the first video focused precisely on students' mathematical thinking, the second video focused on the classroom. They found that teachers in each group showed overall development in their noticing practice. When examined more closely, it was seen that while the teachers in the first group became more focused on students' thinking, the teachers in the second group focused on various approaches, such as classroom climate or pedagogy. Thus, the study showed that video types played a significant role based on the purpose of the study (van Es & Sherin, 2006). Written artifacts might be practical, yet videos might be more helpful if the aim is to delve into the details of students' mathematical thinking (Goldsmith & Seago, 2011, p.184). The primary purpose of using micro-case video clips in the present study was to develop prospective teachers' noticing of students' mathematical thinking in a particular mathematical domain. The Method chapter includes in detail a brief description of micro-case video clips, the selection criteria of video clips, and their sequence.

In this study conducted with prospective teachers, the video-case enhanced learning environment offered a context for investigating the development of prospective teachers' noticing of students' mathematical thinking in line with the Professional Noticing framework (Jacobs et al., 2010). The implementation process was designed in the context of key understandings of proportional reasoning. Below are details about Jacobs and colleagues' framework utilized in this study as a part of the developmental trajectory in the domain of the proportional reasoning-oriented noticing process.

2.2. Professional Noticing of Students' Mathematical Thinking

The present study focused on the development of prospective teachers' professional noticing of students' thinking in the domain of proportional reasoning. We frame our view of noticing within the *Professional Noticing of Children's Mathematical Thinking framework* (Jacobs et al., 2010). They characterized the construct of Professional Noticing as a combination of three practices: *attending* to students' strategies, *interpreting* students' understandings, and *deciding how to respond* based on students' understandings (Jacobs et al., 2010, p.172). They clarified that the inclusion of the intended response in their description of noticing emphasizes the viewpoint that all three processes are conceptually linked to one another. Jacobs and colleagues (2010) applied two assessments based on two artifacts: written student works and videos. Their aim was to analyze if a group of elementary teachers differed in professional noticing.

Attending was described as how teachers attend to noteworthy mathematical details. Professional noticing demands the skill to attend to mathematical noteworthy elements or details of children's strategies, and these details could shape a teacher's instruction (Jacobs et al., 2010). Attending to relevant and significant details while ignoring the irrelevant details is a key feature of this component. In other words, deciding on which details need attention and which details can be ignored requires proficiency in noticing (Erickson, 2011; Sherin & van Es, 2009). Making sense of students' understanding depends on understanding the details of students' strategies (Carpenter et al., 1999). In the present study, attending was characterized as the extent to which prospective teachers attended to the noteworthy mathematical details of students in the video clips.

Interpreting was characterized as a reflection of children's mathematical ideas. Goodwin (1994) refers to professional vision as a "way of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (p. 606). Sherin and van Es (2009) identify this expertise as knowledge-based reasoning, which consists of teacher interpretations of what they notice in students' understanding (p. 26). They point out that how teachers interpret what they notice is as significant as what they notice. Moreover, Jacobs et al. (2010) point out that teachers' interpretations

should be consistent with students' performance in the problem and the existing research about students' mathematical understandings in a particular domain. Interpretive stance has been seen as fundamental for reflecting and learning from teaching ability (Sherin, 2002). This expertise involves seeking classroom events to figure out what happened and to understand students' thinking or how this thinking shapes the following action (Sherin, 2002).

In this study, we were interested not only in what prospective teachers attend to but also in how and the extent to which prospective teachers make sense of students' mathematical thinking they attend to. Therefore, as Jacobs and her colleagues characterize, we refer to interpreting as the extent to which prospective teachers' reasoning is consistent with the details of students' mathematical understandings, including the specific child's strategies and the research on children's mathematical development.

Deciding how to respond based on students' understandings was characterized as the reasoning teachers apply to decide how to respond. They describe this expertise as the degree to which participants' responses were associated with students' understanding of particular situations and the existing studies about students' mathematical development (Jacobs et al., 2010). This expertise indicated the intended response, not the real execution of the response (Jacobs, 2011, p. 99). In the present study, as Jacobs et al. (2010) characterized, we described deciding how to respond as the degree to which prospective teachers' responses were linked to students' mathematical thinking and understanding of the mathematical concept in the video clips.

According to Philip et al. (2017), these three practices are "highly interrelated and often occur seemingly simultaneously" (p. 114). For instance, when a student's response reveals an unexpected idea to the teacher, the teacher may ask that student or other students a follow-up question(s) and may revise and change the lesson flow. This scenario shows that attending to the student's strategies and interpreting the student's thinking are not practically separable. The teacher simultaneously begins planning a response while interpreting the student's understanding (Philip et al., 2017). Overall, teachers can decide how to respond based on students' mathematical understandings

after attending to students' strategies and interpreting students' understandings as demonstrated in their strategies (Jacobs et al., 2011). These interrelated skills should be practiced together to scaffold professional teacher development (Grossman et al., 2009). Schoenfeld (2011) asserts that "what you see and do not see determines what you do and do not do" (p. 228) and underlines that teacher noticing is crucial to their decision-making and subsequent actions. Thus, studies (e.g., Mason, 1998, 2002; van Es & Sherin, 2002) have emphasized the significance of teacher noticing in terms of improving teaching practices. Studies argued that teacher noticing is trainable; hence, professional teacher noticing must be promoted as a holistic set of these three components through teacher education programs (Lee et al., 2018).

Jacobs et al. (2010) asked teachers to write in response to three prompts related to three noticing skills. They scaled teachers' responses at three levels (attending, interpreting, and responding) based on the evidence they provided about students' mathematical understanding:

Attending prompt:To evaluate teachers' expertise in attending to students' mathematical strategies, they asked, *"Please describe in detail what you think each child did in response to this problem*." A three-step process was followed while coding the responses. First, the noteworthy mathematical details were defined. Secondly, teachers' responses were examined to determine whether they attended to most mathematical details or only a few (Jacobs et al., 2010, p.179).

*Interpreting prompt:*To evaluate teachers' expertise in interpreting students' mathematical understandings, they asked, "*Please explain what you learned about these children's understandings*" (Jacobs et al., 2010, p.179)

*Responding prompt:*To evaluate teachers' expertise in deciding how to respond based on students' mathematical thinking, the researchers asked, "*Pretend that you are the teacher of these children; what problem or problems might you pose next?*" (Jacobs et al., 2010, p.179).

In this study, prospective teachers were asked to respond to the following prompts and the coding scale in the context of proportional reasoning in order to have insights into the progress that prospective teachers were making with respect to their skills of noticing student mathematical thinking:

(a) Please describe in detail what you think this student in the video clip did in response to the problem *(Attending prompt)-limited evidence and robust evidence,*
(b) Please explain what you learned about this student's understandings in the video clip *(Interpreting prompt)- lack of evidence, limited evidence and robust evidence,*
(c) Pretend that you are the teacher of this student; what instructional action(s) might you pose next *(Deciding how to respond prompt)- general, case-specific-undetailed and case-specific-detailed*

Table 2.1 displays the definitions of each noticing skill utilized in the current study based on Jacobs et al.'s (2010) work. This study with prospective teachers demonstrates how a video-case based learning environment can be integrated into development of noticing skills, guided by a noticing framework and specific prompts. In this respect, we can investigate the development of prospective teachers' noticing of students' mathematical understandings in the specific mathematical domain of proportional reasoning. To this end, I designed an intervention providing scaffolding activities such as viewing and reflecting upon video clips of clinical interviews (i.e., micro-case video clips) to promote prospective teachers' professional noticing of students' mathematical thinking. This intervention followed the Professional Noticing framework as a developmental trajectory. The further details of three interrelated skills, the coding, and the video-enhanced intervention process were explained in the Method section.

Table 2.1 Developmental trajectory of professional noticing (Jacobs, Lamb, & Philipp, 2010)

Noticing Skills	Explanation
Attending to students' strategies	This skill includes the description of mathematically noteworthy elements and consistencies or inconsistencies in students' mathematical actions in the context of proportional reasoning.
Interpreting students' understanding	This skill includes both highlighting noteworthy mathematical elements and the practice of making sense of students' understanding with the reasons and evidence that underlie students' understanding in the context of proportional reasoning. - Making sense of the reasons of what students did understand - Making sense of the reasons of what students did not understand
Deciding how to respond on the basis of students' understanding	This skill includes next instructional actions based on each student's understanding in the context of proportional reasoning. The rationale of selected instructional actions based on students' understandings is provided for existing situation.

2.3. The Development of Proportional Reasoning

Proportional reasoning is considered to be a keystone of students' mathematical development. Lesh et al. (1988) describe proportional reasoning as "a sense of covariation and multiple comparisons, and the ability to mentally store and process several pieces of information" (p. 93). Thus, proportional reasoners can sense covariation and compare quantities (Felding-Wells et al., 2014). In order to think proportionally, students require a highly developed understanding of essential elements such as multiplication and division, fractions, and decimals to build a connection to advanced high school mathematics like algebra, geometry, probability, and statistics (Hilton & Hilton, 2016; Lesh et al., 1988). Besides, proportional reasoning is essential for other subjects such as geography and science (Akatugba & Wallace, 2009). Studies have shown that understanding scientific concepts such as density, concentration, speed, and pressure—typically encountered in senior secondary school science curricula—requires the ability to reason proportionally. In this vein, NCTM Curriculum and Evaluation Standards (1989) highlighted that "the ability to reason proportionally develops in students throughout grades 5–8. It is of such great importance that it merits whatever time and effort that must be expended to assure its careful development" (NCTM, 1989, p. 82). On the other hand, studies have shown that "there is evidence that a large segment of our society never acquires fluency in proportional thinking" (Hoffer, 1988, p. 285).

Torre et al. (2013) summarized proportional reasoning problems as (a) missing-value problems, (b) comparison problems, and (c) qualitative problems. Three quantities in missing value problems are given, and the fourth quantity is asked. In comparison problems, two ratios are given, and students are asked to make a comparison based on the context (e.g., best buy problems). In qualitative problems, students are asked to investigate the effect of varying quantities in a the proportional reasoning context r(i.e., increasing the orange juice makes the taste of the mixture stronger). Lamon similarly categorized the proportional reasoning problems as follows: (a) well-chunked measures, (b) part-part-whole, (c) associated sets, and (d) stretchers and shrinkers. First, well-chunked problems refer to the third quantity which includes the comparison of two measures and the relation between the two quantities that create the rate

(Lamon, 1994, p.42). Part-part-whole problems include the ratio of the glasses of water/orange juice to the glasses of a mixture (part-whole) or the glasses of water to the glasses of orange juice or vice versa (part-part). Associated sets problems provide an ill-defined relation between given quantities (e.g., aliens and spaceships). Finally, stretcher-shrinker problems, similar to qualitative problems for Torre et al. (2013), focus more on 'scaling up-scaling down' in the geometry and measurement context. While Torre et al. (2013) focused on the problem structures, Lamon (1993) highlighted the problem contexts. Yeong and colleagues (2020) state that task types influence students' performance in proportional contexts. In this regard, various types of tasks (e.g., part-part-whole relations, integer-non-integer relationships) should be incorporated into instructions to reveal students' understanding. For instance, students tend to apply additive reasoning on non-integer ratios and multiplicative reasoning on problems including integer ratios (Van Dooren et al., 2009). Problems containing composed or singleton units are significant for improving students' proportional reasoning. Lamon (1993) claims that scaling problems are more complicated than others. On the other hand, Alatorre and Figueras (2005) state that mixture problems (i.e., part-part-whole problems) are more difficult for students than other proportion problems (i.e., well-chunked problems).

Freudenthal (1983) points out that elements should support the critical elements of the learning area to promote students' understanding. Based on this statement, Lamon (1995) states that for the development of proportional reasoning, it is essential to support the fundamental mathematical components of proportional reasoning in the first level (i.e., absolute and relative thinking, covariance, and invariance, ratio appropriateness) by constructing proportional reasoning activities of the second level (i.e., relationships, unitizing, and splitting). In this study, five proportional reasoning tasks were administered. Previous studies revealed that proportional reasoning develops along a path beginning with additive and multiplicative reasoning. Even though middle school students could be unaware of the multiplicative relationships between the variables, they can construct different unit types. At the upper level, students can investigate the multiplicative relationships by working with within-between ratio and part-part and part-whole relationships (e.g., Carpenter et al., 1999; Ayan-Civak, 2020; Steinthorsdottir & Sriraman, 2009; Wright, 2014).

In this study, Carpenter and colleagues' developmental framework (1999) was used to choose and implement particular tasks. We proposed that proportional reasoning tasks be followed in instructional sequences, starting with additive and multiplicative reasoning to reveal students' thinking while they were individually solving the proportion word problems, and continuing with unitizing and norming and lastly with proportional relationships (Carpenter et al., 1999; Ayan-Civak, 2020; Lamon, 1995; Lobato & Ellis, 2010; Wright, 2011, 2014). These five proportional reasoning problems were selected based on the developmental framework created by Carpenter et al. The details of this framework are given in the Method section. The five tasks were given under three key understandings: *additive and multiplicative reasoning, unitizing and norming, and proportional relationships.* The details of key understandings of proportional reasoning are introduced below.

2.3.1. Additive and multiplicative reasoning

A substantial body of studies has concentrated on multiplicative reasoning (Behr, Lesh et al., 1983; Clark & Kamii, 1996; Fuson, 1992; Hackenberg, 2010; Mulligan & Mitchelmore, 1997; Nesher, 1988). Multiplicative reasoning is described in many ways. Wright (2005) described it as "making multiplicative comparisons between quantities" (p.363). Confrey's (1994) perspective is based on the construct of splitting, which she described as "an action of creating equal parts or copies of an original" (p.300). Alternatively, Steffe (1994) identifies it as coordinating "two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit." The construction of composite units is important for proportional reasoning (Lamon & Ellis, 2010). Throughout the present study, as with others who study proportional reasoning, we drew upon Steffe's (1994) definition of multiplicative reasoning . After all, the present study focuses on child's multiplicative schemes.

The development of multiplicative thinking and, more specifically, the transition from additive to multiplicative thinking have been the focus of considerable research (e.g., Clark & Kamii, 1996; Dooren et al., 2010). Understanding the distinction between addition and multiplication and the situations in which either operation is suitable

requires a certain level of mathematical maturity. Students who cannot distinguish between the difference may apply additive strategies for multiplicative situations since they are the most at ease with addition. Also, additive strategies require a lower level of sophistication in the cognitive schema of operation (e.g., Harel & Behr, 1995; Orrill & Brown, 2012). It has been extensively documented that students use additive strategies as familiar and comfortable methods for solving multiplicative problems (Jacobs & Willis, 2001; Lamon, 1993, 1994; Resnick & Singer, 1993). Specifically, "early preference for additive solutions to proportion problems" is a consistent finding that has been reproduced across much research (Resnick & Singer, 1993, p. 123). In this vein, additive strategy becomes the "default" strategy, and multiplicative strategies and essential building-up methods only emerge after systematic instruction (Dooren et al., 2009, p.191).

Students tend to use proportional strategies for all missing-value format problems, even when they are not appropriate (Van Dooren et al., 2009). Specifically, students in higher grades of primary education frequently solve problems multiplicatively, which requires additive answers (Cramer et al., 1993). For instance, for the problem, "Sue and Julie are running equally fast around a track. Sue started first. When she had run nine laps, Julie had run three laps. When Julie completed 15 laps, how many laps had Sue run?", 32 out of 33 pre-service teachers adopted an incorrect multiplicative approach (Cramer & Post, 1993, p. 344). Dooren et al. (2010) simultaneously focused on the erroneous use of additive and multiplicative strategies at primary level students. They noticed that the use of additive strategies in proportional situations decreased from the third to sixth grades, whereas the use of multiplicative strategies in additive situations increased. Their results demonstrate a shift from the early years of primary school, when additive strategies were applied practically everywhere, to the latter years of primary school, where proportional strategies were applied almost exclusively (Fernández et al., 2012). Understanding the nature of multiplicative thinking in proportional situations is one of the most challenging skills for children (Lamon, 2012). Even though additive thinking is seen as crucial for the progress of proportional reasoning, it also impedes the progress of proportional reasoning (Ayan & Isiksal Bostan, 2018).

There is ample evidence that students of various grades are inclined to use erroneous strategies for both proportional and nonproportional problems (e.g., Hart, 1981; Kaput & West, 1994; Karplus et al., 1983; Lesh et al., 1988; Tourniaire & Pulos, 1985). The research illustrates the challenge students have in identifying nonproportional situations from proportional situations (De Bock et al., 2007; Fernández et al., 2008; Van Dooren et al., 2005, 2008). For example, students in their early years tend to apply erroneous additive strategies for proportional problems (Ben-Chaim et al., 1998; Duatepe et al., 2005; Fernandez et al., 2012; Harel et al., 1994; Hart, 1981; Karplus et al., 1983; Misailidou & Williams, 2003; Tourniaire & Pulos, 1985). Inversely, higher-level students are inclined to use multiplicative strategies for additive problems (Fernández et al., 2008; Modestou & Gagatsis, 2007; Van Dooren et al., 2009).

In proportional reasoning literature, one of the concerns is that the ratio type influences the application of erroneous additive procedures. One of the most common difficulties of proportional reasoning problems is application of additive reasoning in multiplicative situations, especially if the numbers are given in non-integer format as task-related factors (Karplus et al., 1983; Lin, 1991; Tourniaire & Pulos, 1985; Van Dooren et al., 2010). Non-integer ratios result in more additive errors than integer ratios (Fernández et al., 2009; Harel & Behr, 1989; Kaput & West, 1994; Tourniaire & Pulos, 1985; Van Dooren et al., 2009). Studies investigated the influence of the non-integer and non-integer ratios between the given quantities on students' responses on multiplicative and additive problems (Van Dooren et al., 2009; Fernández et al., 2009). They found that students tended to apply proportional strategies to problems including integer ratios and more additive strategies to problems including non-integer ratios (Fernández et al., 2012; Van Dooren et al., 2009).

Furthermore, Son (2013) examined preservice teachers' responses to students' errors (i.e., additive reasoning) in ratio and proportion, specifically in finding missing lengths in similar shapes. A growing body of studies has focused on additive reasoning (e.g., Son & Sinclair, 2007; Steven & Stigler, 1992); however, different from other studies, Son (2013) investigated how prospective teachers analyze additive reasoning in similar tasks that integrate proportionality and geometry. She found that prospective teachers with strong knowledge of ratio and proportion identified the errors stemming from

'procedural misunderstanding' (Son, 2013, p. 65). The study showed that a lack of ability to describe the mathematical elements (i.e., errors) of the problem contributed to the inability to respond appropriately to students' understanding. Studies indicated that familiarity with problem types and overuse of specific types of problems in the context of proportional reasoning led students to use multiplicative strategies erroneously (Karplus et al., 1983). For instance, missing-value problems have been considered to be one of the significant ways to teach multiplicative reasoning throughout primary and secondary mathematics education (e.g., van Dooren et al., 2009; Degrande et al., 2017). Fourth graders typically encounter missing-value problems (Cramer & Post, 1993) or, as Vergnaud (1988) called them, "rule-of-three problems." This type of problem consists of three given values. The fourth one must be found by defining the multiplicative relation between two given values and applying the multiplicative relation to the third given value (Kaput & West, 1994; Vergnaud, 1988). The overuse of missing value problems causes application of erroneous strategies, such as using multiplicative strategies for all problems.

Therefore, it is understood that the lack of distinction between additive and multiplicative reasoning constitutes one of the obstacles to the development of proportional reasoning in students and teachers. The existing literature shows that students tend to apply multiplicative strategies for additive problems, specifically integer ratios. Thus, in the first task, we considered 'the lack of distinguishing between additive and multiplicative reasoning*' as* critical elements to analyze the prospective teachers' professional noticing of students' thinking.

2.3.2. Unitizing and Norming

Students conceptualize the unit and unit types for developing proportional reasoning (Lamon, 1994). For instance, if children only see a round pizza, they will attempt to divide a round pizza into four for their sharing activities. Why not a rectangular pizza? Do children divide a rectangular pizza into four? Children must understand that their representation will not change the answer (Lamon, 1994). A further reason for altering the unit type is that it is psychologically different to split a set of hard candies into three equal-sized groups vs. a rectangle pizza into three equal-sized pieces. Studies

show that different types of units present obstacles for students in different grades (Lamon,1994). For instance, a package of cookies and a package of candies are not identical for students because students' thinking about the problem is affected by whether they can see all the components under consideration. Herein, students must experience working with various unit types without limitation (Lamon, 1994).

Lamon (1994) describes *unitizing* as "the ability to construct a reference unit or a unit whole" (p.92) and *norming* as reinterpreting a situation in terms of that unit (Freudenthal, 1983; Lamon, 1994). The unitizing process begins with early childhood activities, continues with subitizing, and expands to counting activities. At the middle school level, one of the most notable distinctions between proportional reasoners and non-proportional reasoners is the ability to decide on the more efficient unit types- singleton units or composite units- and to choose which unit to use more appropriately within various composite units (Lamon, 1992, 1993). Lamon gives some examples of constructing units; let us think about a box containing 24 one-units (singleton units), 1 twenty-four-unit (composite unit), 2 twelve-units (composite units) or 3 eight units (composite unit). Generating and operating with progressively complicated unit structures is essential for developing more sophisticated reasoning. Several studies have emphasized the significance of composite units in developing multiplicative reasoning (Steffe, 1988; Steffe & Cobb, 1988; Steffe et al., 1983). Specifically, constructing composite units is the core of multiplicative reasoning to create and work with them (Battista & van Auken Borrow, 1995; Lamon, 1994). Steffe (1988) argues that understanding composite units is essential for students to make sense of multiplication. He claims that "for a situation to be established as multiplicative, it is necessary to at least coordinate two composite units in such a way that one of the composite units is distributed over elements of the other composite units" (Steffe, 1988, p. 19). While keeping the unit nature of components, the set is seen as a countable unit. For instance, the question "if there are six groups of 4 blocks, how many blocks are there?" encourages students to coordinate two number sequences called *iterable composite units* (Singh, 2000, p. 273). Thus, the student counts; 1–4, 2–8, 3–12, 4–16, 5–20, and 6-24. Iterating is a type of quantity distribution: one quantity across another quantity.

Kaput and West (1994) worked with sixth graders on missing value problems and proposed a model for the development of proportional reasoning. The "unit factor approach" is a part of this model. The unit factor approach was observed as a natural process of division strategies. They conducted teaching experiments and introduced three competent informal proportional reasoning levels. The unit factor approach is one of these levels students demonstrate before formally being taught in schools. In that study, they pointed out that initially, students must determine the variables (the quantities that vary). They stated that "the solver must then be able to form groups or segments that are the referents for the incrementing quantity steps and finally must be able to coordinate the two types of groups or segments to coordinate the dual incrementing acts" (Kaput & West, 1994, p. 246–247). Although reducing the ratio's denominator to 1 for the unit factor approach (i.e., singleton units) looks easy, there is a vital necessity to understand how students construct the meanings of the role of divisor and dividend (Kaput & West, 1994). It is significant that "once the two initial units are matched, a decision needs to be made regarding which quantity's unit size will act as the divisor and which will act as the dividend in forming the quotient that will be the unit factor" (Kaput & West, 1995, p. 250). The study showed "considerable confusion concerning the numerator/denominator roles of the two quantities comprising a rate" (Kaput & West, 1995, p.275). For instance, students who cannot conceptualize the unit factor approach (i.e., singleton unit) show a tendency to use a divisibility relationship between given quantities without checking which one acts as the divisor and which one as the dividend (Kaput & West, 1994). The unit factor approach regarding divisibility adjustments appears to entail a rate concept. Ultimately, the unit factor represents a rate. Even though the fraction symbol has multiple meanings, the study represents the ratio meaning. Thus, if students grasp the meaning of the fraction symbol (i.e., a per b; per 1), they can distinguish between the known and unknown quantities by using the unit size of the known quantity acting as the divisor (Kaput & West, 1994).

Although the unit factor approach containing a strong intuitive base should be formalized as an applicable algorithm for all situations (Noelting, 1980), it will not be appropriate to apply the unit factor approach in some cases (Lamon, 1994). Several factors related to context and the number structure of proportional reasoning problems

significantly contribute to students' performance. These number structure factors are the divisibility of given quantities (i.e., integer-non-integer ratios) and the number and numerical complexity (Tourniaire & Pulos, 1995). For instance, students were more likely to apply additive strategies for multiplicative problems when the given numbers formed non-integer ratios (van Dooren et al., 2010). Students can reinterpret the situation concerning the new unit (i.e., composite unit).

Composite unit coordination is the key to students' conceptualization of multiplicative reasoning conceptualization (Steffe, 1988). "Once students can iterate composite units, they can extend their multiplicative thinking to ratio situations" (Battista & Van Auken Borrow, 1995, p.3). This understanding involves taking a group of singleton units while keeping the nature of their relationship (Battista & Van Auken Borrow, 1995). For instance, a student is asked, "If there are five boxes of 4 chocolate cookies, how many chocolate cookies are there?" If the student can coordinate the number sequences (i.e., one box is 4, 2 boxes are 8, 3 boxes are 12, 4 boxes are 16, and 5 boxes are 20), then s/he can conceptualize the iterable composite unit. Creating composite units and working with them is the key to covariation and building up (Battista & van Auken Borrow, 1995; Behr et al., 1992; Lamon, 1994; Steffe, 1988). This procedure involves creating a unit of units, treating it as a single entity (i.e., a composite unit), and working with the composite unit by keeping track of the number of times it is iterated. In this vein, the iterating composite unit is described as taking the set of unit items as a new unit (i.e., composite units) and iterating this new unit by keeping the nature of the relationship of the quantities (Steffe, 1994).

The solution procedure appears to be revealed in anticipation of an appropriate unit. Singh (2000) points out that "multiplicative situations in proportional reasoning require an appropriate schema of the unit before solution procedures can be implemented" (p.289). According to Singh (2000), the anticipation of composite units needs three critical components:

1. In order to make sense of ratio and proportion problems, the student needs to conceptualize the iteration of the composite unit.

2. The student needs to sufficiently understand the meaning of multiplication and division to recognize the connection in the iteration process.
3. The student needs to abstract the iteration process adequately to reflect on it and reconceptualize the iteration in terms of that unit.

This study implies that students should not be presented with the unit strategy until they understand unit coordinating schemas (i.e., unitizing, unit ratios, iterating of unit ratios, composite units, and iterable composite units). This skill can be achieved by posing problems containing unit types to develop unit coordination schemas of students (iterating ratio units) (Singh, 2000). According to Behr et al. (1992), this skill helps students improve "flexibility in unitizing and reuniting quantities" (p. 630).

Buforn et al. (2022) concentrated on prospective teachers' skill to notice students' understanding of the unitizing process as a part of proportional reasoning. They found that more than half of prospective teachers (65 out of the 91) applied singleton units for solving proportional problems. On the other hand, prospective teachers who applied additive reasoning as an erroneous strategy or incorrectly used unit ratio had difficulties with the unitizing process. Furthermore, pre-service teachers who identified the unitizing process as a critical element of the task and recognized the characteristics of students' reasoning proposed specific instructional actions with a conceptual focus (Buforn et al., 2022).

To sum up, the conceptualization of unitizing and norming is one of the crucial processes for developing more sophisticated reasoning for the ratio and proportion domain (e.g., Lamon, 1994; Steffe, 1994). Thus, we considered the critical ideas and potential difficulties in the second task, namely the Grocery and Balloon problems in this study, to analyze the prospective teachers' professional noticing of students' thinking.

2.3.3. Proportional Relationships

Understanding of proportional relationships as another critical component has been seen as a critical part of higher-order reasoning that allows young students to make

judgments based on proportional reasoning (Piaget & Inhelder, 1975). Experiments on proportional reasoning in children involve several tasks, ranging from qualitative judgments (such as more or less, further or nearer) to quantitative interpretations using numbers and comparison problems in a different context (i.e., scaling or mixture problems) (Ayan & Isiksal, 2019; Bruner & Kenney, 1966; Cramer & Post, 1993). These studies share a common theme: to handle proportional problems, children must assess second-order relations, to quote a line from Piaget and Inhelder (1975), "relationships between relations". Young children's early interactions with ratio and proportion involve analyzing the relative size relationships between the pieces of an object or between other objects on a visual and intuitive level. Since proportional reasoning concerns the relationship between quantities rather than pure numbers, it is crucial that thinking about relationships occurs well before symbolic instruction so that children understand there is more to a situation than simply extracting the numbers and operating blindly with them (Lamon, 1999). Therefore, it is vital to help children build on their intuitive knowledge by analyzing different proportional contexts and making relationships more explicit.

The quantities, a whole and two parts, are associated with proportions expressed as ratio relationships, which can be represented by two distinct schemas: a part-whole ratio and a part-part ratio (Noelting, 1980; Singer & Resnick, 1992). These relationships are described as "part-whole schemas represent the relationship of a part to a whole, while part-part schemas represent the relationship of two parts with each other" (Singer & Resnick, 1992, p. 231). For example, there are ten marbles; six are red, and four are black. The relationship between marbles can be represented as part to the whole (red with whole or black with whole) or part to part (red with black or black with red). Research on proportional reasoning has employed problem contexts that implicitly encourage children's adoption of part-whole or part-part strategies without specifically analyzing these aspects of children's reasoning process (Singer & Resnick, 1992). Part-part reasoning is required for 'every' relationship, and part-whole reasoning is required for 'out of' relationships (Lamon, 1995). For instance, the relationship between the number of boys and the number of girls in a classroom provides a part-part relationship (i.e., for every situation). The relationship between the number of girls in a classroom and the whole class provides a part-whole

relationship (i.e., out of). Studies show that some types of problems invite students to make judgments about part-part and part-whole reasoning (e.g., Karplus et al., 1983; Lamon, 1995; Noelting, 1980a, 1980b; Singer & Resnick, 1992). For example, mixture and geometrical scaling problems highlight the part-part-whole and within-between ratios. The most widely known mixture problem is Noelting's Orange Juice Puzzle (1980), including comparing orange juice and water in mixtures. Mixture problems are different from other rate problems. First, the ratio components produce a new object (e.g., a red and blue paint mixture makes purple paint) or an altered object (e.g., less water and more orange juice make a strong taste of the mixture). Moreover, they demand an understanding of what occurs when two elements are combined.

Nonetheless, difficulties regarding the part-part-whole relationship have received less consideration. Mixture problems contain a part-part-whole relationship in nature. However, classroom instruction or mathematics textbooks primarily focus on part-part relationships (Kilpatrick et al., 2001). Spinillo and Bryant (1991) state that early-grade students use part-part and "half/doubling" relationships rather than part-whole relationships. On the other hand, proportional reasoning must include a part-whole relationship which is not explicit in a ratio of a part-part relationship. Most research focuses on the part-part ratio without considering the part-part-whole relationship (Singer & Resnick, 1992; Yeong I et al., 2020). This limited perspective on ratios may lead to difficulties when students confront part-whole relationships (Yeong I et al., 2020). Furthermore, most textbooks provide examples of part-part and part-whole relationships as decontextualized elements without explaining why part-part and part-whole relationships are represented with the same notation. This situation also necessitates that students determine what each term in a ratio represents within the given context. For example, there are three glasses of orange juice and two glasses of water. The ratio represents the relationship between orange juice and water. It is interpreted as "for every two glasses of water, there would be three glasses of orange juice," or as "for every glass of water, there would be 3/2 glasses of orange juice." Prior studies show that students and teachers have difficulties comprehending this contextual relationship to make inferences about proportional relationships (Yeong et al., 2020).

Children's visual perceptual reality is a starting point for discovering the relationships between quantities (Streefland, 1985; Van den Brink & Streefland, 1979). Children's early experiences of ratio and proportion involve a visual and intuitive evaluation of the relative size relationships between the features of an object or between different objects. For instance, most preschoolers probably can intuitively determine if a drawing or picture looks to be scaled or enlarged correctly or incorrectly (Harel et al., 1992). That is, they can easily evaluate the distortion of images (Lamon, 1995). Children employ self-invented strategies that are primitive, content-bound, and based on adding, doubling, and halving (Hart, 1984). Even though students intuitively understand the relationships, they require formally recognizing the relationships of similar geometrical shapes. Geometric similarity problems, including stretcher/shrinker tasks, are seen as difficult problems for early sixth graders (Kaput & West, 1994; Lamon, 1993). These problems are proportional reasoning problems in that students multiplicatively compare the sizes of objects. Figures need to be identical in shape but not necessarily of the same size to be considered similar. If all pairs of corresponding sides are in proportion and all of the corresponding angles are of equal measure, then the two figures are similar in form (Cox & Lo, 2014). Studies report that students apply some strategies to differentiate between figures: appearance-type strategy (i.e., alike/different shape), length-type strategy (i.e., corresponding the length comparison/constant gap strategy), or relationship-type strategy (i.e., the constant difference/the constant ratio) (Cox & Lo, 2014). These various strategies show that it is essential to consider how they reason about geometrical similarity (Son, 2013). Angles of similar figures are congruent, but not always the sides. They refined this notion of similarity to the extent that, for example, similar figures match exactly when enlarged or shrunk since their corresponding sides are proportional to a scale factor. Then, students address similarity tasks by establishing a proportion utilizing fractions that link corresponding lengths of shapes or length relationships within a shape. (Son, 2013). Students gain a conceptual and procedural understanding of similarity through similarity tasks, thereby integrating proportionality and geometry (NCTM, 2000). In this vein, Son (2013) points to the critical points to find a missing value in similar figures:

(a) understanding the concept of similarity,
(b) recognizing the proportionality embedded in similar figures by comparing lengths and widths between figures (between ratio) or by comparing the length to width within a rectangle (within ratio) or determining a scale factor, called the conceptual aspect of similarity,
(c) representing the relationship between two similar figures using a ratio, a proportion, or a scale factor (p. 52).

As stated earlier, while explaining the relationship between quantities in scaling problems, researchers used two types of ratios: *within ratios* and *between ratios*. In proportional contexts, within and between ratios have various definitions which are almost opposite. Some researchers define within ratio as the relationship between different measure spaces, such as the ratio between orange juice and water in each drink, and between ratios *as* the relationship within the same measure spaces, such as the ratio between the number of glasses of orange juice and water between two mixtures (e.g., Karplus et al., 1983; Lamon, 1994). Noelting indicated the relationship between quantities on the orange-juice task. He pointed to the within ratio as the ratio between orange juice and water in each mixture and the between ratio as the ratio between the orange juice and water in two mixtures (Noelting, 1980a,1980b). Lamon (1996) identified within strategy as "a student is said to be using a within strategy or a scalar method when he or she equates two internal or within measure space ratios and uses the sameness of scalar operators to determine the missing term" (p. 95). In addition, she explained between ratios strategy as "a student is using a between strategy or a functional method when he or she equates two external or between measure space ratios and relies on the functional relationship between the measure spaces to find the missing term" (p. 96)

Vergnaud (1994) represented the relationship as a comparison between the same measure space (scalar ratios) and a different measure space (functional rates). As seen in Figure 2.1 below, the context, including the number of cakes and the corresponding cost, indicates that the scalar ratios mean "How many 3-cakes (composite unit) go into 12 cakes?". The answer "there are four units of 3-cakes" shows the scalar ratios (within quantity types), and then the corresponding cost is multiplied by 4, which is called the

scalar factor, to find the missing value. Besides, functional rates (between quantity types) lead to considering the relationship between the cakes and cost. The relationship between the number of cakes and cost is 1:2, so the cost should be multiplied by 2 to find the cost of twelve cakes.

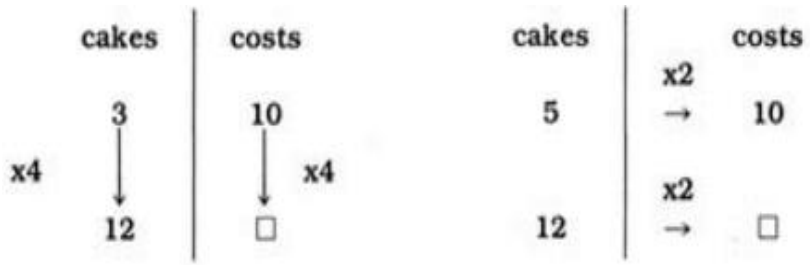


Figure 2.1 Scalar ratios and functional rates (Vergnaud, 1994, p.51)

Similarly, Freudenthal (1973) identifies ratio as the relationship within a system and between ratios as the relationship between two systems. Lamon (2007) eliminates this confusion by using different terminology, "within or between systems" or "within or between measure spaces" (p. 634). Cramer (1993) emphasizes that students can notice both within and between ratios in proportional situations, "knowing that both within and between relationships exist in students' alternative strategies." (p. 163). In order to avoid confusion in this study, we use the definition of these terms, as Noelting and Lamon defined. For instance, consider the following problem, which we used in this study:

> Derya, Ahmet, and Kagan bought three balloons and paid 1.5 liras. How much do they need to pay for balloons for their class of 24?

Within and between ratios leads to recognizing the relationship between the two multiplicative comparisons. For instance, buying three balloons for two dollars illustrates the same relationship as buying nine for six dollars. Although the quantities change in each case (i.e., the number of balloons and the total price change), the relationship remains the same (i.e., one balloon's price). While quantities including a ratio change together, which is called *covariance*, their relationship remains constant, which is called *invariance* (Kaput & West, 1994; Lamon, 1995). In this regard, "all at the same time, something changes, but something else does not change" (Lamon, 1995,

p. 176). This statement indicates that students should consider covariation and invariation in a given case (Lamon, 1995). As Lamon (2007) stated, "Proportional relationships involve one of the simplest forms of covariation. Two quantities are linked to each other so that when one changes, the other also changes precisely with the first quantity" (p. 648). According to Thomson and Carlson (2017), covariational reasoning is most prevalent when an individual considers monitoring multiple quantities' values simultaneously. For example, students should recognize that both ratios are linked, and the second ratio (i.e., nine balloons for six dollars) is three sets of the first ratio (i.e., three balloons for two dollars). The quantities composing a ratio of price to balloons address the unit price (i.e., unit ratio/singleton unit).

Scaling problems and mixture problems play an essential role in understanding *part-part-whole* and *invariant-covariant* relationships; however, studies show that students are less successful in these tasks (Kaput & West, 1994). Numerous characteristics stated in the research on proportional reasoning may explain why these kinds of tasks, in particular, are more challenging and do not appear to be areas where proportional reasoning can be utilized by students (Lamon, 1993). For instance, the numbers used in the task are critical factors affecting students' strategy choices (Kaput & West, 1994; Noelting, 1978). In this respect, Lamon (1994) claims that students were able to apply the scalar method or between ratios strategy when one quantity was a scalar multiple of another for the following problem:

> *If I can make five team shirts with 7 yards of material, how many yards of material will I need to make a team shirt for each of fifteen children on the soccer team? (p.95)*

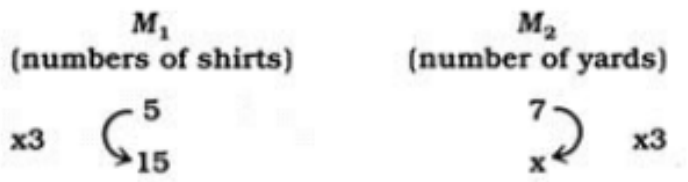


Figure 2.2 The scalar method for finding a missing value in a proportion: an instance of norming (Lamon, 1994, p.95)

Students could see the relationship: three as a scalar operator transforms the five into fifteen. Consequently, seven should be multiplied by the same scalar operator to get twenty-one (Lamon, 1994). Although between strategy is a more natural and preferred strategy for students, the within strategy is more appropriate to illustrate the linear relationship in the proportions (Karplus et al., 1983). As in the example, there is a relationship (i.e., between strategy) between five shirts and seven yards of fabric. In this problem, five shirts were made with seven yards of material. Therefore, one and yards of fabric should be used to make one shirt (i.e., non-integer scale factor), and the ratio of the number of yards to the number of shirts gives the constant ratio (i.e., unit ratio). In the similarity task, students who do not conceptualize or ignore the constant ratio between the width and length of the rectangle fail to consider how they can find the missing value if the quantities do not have an integer relationship (Lamon, 1993). Noelting (1980a, 1980b) conducted a study with children to investigate their comparison of mixtures of varying numbers of glasses of orange juice and water. The findings show that when the amount of orange juice is equal in both glasses, even younger students can compare the two mixtures. They concentrated on the amount of water in two mixtures without making any computation.

On the other hand, even older students (aged 10-12) were not able to make a comparison of the two mixtures when the quantities did not have a scalar relationship (i.e., five glasses of orange juice and two glasses of water, and seven glasses of orange juice and three glasses of water). Since students were not able to solve this type of problem, which contains the non-integer relationship between quantities (Fernández et al., 2011; Noelting, 1980a, 1980b; Rupley, 1981), the types of proportional reasoning required in a task are likewise determined by the numbers utilized in the activity (Noelting, 1980a, 1980b). Thus, in the fourth and fifth tasks of this study, we considered '*proportional relationships such as part-part and part-whole relationships and invariant and covariant relationships"* as critical elements to analyze the prospective teachers' professional noticing of students' mathematical thinking.

2.4. Prospective Teachers' Professional Noticing of Students' Thinking in the Domain of Proportional Reasoning Through a Video Case Enhanced Learning Environment

The ability of mathematics teachers to recognize the essential elements of a teaching environment and to interpret them from students' stances to decide on instructional actions is a crucial part of teaching practice (Mason, 2002). Teachers must possess various knowledge and skills to assist students in developing multiplicative reasoning and equip them with sophisticated ideas involving proportional reasoning (Llinares, 2013). In this sense, teacher noticing shapes teachers' perspectives on students' understanding (Jacobs et al., 2010; Sherin et al., 2010). The ability to be attuned to students' understanding of mathematics is a specific feature of the teacher's noticing. Understanding and analyzing students' mathematical thinking requires "reconstruction and inference" of their understanding based on what they say or write. The teacher's ability to recognize students' mathematical thinking is more than simply indicating whether their replies are correct. It entails determining in what ways and how the students' answers are or are not relevant from a mathematical learning perspective (Llinares, 2013; Wilson et al., 2013).

The National Council of Teachers of Mathematics (NCTM) discusses that the development of proportional reasoning is so significant that "it merits whatever time and effort that must be expended to assure its careful development" (NCTM, 1989, p. 82). However, it is rarely explicitly stated in curriculum materials, and many teachers do not have sufficient specific content knowledge to teach proportion and ratio (Ayan, 2020; Hilton et al., 2016; Lobato et al., 2011; Sowder et al., 1998). Elementary school curricula do not emphasize multiplicative structures and frequently point out multiplication and division as an extension of addition and subtraction. Specifically, the distinctions between addition and multiplication have not been stressed adequately (Behr et al., 1992; Hilton et al., 2016; Sowder et al., 1998). Thus, there should be more focus on teaching the critical points of proportional reasoning at elementary and middle school levels to improve students' understanding (Lamon, 2005). For teachers to assist their students at the elementary level in developing proportional thinking, they must comprehend the characteristics of proportional reasoning and be aware that it is

developmental, arises gradually, and evolves over several years (Thompson & Bush, 2003). Teaching students to answer proportional problems solely through the cross-product method, like an algorithm, does not enhance students' proportional thinking skills (Thompson & Bush, 2003). In order to support students, teachers must be able to reason proportionally and be aware of how students reason with ratios.

Several studies primarily concentrate on the difficulties experienced by upper-level students in ratio and proportion. Researchers have primarily centered on teachers' content and pedagogical knowledge (Ben-Chaim et al., 2002) and teachers' interpretations of students' solution strategies in the context of ratio and proportion (Hines & McMahon, 2005). However, a few studies have revealed that teachers might have the same experience and difficulties in proportional situations as their students, such as being unable to make a distinction between additive to multiplicative reasoning (Lobato et al., 2011; Orrill & Brown, 2012**;** Sowder & Philipp, 1995**;** Sowder et al., 1998). For instance, teachers frequently struggle to comprehend the distinction between additive and multiplicative situations and lack strategies to assist their students in recognizing or solving these kinds of problems (Behr et al., 1992). Sowder (2007) claims that while teachers may know mathematical methods and procedures, they may lack conceptual knowledge and reasoning skills. Even teachers with a strong background in this field may fail to teach their students the fundamental concepts of multiplicative reasoning (Thompson & Thompson, 1996). Thus, how to describe what teachers know and how to implement this knowledge in practice has become an important topic for researchers and mathematics educators (Llinares, 2012).

Numerous studies have spent the last two decades attempting to comprehend teachers' noticing better and focusing on promoting the development of teacher noticing (Jacobs et al., 2010; Schack et al., 2017; Sherin et al., 2011; van Es, 2011). One of the most fundamental goals of teacher preparation is to provide teachers with the required knowledge and skills to keep learning and developing over time; there should be a careful evaluation of what this knowledge and skills involve (Hiebert et al., 2007). Prior studies suggested that classroom experience improves the ability to notice, which, unfortunately, prospective teachers do not have. Even though prospective teachers do not have classroom experience, they can notice, and noticing ability can

be cultivated (Males, 2017). In order to investigate and develop prospective teachers' and in-service teachers' noticing skills, researchers have utilized different tools such as students' written work (e.g., Fernandez et al., 2013; Jacobs et al., 2010) or different video clips of classroom instruction (e.g., Jacobs et al., 2010; McDuffie et al., 2013) or one-to-one students interview, teachers' live classrooms, and guided reflection (Fernandez et al., 2013; McDuffie et al., 2013; Star & Strickland, 2008). Fernández and colleagues (2013) defined how PSTs attended to and interpreted students' mathematical reasoning when examining students' written work, including proportional and non-proportional tasks. They generated four level descriptors to characterize the professional noticing of students' mathematical thinking in ratio and proportion. Castro Superfine and Li (2011) examined the change in preservice teachers' knowledge and ability to analyze students' thinking using a video case curriculum. The results show that the video case curriculum limited prospective teachers' ability to analyze students' thinking. Moreover, some research indicated that prospective teachers' noticing ability could be promoted using video clips (Fernandez et al., 2012; Star & Strickland, 2008).

Recent research has highlighted developing prospective teachers' ability to notice primarily based on watching and discussing videos of practice (Schack et al., 2013; Star & Strickland, 2008; van Es & Sherin, 2002; van den Kieboom et al., 2017). For instance, van Es and Sherin (2002) investigated how video clips of mathematics classroom instruction promote prospective teachers' noticing skills. The findings showed that prospective teachers changed their noticing skills after watching chosen video clips from just providing a chronological description to attending to and interpreting the noteworthy mathematical details of teaching and learning.

Fernandez et al. (2012) aimed to characterize the development of professional noticing of students' mathematical thinking in an online context. In the study, prospective teachers worked individually or collaboratively on proposed tasks (i.e., written students' work, theoretical papers about related topics, and video clips). Then, they shared their ideas in an online discussion environment. They found that participation in online discussions allowed prospective teachers to "spend more time constructing their arguments and also deliver a higher degree of joint elaboration and construction

of interpretations" (Fernandez et al., 2012, p. 748). From this perspective, the findings suggest that an online discussion environment in which they can share and defend their interpretations of students' mathematical thinking based on students' written work and video clips supports the development of prospective teachers' noticing (Strickland, 2008).

Studies on how teachers-to-be learn to notice things have shown that different video types affect how teachers-to-be learn to notice things (Superfine et al., 2017). The videos used for teacher education have different purposes and foci, stressing specific aspects of learning and teaching while hiding others. For instance, recent research indicates that classroom videos can capture the complexity of classrooms. In this way, classroom videos have been used to encourage and record the noticing of prospective teachers in a way that is more like what happens in classrooms. Earlier work by Sherin and van Es (2005) revealed that after analyzing classroom videos of their teaching methods, both in-service and preservice teachers changed what they noticed and how they discussed what they noticed. While in-service teachers changed their focus from the teacher's statements in the video to the students' statements, in-service teachers changed their focus from the chronological events in a class to specific events during the lesson. Overall, what Sherin and her colleagues found showed that watching class videos could change how prospective and in-service teachers act in the classroom (Sherin & Han, 2004). Therefore, in this study, we used video clips as contexts for providing opportunities to develop prospective teachers' noticing skills.

On the other hand, even though classroom videos encapsulate the classroom events, prospective teachers tend to focus on teachers' or students' behavior or classroom management and struggle to pay attention to students' mathematical thinking (Star & Strickland, 2008). Before prospective teachers can make sense of classroom features, Star and colleagues (2011) argue that they must first learn to pay attention to relevant elements. Although classroom videos can help prospective teachers to develop their noticing skills (Santagata et al., 2007; Star& Strickland, 2008), they also focus on the classroom environment rather than students' mathematical thinking (Superfine et al., 2015). Star and Strickland (2008) investigated what prospective teachers notice when watching a video and whether their attending skill, as the first component of

professional noticing skill, can be developed. They pointed out that when preservice teachers are not very observant of content, it is difficult for videotaped courses to act as a tool to promote prospective teachers' ability to notice. Early studies indicated that novice teachers were more likely to show a tendency to notice less related details, such as general teacher and student actions and classroom materials, rather than significant pedagogical actions or students' mathematical thinking (Sherin & van Es, 2005; Star & Strickland, 2008; Star et al., 2011). Researchers reported that merely watching classroom videos may not guide prospective teachers to notice what they attend to and interpret what mathematics educators aimed prospective teachers to notice (Star & Strickland, 2008).

van Es and Sherin (2006) investigated how specific types of video-based professional development support prospective teachers' noticing skills. Their study took place in two different video club contexts, designed to support teachers' focus on students' mathematical thinking and discuss teaching and learning issues. For the first video clip - Mapleton -, the researcher chose classroom video excerpts in which students' thinking was prominent. While teachers who participated in the first video club -Mapleton- concentrated more on students' thinking, as they attended to students' strategies and interpreted students' mathematical thinking, their noticing became more focused.

In contrast, teachers in the second video club-Wells Park Video Club-, containing teaching and learning issues, adopted a broad and general perspective. In other words, Wells Park Video Club concentrated less on student thinking compared to Mapleton video clubs. Even though van Es and Sherin (2006) pointed out that both approaches were significant, the apparent narrowing of viewpoint among the Mapleton participants reflects a crucial objective of mathematics education reform: for teachers to pay close attention to students' mathematical thinking.

Krupa et al. (2017) and Teuscher et al. (2017) recommended interventions to develop novice and experienced teachers' skills in noticing students' mathematical thinking. Krupa and colleagues (2017) utilized video recordings of a task-based interview with a student to investigate the development of teachers' noticing skills. The findings indicate that prospective teachers provided evidence of increased attention to and

interpretation of students' mathematical thinking but a slight change in responding. On the other hand, Teuscher and colleagues (2017) examined the effects of the length of video analysis on prospective teachers' noticing of students' mathematical thinking in their instruction. They claimed that long-term video analysis greatly impacted PSTs' professional noticing abilities in real-time. These results indicate a distinction in the cognitive demands of the three components of professional noticing and the requirement for a different method to enhance the ability of PSTs in the third component (i.e., responding) (Criswell & Krall, 2017). Therefore, in the current study, I centered on student interviews in video clips to advance prospective teachers' noticing of student thinking.

Furthermore, Ulusoy and Cakiroglu (2020) explored prospective teachers' noticing students' mathematical understandings in micro-case videos. They used micro-case video clips to support prospective teachers' noticing skills. They described the micro-case video clips as "educational video clips that involve a collection of specifically selected and edited cases related to an individual learners' thinking process on a particular content-related task or a problem situation in a non-classroom environment" (Ulusoy& Cakiroglu, 2020, p. 255). They found that in the group discussion of micro-case video clips, prospective teachers showed progress in perspective teachers' professional noticing of students' mathematical thinking. In this vein, I also utilized micro-case video clips, including problem-solving interviews with one student, to investigate prospective teachers' professional noticing of students' mathematical thinking.

In previous research, video clubs did not focus on a specific content domain (e.g., Sherin and van Es, 2009; others). However, focusing on different mathematical domains is necessary to analyze prospective teachers' noticing skills (Amador et al., 2022; Fernández et al., 2013; Walkoe, 2015). Blomberg et al. (2011) investigated differences in noticing in several fields, such as mathematics and social science, but they did not explore the details of noticing in a single content domain. Thus, the findings of recent studies revealed that teachers' noticing skills are domain specific, so teachers' noticing skills should be examined in various domains to determine where improvements are necessary (Ivars et al., 2020; Kaider et al., 2021). Based on these

results, the present study focused on the prospective teachers' noticing of the domain of ratio and proportion. Proportional reasoning is a significant and complex domain of mathematics. It has been known that proportional reasoning is challenging for students and teachers (Hilton & Hilton, 2019; Lamon, 2007). According to Amador et al. (2022), "Teachers need specific support to advance their own proportional reasoning content knowledge to make instructional decisions that are responsive to how students reason about proportions" (p.2). Although there are studies on teachers' noticing of students' thinking in the context of ratio and proportion, they mainly concentrated on in-service teachers and a particular content (i.e., additive-multiplicative thinking), and researchers used classroom videos (Amador et al., 2022; Fernandez et al., 2013; LaRochelle, 2018; Llinares, 2013). The literature on proportional reasoning and prospective teachers' noticing of students' understandings helped us define the key understandings of proportional reasoning. Carpenter and colleagues' developmental trajectory of proportional reasoning allowed the researcher to uncover students' thinking during problem-solving.

Summary

Professional noticing of students' mathematical thinking has been a promising approach for supporting prospective teachers' growth in professional development (Jacobs et al., 2010; NCTM, 2000; Sherin et al., 2011). Studies have shown that noticing cannot emerge and develop naturally but can be learned (Star & Strickland, 2008; Stockero, 2008). Even though prospective teachers face difficulties in learning it (Stockero, 2008; Peterson & Leatham, 2009), studies further show that with support, prospective teachers can improve their noticing skills (Fernandez et al., 2012; Goldsmith & Seago, 2011; Jacobs et al., 2010). Therefore, there is a need for professional development programs that promote prospective teachers' skills to notice students' mathematical thinking (Amador et al., 2021). Students' mathematical thinking can serve as a constant and coherent professional development source for prospective teachers, but only if they learn to notice students' thinking. In this respect, student thinking is vital for researchers of professional noticing (Jacobs et al., 2011; Santagata, 2011).

In order to promote prospective teachers' noticing skills, I framed my view of noticing with Jacobs and colleagues' framework for specialized teacher noticing—professional noticing of children's mathematical thinking (Jacobs et al., 2010). They narrowed their focus and concentrated on professional noticing of elementary students' mathematical thinking with three interrelated components: "attending *t*o children's strategies, *i*nterpreting children's understanding, and deciding how to respond on the basis of children's understandings" (p. 172). They highlighted that teachers' decision-making process was constant, so they must attend to and interpret students' thinking to be able to respond (Sherin et al., 2011).

While video clips afford productive discussions on noticing, researchers aim to fully understand the nature of teacher noticing in a video-case-based context (van Es & Sherin, 2008). Early results of such studies found that while watching videos, teachers, specifically prospective teachers, notice various details (e.g., classroom management, general teacher and student actions) rather than noteworthy details in student thinking. To address this issue, researchers have examined what kind of video clips should be used and how the video-case-based learning environment influences the development of teachers' noticing of student thinking. Utilizing one-on-one student interview video clips rather than classroom videos has enabled prospective teachers to learn how to attend to and interpret students' mathematical thinking (McDuffie et al., 2013; Schack et al., 2013; Star & Strickland, 2008). In this respect, the promising studies conducted with prospective teachers to promote noticing abilities have highlighted micro case video clips as a powerful tool for teachers' professional development, so mathematics educators suggested the use of selected video clips to assist in developing teachers' noticing of student thinking (Ulusoy, 2020; Ulusoy & Çakıroğlu, 2018). Micro-case video clips depicting only a single student engaging in the problem-solving process have allowed prospective teachers to attend to students' strategies by eliminating the complexity of classrooms (e.g., Krupa et al., 2017; Teuscher et al., 2017). Such studies share evidence that in a sustained and intense experience centered on elucidating student mathematical thinking through video cased-based analysis, prospective teachers enhanced their noticing skills. Therefore, these studies indicate a transition of professional noticing from general to context-specific skills (Amador et al., 2020; Criswell & Krall, 2017).

Researchers have emphasized the importance of working on particular mathematical domains to analyze prospective teachers' noticing of students' mathematical thinking (Amador et al., 2022; Fernández et al., 2013; Walkoe, 2015). Even though these studies highlight a need for further examining the professional noticing of prospective teachers in well-defined mathematical areas, existing studies have been limited, specifically in the domain of proportional reasoning (e.g., Amador et al., 2022; Fernandez et al., 2013; LaRochelle, 2018; Llinares, 2013; Son, 2013). Therefore, it is essential to support prospective teachers in noticing student understanding of proportional reasoning, which is a significant and complex domain of mathematics (Lamon, 1994; Lobato & Ellis, 2010). For this reason, in this study, proportional reasoning was chosen as a content domain to support prospective teachers' professional noticing of students' thinking.

To summarize, research has shown that prospective teachers' professional skills of noticing students' mathematical thinking can be developed using micro-case video clips (e.g., Ulusoy, 2020). In light of previous studies on teacher noticing and proportional reasoning, the present study aimed to examine the development of professional noticing of students' mathematical thinking in the context of proportional reasoning through a video case-enhanced environment. This study also aimed to contribute to the existing literature about how micro-case video clip viewing advances prospective teachers' professional noticing of students' thinking across Jacobs and colleagues' (2010) framework.

CHAPTER 3

METHODOLOGY

This study examines the development of prospective teachers' professional noticing of students' mathematical understanding in the context of proportional reasoning throughout video case-based teaching experiments. This chapter began with a discussion of why classroom teaching experiment methodology was selected. Then, the detailed information about the context of the study, participants, and the data collection tools were given. Next, considering the classroom teaching experiment characteristics, the teaching experiment planning and implementation process were discussed. Finally, the issues related to trustworthiness and limitations were outlined.

3.1 Design of the Study: Teaching Experiment Methodology

Within the existing studies in mathematics education, teaching experiment has been applied as a different methodological approach (e.g., Steffe et al., 2000; Cobb, 2000; Simon, 2000). The main aim of conducting teaching experiment methodology is for researchers to provide firsthand experience about students' mathematical learning and reasoning (Steffe et al., 2000). For instance, in the case of the constructivist teaching experiment, the primary purpose is to influence the cognitive restructuring of students while interacting with the researcher one-to-one during the process (Cobb & Yackel, 1996, p. 177). Even though the constructivist teaching experiment was derived from Piaget's clinical interviews and focuses on students' mathematical knowledge, it is more than a clinical interview (Steffe et al., 2000). Cobb and his colleagues extended this approach to investigate students' and teachers' learning in the alternative classroom contexts developed in collaboration with teachers (Cobb, 2000, p. 312). In this study, classroom teaching experiment methodology was conducted to study with prospective

teachers and to document the development of prospective teachers noticing skills for proportional reasoning.

Classroom teaching experiment has seen an extension of constructivist teaching experiments involving individual construction and social interactions. Previous studies show that learning involves both individual construction and social interaction processes, and social interactions have been seen as catalysts for students' mathematical development (Cobb & Bauersfeld, 1995; Cobb & Yackel, 1996). A classroom where social interaction occurs is a primary learning environment for students, teachers, and researchers because teachers and also researchers can make sense of noteworthy mathematical events and understand students' perspectives in the classrooms (Ball, 1993b: Cobb et al., 1990). Therefore, the teaching experiment sessions as the social context were designed to develop prospective teachers' noticing of students' mathematical understandings through interaction with each other and the researcher.

As a theoretical stance, the emergent perspective guides classroom teaching experiments (Cobb, 2000; McClain, 2002). According to this perspective, learning is characterized as a process of active individual and joint reconstruction that reflexively relate to each other (Cobb et al., 1994). More specifically, the emergent perspective coordinates students' mathematical development with the social process in the social context of the classroom (Cobb, 2000; McClain, 2002).

In the light of the emergent perspective, classroom teaching experiment shows the significance of examining individual students' mathematical actions that are situated in the social world of the classroom in which they participate (Cobb, 2000, p., 310). The classroom teaching experiment in this study sought to investigate the development of prospective teachers' noticing of students' understandings while interacting with each other through group discussions in the classroom. During this classroom teaching experiment, as a facilitator, I engaged prospective teachers in individual video clip analyses and group discussions about students' mathematical understanding of the video clips.

Another significant characteristic of a teaching experiment involves sequence of teaching episodes (Steffe, 1983). This sequence includes a teacher-researcher, one or more students, a witness-researcher of teaching episodes (in some cases), and data recording (Steffe et al., 2000, p. 274). Essentially, teaching experiment as a dynamic process has a series of teaching sessions and individual interviews during an extended period (Cobb & Steffe, 1983).

Considering the aims of this study and the nature of the classroom teaching experiment, I had interactions with participants for about nine weeks. Moreover, as seen in Table 3.6 below, I kept our communication till they sent us their last version of the lesson plan.

3.2. Context of the Study

3.2.1. Broad context of the study

For this study, an elective course was offered for fourth-year prospective teachers in Elementary Mathematics Education Program (EME) at a public university in Turkey. The details about this program are given below.

The elementary mathematics education program is a four-year undergraduate program that provides education to those who will work as elementary mathematics teachers in middle schools. As requirements of this program, in the first two years, students have to take content courses (e.g., General mathematics, calculus, discrete mathematics, linear algebra). In the following two years, students need to take methods courses, school experience, and practicum courses. The EME curriculum is given in Appendix C. The method course is given in the third year. The book titled "Elementary and middle school mathematics: Teaching developmentally" by Van de Walle, Karp, and Bay-William (2013) is used as a guide for students in the teaching and learning process. During two terms, prospective teachers discuss each mathematical topic, including proportional reasoning, with the course instructor based on content knowledge and pedagogical content knowledge of related topics. The mathematics teaching methods courses provide great opportunities to make sense of the

mathematics teaching and learning process for students (ages 6-14). Then, prospective teachers apply their teaching and learning designs at middle schools during practicum courses in the fourth year. In the first term of the practicum courses, prospective teachers observe the teacher and students in the schools in their natural teaching and learning environment. In the second term of the fourth grade, prospective teachers observe the class and have teaching practices under the guidance of class teachers and course instructors.

On the other hand, there is no course in the mathematics education program to provide direct and specific experience about students' understanding and thinking in any subject domain. However, prospective teachers are required to examine students' understanding of specific mathematical topics. In light of these requirements in the mathematical education program, my supervisor, my co-adviser, and I decided to open an undergraduate elective course named "Projects in Elementary Science and Mathematics Education" during the spring semester of the 2018-2019 academic year in Ankara, Turkey. I provided details of this elective course and the participants below.

3.2.2. The context of the “Projects in Elementary Science and Mathematics Education” course

In order to develop the noticing skills of prospective teachers of students' mathematical thinking in the domain of proportional reasoning, an elective course named "Projects in Elementary Science and Mathematics Education" for fourth-year prospective teachers was opened and designed. This course took fourteen weeks during the spring semester of 2018-2019 and nine weeks’ of the course were used for this study. The course catalog described this course as providing students video clips one-to-one to investigate students' mathematical understandings based on attending, interpreting, and responding to provided task approaches. Specifically, this course provided a unique chance for prospective teachers to examine students' work on proportional reasoning tasks.

In order to answer my research questions, lesson plans about ratio and proportion were first requested to collect data about their content knowledge and knowledge of students' mathematical thinking. Then, individual interviews with prospective teachers

were conducted. This data was utilized while creating the tentative plan for the course. The details of the interview protocol are given later. The lesson plan and pre-interview data were used to examine PST's prior knowledge of ratio, proportion, and anticipations about students' understandings. The data indicated that PSTs had difficulties providing the next instructional actions not only in the case of students' incorrect responses but also in their correct ones. The PSTs had provided too general or no responses to incorrect solutions. Interestingly, interviews indicated that the perspective teachers did not provide the next instructional actions to the correct solutions for students either. Therefore, the correct and incorrect (or consistent and inconsistent) responses were chosen as criteria for selecting the students' video clips based on the data from individual pre-interviews.

The pre-assessment was administrated at the beginning of the 3 weeks of elective course with the intervention taking place approximately. Before starting the course, I conducted pre-interviews with prospective teachers, and then asked them to prepare a lesson plan. Firstly, prospective teachers watched video clips at the beginning of the course sessions. Then, they were asked to write reflection papers about the micro-case video clips (MCVs) individually each week. In the following sections, I explained the details of micro-case video clip production. In the reflection papers, the PSTs wrote what they noticed about students' mathematical understanding and how they interpreted the student's understanding in the video clips. Lastly, the PSTs proposed next instructional actions by considering students' understandings in the video clip. Each clip had key issues for understanding important relationships in the context of proportional thinking. A detailed description is given in Table 3.3. After writing the reflection papers, the researcher-facilitator initiated the group discussion about the week's MVCs. In order to facilitate group discussion, a framework was developed based on van Es et al. (2014). This framework was used to create productive group discussions concerning the complex nature of prospective teachers' learning. Four practices in this framework helped accomplish the goal of professional noticing of students' mathematical understanding. Based on this framework, we started (1) orienting the group to the video analysis task (e.g., What did you notice in the video? What did you find interesting about students' understanding?), (2) sustaining an inquiry stance (e.g., Why did you find this moment interesting/important? Can you tell

more about why this student thought in such a way? (I am not sure I can understand your idea, can you explain more?), (3) maintaining a focus on the video (e.g., What would be your suggestion for this student?, What kind of instructional action(s) you provide for this student?), and (4) supporting group collaboration (e.g., You said that this student would be familiar of finding one banana as a procedural. This makes sense. Do you agree with your friend, and why? Is there any different interpretation in the group?). During group discussions, prospective teachers could watch the video clips whenever they needed to remember. A similar procedure was followed for each course.

At the end of each discussion, prospective teachers were also asked to make revisions to their lesson plan if necessary. More specifically, the revisions involved the following questions:

- Please explain in detail if you made any change on your task design after Teaching Experiment session?
- If you made some revisions, can you give some specific examples that made you change on your previous work?
- If you did not make any revision, can you explain why you did not?

The first question helped us see PST's changing thoughts about noteworthy details and multiple interpretations of students' mathematical understandings after the group discussions of video clips in the week. The second question helped us to see how PSTs develop alternative instructional actions based on students' understandings regarding students' misconceptions and conceptions about ratio and proportion. The PSTs made revisions to their lesson plan (later, we called task designs) as they engaged in micro-case video clip analysis and group discussion sessions. These revisions of the lesson plans led us to learn what and how group discussions influenced the PSTs' noticing of students' understandings.

As a result, the current study aimed to examine the developmental process of prospective teachers' noticing skills about proportional reasoning within a video case-based approach using the teaching experiment methodology. Below, I share the details of the data collection tools and the data collection process.

3.3. Participants

This study was conducted with six prospective middle school teachers in a public university in Ankara, Turkey, for nine weeks. The fourth-year prospective teachers who completed the courses of Methods of Teaching Mathematics 1, Methods of Teaching Mathematics 2, and School Experience were invited to participate in the study. As a result, five female and one male fourth-year prospective teachers took the elective course we offered, and we carried out this study with these six participants. We determined some criteria for this course to make sure that participants had knowledge of proportional reasoning as a chosen subject and had experience with students' understanding. Prospective teachers completed the required courses of Methods of Teaching Experience 1-2 and School Experience. Some information about the participants is given in Table 3.1 below.

Their calculated cumulative grade point averages (Cum-GPA) were between 3.12 and 3.37. More specifically, two participants' Cum-GPAs were above 3.20 out of 4.00, and four PSTs' Cum-GPAs were between 3.12 and 3.37. The distribution of participants' Cum-GPAs shows that the participants in this classroom teaching experiment consisted of prospective teachers with similar academic achievement levels.

Table 3.1 Participants' academic background

Pseudonyms of the participants	Gender	Methods of Teaching Mathematics 1	Methods of Teaching Mathematics 2	School experience	Cum-GPA*
PST1	Female	BA	AA	AA	3.19
PST2	Female	BA	AA	AA	3.12
PST3	Female	BA	AA	AA	3.17
PST4	Male	BA	AA	AA	3.37
PST5	Female	BA	AA	AA	3.12
PST6	Female	BA	AA	AA	3.29

3.4. Data Collection Tools

In order to get detailed information from the prospective teachers, different data collection tools were used (see Table 3.2). The data collection procedure involved

"multiple source information" by Creswell (2014). This study's primary data collection tools were *pre-post interviews, individual video analysis reflection papers, video-based group discussions, PSTs' initial and revision lesson plans, and the researcher's field notes*. The following table indicates the teaching experiment data collection process of the current study. The details of each data source were explained in the following sections.

Table 3.2 Teaching experiment data collection process

Data Collection Phases	Data Collection Tools
Pre-intervention	Pre-clinical interviews Initial lesson plans
Video-case enhanced intervention	Individual analysis and reflection on video clip-based reflection papers Video-clip-based group discussions
Post-intervention	Revised lesson plans Post-clinical interviews

*Additional field notes were made as teaching experiment progressed

3.4.1. Pre and Post individual clinical interviews

Clinical interviews are a significant data collection technique providing unique and direct information via face-to-face interaction between participants and the researcher (Clement, 2000). The clinical interview process with children was described as follows: "The child interview is a face-to-face interaction of bidirectional influence, entered into to assess aspects of the child's functioning that have relevance to planning, implementing, or evaluating treatment" (Hughes & Baker, 1990, p. 4).

Clinical interviews enable researchers to elicit and document the participants' thinking naturally and gain in-depth information on conceptual understandings by engaging participants in the discussion of the tasks (Clement, 2000; Ambrose et al., 2004). In this study, I had two aims to conduct pre-post clinical interviews: (a) to select case-based video clips (e.g., what will be the key conceptual aspects of additive and

multiplicative strategies), and (b) to observe the developmental process of prospective teachers' noticing skills through teaching experiment sessions.

In this regard, both clinical pre and post interviews used the same task problems and similar probing questions. The clinical pre-interviews took nearly one hour, but the clinical post-interviews took nearly two hours. As suggested, pre-post clinical interviews were conducted in a private location in the School of Education Faculty with only the participant (respectively, the first week of the teaching experiment and the ninth week of the teaching experiment) and the researcher. All the clinical interviews were recorded.

As mentioned, there were five problems in the task. For each problem, we asked three questions considering the prospective teachers' existing mathematical knowledge about ratio and proportion and also prospective teachers' knowledge about students' mathematical thinking while solving ratio and proportion problems. Therefore, we used the same task problems and prompting questions in both pre and post clinical interviews: Can you solve this problem, please? Are there any different solutions that students might use? What will be your next instructional actions for the students who use the alternative solutions as you told? What are the possible difficulties that students might have with this problem? Imagine one of your students did not solve this problem, what would be the reasons for this and how would you overcome this possible difficulty/misconception? We used the same tasks in both pre-interviews and post-interviews. The details of the preparation process and the purposes of each task are explained below.

3.4.2. Initial lesson plans and revised versions

In this study, the data from the PSTs' initial lesson plan was significant in the teaching experiment process. The teaching experiment process was designed in two phases: preparation and implementation. First, the PSTs' initial lesson plans were used to capture prospective teachers' knowledge about ratio and proportion and to choose video clips for teaching experiment sessions. Then, they were asked to make revisions in lesson plans if they needed to do. Although prospective teachers prepared the lesson plan at the beginning of the teaching experiment, the nature of this study's teaching

sessions led them to change their lesson plan to task design. The lesson plan revisions helped us understand why they needed to make revisions on the previous task designs and what changes they made for the following teaching experiment sessions (e.g., video clip selection and key issues to discuss). Therefore, the revisions were crucial for this study to follow the development of the PSTs' professional noticing skills of students' mathematical understanding of proportional reasoning during the teaching experiment implementation process.

3.4.3. Video based reflection papers

Reflection papers were another data source of the current study. The prospective teachers were asked to write a reflection paper after watching each micro-case video clip in the individual video analysis process. Reflection papers were significant to understand the PSTs' noticing skills and students' thinking in the domain of proportional reasoning in each video clip. I followed the questions of Jacobs et al. (2010) to reveal the PSTs noticing skills. They wrote what they noticed about students' understanding (e.g., unit rate or build-up strategy) and how they interpreted the student's understanding in the video clips. Then, I asked the PSTs to provide the next instructional actions considering the student's understanding of ratio and proportion in the video clip. An example of a reflection paper was given in the Appendix B. The questions for the reflection paper were as follows:

- What did you notice in the video clip about student's strategy in response to the problem?
- Can you describe in detail what you think this child did in response to this question?
- Can you explain in detail what you learned about this student's understandings?
- Pretend that you are a teacher of this student. What will be your next instruction action (e.g. posing questions) for this student? Why?

3.4.4. Video based group discussions

As stated before, the social constructivist theory was used in this study. This approach emphasizes how individuals construct knowledge and understanding within the social context of learning environments. In line with this approach, group discussion provided opportunities to attain more interaction on different thoughts and sometimes shared meanings than individuals working alone. Van Es et al. (2014) offered a framework for facilitating video-based discussion as a part of professional development. As a facilitator, I was guided by this framework for engaging prospective teachers. I gave details on the facilitation of video-based group discussions in following sections.

3.4.5. Field notes

Bogdan and Biklen (2007) described field notes as "*what the researcher hears, sees, experiences, and thinks in the course of collecting and reflecting on the data in a qualitative study*" (p.118-119). As a facilitator-researcher, I took field notes as a reminder of what was discussed in teaching sessions. These field notes helped me keep track of my understanding of prospective teachers during video-based group discussions and get information about the prospective teachers' aspects of the student's understanding in the week's video clip. Further, as Bogdan and Biklen (2007) suggested, we used the previous week's field notes for my preliminary analysis for next week's session. Therefore, we took some descriptive notes, such as gestures, and reflective notes, such as what the participants might think or refer to and why she named a concept in the video clips, as noteworthy details after each teaching session.

3.5. Teaching Experiment Process Planning

Before conducting video-case based teaching experiment sessions, I followed three steps as in Figure 3. I described the details about each step in the following.

3.5.1. Micro-case video clip production

In this study, I collected videos that involved critical details related to the student's understanding of proportional reasoning. The prominent characteristics of video clips

are as follows: (a) video clips were purposefully designed, selected, and edited; and (b) video clips provided one-on-one student interviews. The details of the case-based video production process are given below.

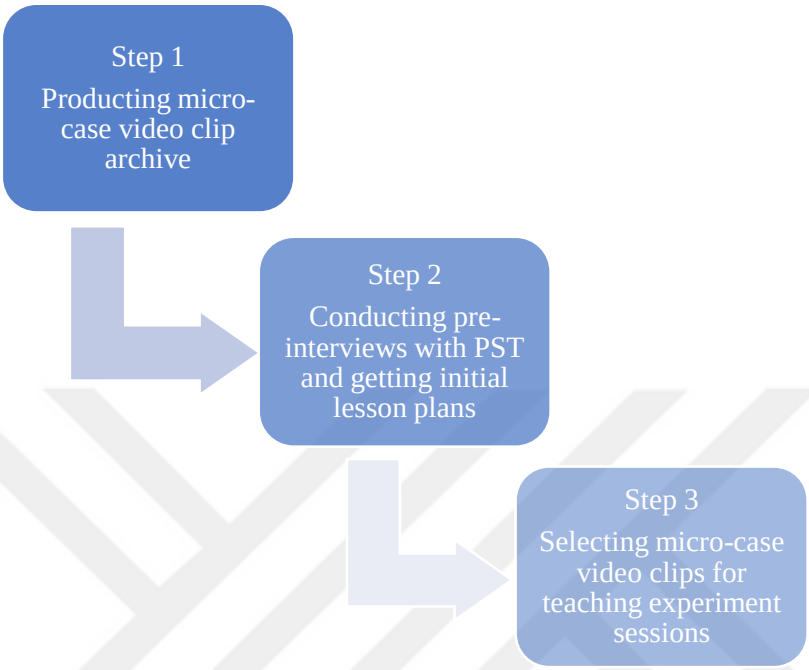


Figure 3.1 Teaching experiment steps in process planning

3.5.1.1. Student interviews for micro-case video clip production

For micro-case video production, I conducted interviews with 7th-grade students in Ankara, Turkey. We decided to conduct interviews with 7th-grade students because the students had mathematics courses about ratio and proportion in the sixth and seventh grades. I asked students and their families to be participants in the study by sending consent forms (see Appendix F). Twelve seventh-grade students from different elementary schools accepted to participate in the study, and we used nicknames for them. Then, we interviewed 12 seventh-grade students in suitable rooms in their houses.

Before the implementation of the task problems, pilot interviews were conducted with three 7th-grade students to determine whether the task problems could be understood and whether the time would be tiring and long for 7th-grade students. After this

piloting, I observed that students were able to keep their attention during the interview sessions, and all task problems could be understood easily. In addition, as Sherin et al. (2009) suggested, we observed students thinking through verbal explanations of students or their written work. Thus, we used two different approaches while conducting interviews to reveal their understanding. First, we just asked students to solve the problem and to explain their reasoning in a way that made sense to him/her as explaining to their friends. We did not say anything about whether the student solved the problem correctly or incorrectly. Then, we subsequently asked them (a) to solve the problem in another way (e.g., hmmm, I got it, but just in case, can you solve this problem in another way? So I might understand better), or (b) to examine another student's solution (e.g., I am confused, because one of my students solved this problem differently than yours. Can you look at it and explain why these are different). Our pilot study analysis showed that these approaches enabled students to explain their reasoning more explicitly.

After piloting, I conducted interviews with 12 seventh-grade students for video clip production. In order to videotape students' problem-solving process, I used a single camera that focused on students' writing while studying with tasks. Each video was between 75-100 minutes. In each video, one of the 7th-grade students was posed with five problems which were the same as the prospective teachers' ones, and they individually worked on them.

The tasks had different problem types about essential understandings of ratio and proportion. These key understandings are as follows: reasoning about additive-multiplicative thinking, reasoning about unitizing and norming, and reasoning about proportional relationships. The details of the problems and the structure of the micro-case video clip production interviews were explained in the following sections.

Problem types in micro-case video clip production interviews

In these task-based interviews, the students were posed five paper-pencil problems as seen in the Table 3.3 below. These problems were adopted from the existing literature. The fourth problem included figures related to the problem context. The five problems

were divided into three categories based on the reasoning strategies as listed above: reasoning about additive-multiplicative thinking, reasoning about unitizing and norming, and reasoning about proportional relationships.

The problems were determined based on Carpenter and colleagues' four-level developmental trajectory that identified levels of development in proportional reasoning (Carpenter et al., 1999). Balloon-Grocery problems and Scaling-Mixture problems were grouped because although they had different contextual structures, they had similar multiplicative relationships. The researcher implemented problems specifically designed to further students' thinking of proportional reasoning. The levels of development in proportional reasoning in Carpenter et al.'s (1999) were as follows:

At Level 1, students mostly focus on additive differences or apply random calculations. At Level 2, students perceive ratio as a single unit, and they construct equivalent ratios by adding the given units additively or multiplying them by a whole number (i.e., integer ratios). However, students at this level cannot perform operations, including partitioning the given ratios (i.e., non-integer ratios). Thus, students at this level mostly use build-up strategies in which a ratio is constructed using repeated addition, multiplication, or a combination of these operations when integer ratios exist.
At Level 3, students perceive ratio as a single unit as in Level 2 and they can perform operations with integer and non-integer ratios. Thus, Level 3 is an extension of Level 2. This level strategy includes a build-up strategy, using repeated addition or multiplication with the reduction of the given ratios (p. 11). For instance, if there is a problem that can be represented by the proportion $\frac{8}{12} = \frac{42}{x}$, a student at Level 3 can build up the given ratio to $\frac{40}{60}$ and reduce the $\frac{8}{12}$ by four to get $\frac{2}{3}$. This student can add $\frac{40}{60}$ or multiply by 21 to get $\frac{42}{63}$ in order to solve the problem. Besides, this student can multiply $\frac{8}{12}$ by a fraction or non-integer value (i.e., 5 $\frac{1}{4}$) to find the missing value. However, a student at Level 2 cannot solve this problem by using reduction or using non-integer values. Figure 3.2 indicates these kinds of relationships in the ratio table below.

Hours	8	16	32	*(+ 8)*	40	2	*(+40)*	42
Km	12	24	48	*(+ 12)*	60	3	*(+60)*	63

Figure 3.2 Build up strategy at Level 3 (Carpenter et al., 1999, p. 11)

At Level 4, students require considering ratios as more than just units and understanding within-between relationships in the proportion of $\frac{a}{b} = \frac{c}{d}$. This approach provides students with flexibility in strategies. Students can describe the relationship between the given quantities and determine the appropriate and efficient strategy (Carpenter et al., 1999).

The problems in the current study were designed based on the learning trajectory of four levels of reasoning proposed by Carpenter et al. (1999). Level 2 and Level 3 were introduced together. The following table shows the framework of the proposed developmental trajectory in the current study.

Twelve students were asked to solve the problems individually and explain their solutions. The word problems were asked randomly to avoid any expectations that the rest of the problems would be about proportional reasoning. The details of these problems and the changes made to the original problems are summarized below. Also, the Turkish version of the problems is available in the Appendix A.

Table 3.3 The framework of the proposed developmental trajectory in proportional reasoning (adopted from Carpenter et al., 1999)

Key understandings	Levels	Description	Problems
Additive and Multiplicative Reasoning	Level 1	Having Limited knowledge including performing random calculations or reasoning additively	Racetrack Problem (Cramer et al., 2005)
Reasoning with Unitizing and Norming	Level 2	Perceiving ratio as a single unit and constructing equivalent ratios by adding the given units additively or multiplying by integer ratios.	The Balloon problem (Lamon, 1993b)
	Level 3	Perceive ratio as a single unit as Level 2 and performing operations with integer and non-integer ratios.	The Grocery problem (Lamon, 1995)
Reasoning with proportional relationships			
Reasoning with within ratios and between ratios	Level 4a	Considering ratios as more than just units Providing flexibility in students' strategies Being able to describe the relationship between given quantities and deciding the appropriate and efficient strategy	The Photograph problem (Kaput & West, 1994),
Reasoning with part-part and part-whole ratios	Level 4b	Considering ratios as more than just units Providing flexibility in students' strategies Being able to describe the relationship between given quantities and deciding the appropriate and efficient strategy	The Mixture problem (Noelting, 1980)

Racetrack Problem: Studies on previously developed learning trajectories in proportional reasoning demonstrated that the development of proportional reasoning begins with a focus on differences (i.e., absolute change) or with no relevant reasoning at all (i.e., non-proportional situations) (e.g., Carpenter et al., 1999; Steinthorsdottir & Sriraman, 2009; Wright, 2014). In this vein, the first problem in this trajectory is related to additive and multiplicative reasoning. The racetrack problem was created by Cramer et al. (1993). The numbers in this problem are given as integer ratios form (4/2 and 6/2). The extensive body of literature states that one of the most common difficulties of proportional reasoning problems is to apply additive reasoning in multiplicative situations, especially if the numbers are given in noninteger format as task-related factors (Karplus et al., 1983; Lin, 1991; Misailidou, & Williams, 2003; Tourniaire & Pulos, 1985; Van Dooren et al., 2010). On the other hand, other lines of studies have shown that students have a tendency to apply multiplicative strategies for additive situations, especially for nonproportional problems given in missing value format (Van Dooren et al., 2008; Van Dooren et al., 2010). Also, the numbers given in this problem are integer (4/2 and 6/2), as seen in another task-related factor in the existing literature (e.g., Kaput & West, 1994; Van Dooren et al., 2010). The original version of the racetrack problem is as follows;

Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?

Figure 3.3 The original version of the racetrack problem, (Cramer et al., 1993)

In light of the previous studies, some changes were made to the original problem. First, the names of people were replaced with Turkish names, and then, the integer numbers were used. No further modification was made to the following problem:

Rukiye ve Neşe bir pistin etrafında eşit hızla koşmaktadır. Koşmaya önce başlayan Neşe, 4 tur koştuğunda Rukiye 2 tur koşmuştur. Buna göre Rukiye 6 tur koştuğunda, Neşe kaç tur koşmuş olacaktır? (Sorunun çözümünü ayrıntılı şekilde yazınız.)

Figure 3.4 The adapted version of the racetrack problem (Racetrack Problem)

The Balloon problem and the Grocery problem: Following Carpenter et al.'s (1999) suggestions in their learning trajectory (i.e., Level 2 and Level 3), two problems related to reasoning with unitizing and norming were proposed. These problems have integer and non-integer ratios. The details of the problems are given below.

The balloon problem of the study was adapted from Lamon (1993b). The problem was used to investigate students' proportional reasoning strategies, such as the unit rate or composite units, and their tendencies to use singleton units or composite units. The numbers in this problem are given as integer ratios form (1,5/3 or 3/1,5). Lamon (1994) worked with sixth-graders to examine their understanding of solving ratio and proportion problems. In this study, four different strategies were mentioned: using composite units, building up strategy (without recognition of functional relationship), unit rate strategy (correct and incorrect application), and scalar operator within measure space (p. 101). Moreover, studies demonstrated how the types of numbers in the problem strongly influence the decisions regarding the appropriate operation (Noelting, 1980a, 1980b). For example, in this case, we used decimals number and integer numbers divided by each other (3/1.5 and 1,5/3) to understand students' conceptual meanings of dividing or multiplying by decimals. The original version of the balloon problem is as follows:

Ellen, Jim, and Steve bought three helium-filled balloons and paid three dollars. They decided to go back to the store and buy enough balloons for their class. How much did they pay for 24 balloons?

Figure 3.5 The original version of balloon problem (Lamon, 1993b)

Some changes were made to the original problem. Firstly, the names were replaced with Turkish names. Numbers in the problem context were changed considering divisibility.

Derya, Ahmet ve Kaan üç tane balona 1,5 TL ödemişlerdir. 24 kişilik sınıflarına balon aldıkları takdirde kaç lira ödemeleri gerekmektedir? (Sorunun çözümünü ayrıntılı şekilde yazınız)

Figure 3.6 The adapted version of the balloon problem (Balloon problem)

The grocery problem was adapted from Lamon (1995). This problem has also been called the best-buy problem, which is typically used for the proportional context and deals with numerical comparisons (exp. Lamon, 1994; Lamon, 1995). According to Lamon (1993), students should not only solve problems including more than just one unit choice but also make their own context-supported decisions for deciding which unit might be better to use. In these contexts, "better" means more economical or more reasonable. Thus, for every problem, the unit approach in which the unit is a 1-unit may not be the best solution (Lamon, 1995). For instance, Lamon (1995) states that most students solve the problem (see Figure 8) by finding the cost of one ounce of each brand of cereal because it is easier to divide the amount of money by the number of items (16 and 12) for each cereal brand, respectively. The original version of the problem is as follows:

16 ounce box of Breakfast Bits cereal costs $3.36. A 12-ounce box of Fruity Flakes costs $2.64. Which is the better buy?

Figure 3.7 The original version of better buy problem (Lamon, 1995)

From the point of these views, we made some changes to the original problem. Firstly, the context of the problem was adapted to be more familiar. The numbers not only give way to see divisibility relationships (find the rate for one and multiply it to get the rate for many) but also give way to see composite units by coordinating multiples of different composite units. The adopted version of the problem is given below.

Ayşe A manavından aldığı 4 adet muza 3,6 TL ödemiştir. Berna ise B manavından aldığı 3 adet muza 3,3 TL ödemiştir. Hangi manavdan muz almak daha hesaplıdır? (Sorunun çözümünü ayrıntılı şekilde yazınız.)

Figure 3.8 The adapted version of the better buy problem (Grocery problem)

Photograph and Mixture problems: Carpenter and colleagues (1999) stated that Level 4 requires considering ratios as more than just units and understanding the relationships in the proportion of $\frac{a}{b} = \frac{c}{d}$. Based on this view, the Photograph and

Mixture problems were chosen. These problems are related to reasoning with proportional relationships (i.e., within-between ratios, part-part-whole ratios), and they were used to reveal students' mathematical thinking. The details about the problems are given below.

The resized photograph problem was adapted from Kaput and West (1994). The problem addressed the ideas of enlarging and shrinking for proportional relationships in pictures. The problems which involve similarity, enlarging, shrinking, or scaling within geometric figures are seen as the most difficult problems for students because there is no clear unit measure, and numbers are not in integer format to identify the proportional relationships easily (Kaput & West, 1994). Another reason for choosing this problem is that iteration or partition of a composed unit might be challenging. For instance, it might be difficult for students to copy and paste the smaller triangle to fit the enlarged one (Laboto & Ellis, 2010). When an item such as a triangle is enlarged or shrunk by keeping all the dimensions/sides' proportions the same, we can talk about the similarity between the original and the enlarged/shrunk one. Hence, the original and the resized copy are congruent (same size, same shape) (Lamon, 2012). The original version of the problem is given below.

The two sides of Figure A are 9 cm high and 15 cm long. Figure B is the same shape but bigger. If one side of Figure B is 24 cm high, how long is the other side?

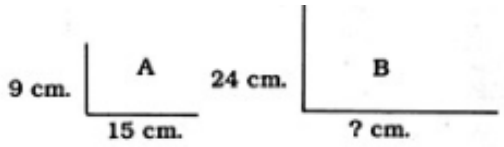


Figure 3.9 The original version of the similarity problem (Kaput & West, 1994, p. 269)

In the light of the existing literature above, we changed the problem. Firstly, we provided the problem within a familiar context (resized photograph). This problem has two photos, and it was stated that the first photo was resized (larger). The first photo had side lengths of 6 cm and 10 cm, and the second photo had a long side length of 30 cm. The numbers were given in integer form. The students were asked to find the unknown short side of the second photo. When we say that the first photo was resized

(larger), we mean that there is a relationship between the first picture and the enlarged one, such that for all of the corresponding dimensions of the first picture, the ratio enlarged size and the original size are equal. The adapted version of the problem is given above.



6 cm

10 cm

30 cm

Aynur, kardeşi Ahmet ile fotoğrafının boyutunu fotokopi makinesinde yukarıda olduğu gibi büyütmüştür. Buna göre büyütülmüş fotoğrafin alanı orijinal fotoğrafın alanının kaç katıdır? (Sorunun çözümünü ayrıntılı şekilde yazınız.)

Figure 3.10 Adapted version of the similarity problem (Photograph problem)

The lemonade concentrate and the similarity problem are related to covariation and invariation in proportional situations. All proportional problems have covariational and constant (invariance) relationships between two quantities. Children can reason about quantities, and the way the quantities change together (co-vary) if adequate experience with appropriate verbal statements and questions is provided (Lamon, 2012; Spinillo & Bryant, 1999). For example, we have orange juice made up of two glasses of orange concentrate and one glass of water as an ordered pair, e.g., (2,1). If a child is trying to find how much orange concentrate there is in this orange mixture, the ratio of orange concentrate to water is 2:1, and there are 24 glasses of orange concentrate. The child can preserve the covariation between the amount of orange concentrate and water as follows: 1-2, 2-4, 3-6… 12 glasses of orange concentrate to 24 glasses of water as a part of the building-up process (Kaput & West, 1994; Thomson, 1994). This strategy shows good reasoning considering the amount of orange concentrate and water that change together. However, most students have never

thought about how they measure the color intensity or orangeness of juice or the crowdedness of an elevator because these characteristics cannot be measured directly (Lamon, 2012, p. 51).

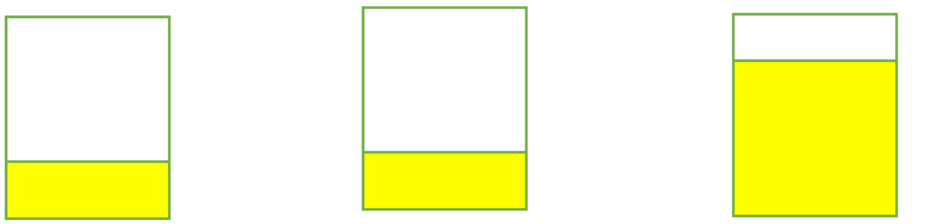
Noelting (1980a; 1980b) studied proportional tasks by comparing two ratios (orange concentrate to water, two mixtures of the same or different taste) rather than finding the desired taste (ratio). It was found that young children could compare the orangeness of these two mixtures successfully: Mixture A (one glass of orange concentrate and two glasses of water) and Mixture B (one glass of orange concentrate and five glasses of water). However, they did not compare the orangeness of the mixtures when they had different amounts of water and orange concentrate (Noelting, 1980a, 1980b).

In addition, Karplus, Pulos, and Stage (1980) focused on the effects of numerical characteristics of proportion problems. For example, in their study, the problems with integral ratios were found to be much easier than those with non-integral ratios. Also, if the divisibility relationship between quantities is masked in proportion problems or no number is evenly divisible by another, the student may not know how to proceed toward a solution (Rupley, 1981). In this regard, Pulos et al. (1980) suggested that students should encode the relationship among the quantities in problems.

We are going to play at making up orange juice for a party. Here (A), aunt Ann has prepared these 3 glasses of orange juice and 1 glass of water; here (B) aunt Beatrice has prepared these 1 glass of orange juice and 3 glasses of water. Now pay attention: which side do you think the drink will have a stronger orange taste? Or do you think both drinks will taste the same?

Figure 3.11 Original version of the orange juice problem (Noelting, 1980a)

We made some changes to the original problems based on the literature review mentioned above. We used lemonade concentrate: Mixture A, Mixture B, and Mixture C. The three mixtures are unequal (do not have the same taste) and have various ingredients. In addition, we considered the type of comparisons involving two ingredients, lemon, and water, that is, within comparison, and corresponding ingredients of two recipes, that is, between comparisons. The number of ingredients leads students to be able to use both types of comparison, as given below:



Limonata A
2 bardak limon suyu
3 bardak su

Limonata B
1 bardak limon suyu
4 bardak su

Limonata C
4 bardak limon suyu
8 bardak su

Esra ve Sinan yukarıda verilen üç farklı limonatanın tadına bakacaktır. Esra ve Sinan hangi limonatadan daha çok limon tadı alacaktır? (Sorunun çözümünü ayrıntılı şekilde yazınız.)

Figure 3.12 Adapted version of the orange juice problem (Lemonade Problem)

3.5.1.2. Micro-case video clip archiving

At the end of interviews with 7th-graders, we had nearly 1000 minutes of videos as raw data. We thought that transcribing all data would be unnecessary and also time-consuming. In this regard, we created an archive with these raw videos to use in teaching experiment sessions. This archive was used as a *video database* we could apply while selecting and organizing the video clips for teaching experiment sessions. Detailed information about case-based video clip selection criteria and strategies are presented below.

Firstly, we needed to evaluate raw videos before implementing them in teaching sessions. As stated earlier, videos are seen as a valuable medium and used to examine students' mathematical understandings (e.g., Goldsmith & Seago, 2011; Sherin et al., 2009). Sherin (2009) stated that video clips lead to productive discussions on student understanding. However, raw videos might have useless and unrelated events. The existing literature shows that there should be specific criteria for video clips that make them a useful resource for teachers while examining students' mathematical understanding (Sherin et al., 2009). These are windows, depth, clarity, and length.

Sherin, Linsenmeier, and van Es (2009) developed a framework in order to characterize the video clips of student mathematical understanding along three

dimensions; (a) the extent to which a video clip provides *windows* into student thinking, (b) *the depth* of student mathematical thinking shown in the video, and (c) *the clarity* of student understanding shown in the video (p. 215). Table 1 provides detailed information about classifying video segments through these dimensions.

Sherin et al. (2009) found that highly in-depth video clips might not lead to productive discussions. For instance, video clips that are low in windows and in-depth lead to little discussion, regardless of the clarity level of the video clip. However, video clips with sufficient depth might lead to productive discussions. Also, in-depth video clips with low clarity provide productive conversations. It has been suggested that clarity should be evaluated with other dimensions. For example, if the clip is high in clarity, it must be high in windows.

Table 3.4 Dimensions for characterizing video clips of student thinking (Sherin, Linsenmeier & van Es, 2009, p. 216)

Dimension	Description	Key issues
Windows	Evidence of students' mathematical thinking in the video clip	Visible student writing Clear verbal explanation
Depth	Substantive student thinking in nature	Routine strategies such as memorization or recall Engage in mathematical reasoning and problem solving
Clarity	Clarity of understanding of student's thinking in the video clip	Make sense of student's mathematical thinking (e.g., "I don't understand what student talking about?:", "Her explanation was very clear.")

In some studies, researchers focused on a video clip's maximum *length* of time. Watching classroom videos might be dull and slow, so videos should be less than ten minutes before people lose their attention (Goldsmith & Seago, 2011; Jawarski, 1990; Sherin et al., 2009). For instance, Jacobs et al. (2011) used 3-minute and 1.5-minute

video clips and van Es (2011) used 5-to-7-minute video segments. In this study, we created 4-7 minute-video segments/clips for teaching experiment sessions from the video database.

Then, we divided raw videos into six segments based on the six tasks, and I identified significant events or idea units for each video segment in chronological order. Notes with time stamps were taken. These notes helped me to have a quick look at what we have while selecting video clips. The notes were descriptive and reflective of students' mathematical thinking. Based on the events and the notes I took and presented above, we re-divided the video segments into 2 to 3 case-based video clips. Table 3.5 enabled us to find noteworthy events, such as invented strategies and our unpredictable misconceptions in students' thinking. Additionally, I applied the second strategy by organizing the notes based on different students' mathematical thinking for the relevant task's same mathematical concepts (e.g., invariant- covariant relationship). In this way, we could compare each student's mathematical thinking.

In the following section, the details of how I used pre-interviews and prospective teachers' initial lesson plans were provided to select and organize case-based video clips for teaching experiment sessions.

3.5.2. Micro-case video clip selection and preparation

Before starting teaching experiment sessions, as a first step, the existing literature and some educational websites (e.g., teaching channels) were reviewed in the planning phase of the teaching experiment. The literature review presented the critical issues that we focused on while creating videos with seventh-graders (e.g., choosing problems in the task and the possible misconceptions). Even though we did research to find videos about ratio and proportion regarding key issues (e.g., unit rate factor), we did not find sufficient and appropriate videos. Therefore, we created our video database. The prospective teachers were asked to prepare a lesson plan (we called it a 'problem-solving task design' after the second week). We wanted to examine prospective teachers' initial lesson plans before clinical pre-interviews. These lesson plans and clinical pre-interviews were critical components of decision-making in

selecting and organizing case-based video clips in the raw videos of the seventh-graders. After analyzing the initial lesson plans and clinical pre-interviews, the video database was narrowed down, considering prospective teachers' needs.

The existing literature, data from the seventh-grade students, initial lesson plans, and clinical pre-interviews conducted with prospective teachers assisted us in determining the most appropriate and useful video clips for teaching experiment sessions. For instance, the literature shows that students could not recognize additive and multiplicative relationships and so, they incorrectly applied additive reasoning in a multiplicative situation. Thus, we used the racetrack problem, which helped us recognize students' thinking about multiplicative and additive situations. The clinical pre-interviews showed that even prospective teachers could not distinguish between additive and multiplicative reasoning.

Moreover, even though prospective teachers applied the correct strategy. They could not anticipate students' possible difficulties/misconceptions and help students overcome their difficulties considering their thinking. Therefore, the first video clip (the racetrack problem) included an incorrect solution for a student who solved the problem using an incorrect additive strategy. The following sections explain all the details about the video clip selection process. Table 3.4 presents the details of the utilized video clips in teaching experiment sessions based on window, clarity, depth, and length by considering Sherin et al. (2009) framework.

3.5.3. Procedures for selecting and organizing micro-case video clips

Before conducting teaching experiment sessions, I gathered data through prospective teachers' initial lesson plans and individual clinical pre-interviews. Based on these data, four-week teaching experiment sessions (W3, W4, W5, W6) were conducted, and each of these sessions lasted approximately 80 minutes. The micro-case sessions were conducted in class at the Faculty of Education. How the micro-case video clips (MCVs) were selected for the subsequent weeks was explained in detail below.

In addition, it must be noted that video clips were first analyzed. Then they were chosen based on prospective teachers' initial lesson plans and individual clinical pre-interviews with another researcher who was informed about the professional noticing of students' thinking and proportional reasoning. The researchers coded video clips based on Sherin et al.'s (2009) framework at the end of this process.

3.5.3.1. Preparation of the MCV1 of the Teaching Experiment Session 1

At the beginning of the teaching experiment, I decided to start with one case-based video clip that involved students' understanding of additive and multiplicative reasoning regarding the existing studies (e.g., Carpenter et al., 1999; Wright, 2004). Developing multiplicative reasoning and being able to discriminate multiplicative relationships from additive relationships in a given context lies at the heart of proportional reasoning development (Sowder et al., 1998). Even though the transition from additive to multiplicative reasoning is seen as critical for proportional reasoning, mathematics teachers in middle school show little attention to this in their courses (Sowder et al., 1998). Another study claims that teachers have difficulty understanding the difference between multiplicative and additive reasoning. For this reason, they will not be able to help students recognize and cope correctly with multiplicative situations (Sowder & Philip, 1995). As our data also showed, many students and even adults are not able to distinguish between additive and multiplicative relationships, and also they misuse them in situations (Lamon, 1994).

Furthermore, Lamon (1995) stated that "ratio and proportion have typically been taught in a single chapter of the mathematics textbook in which symbols are introduced before sufficient groundwork has been laid for students to understand them" (p.167). Studies indicate that students need to work on additive and multiplicative reasoning at the beginning of grade 3 (Kaput & West, 1994). However, Turkish textbooks have not provided examples that discriminate additive situations from multiplicative situations. Most of the textbooks focused on prescribed algorithms, such as using cross-product (Lamon, 1995).

The micro-case video clip for Session 1 (MVC1) was selected based on the student's multiplication approach to the additive situation. The student in the video clip applied a multiplicative strategy, but the problem required additive reasoning. Prospective teachers expected this answer as they explained in the pre-interviews. Therefore, to reveal the student's thinking, we gave him a correct solution by saying: "Hmm, I am confused because one of my friends solved it differently. Can we look at why this solution was different?". After examining the given solution, he reconsidered his solution, and he realized that he had made a mistake. Then, he explained to us why his strategy was wrong. MVC1 lasts for 7.47 minutes and involves student failure of distinguishing multiplicative relationships from additive relationships. This video clip was coded as high in window, high in depth and high in clarity into students thinking (Sherin et al., 2009).

3.5.3.2. Preparation of MCV2 and MCV3 for Teaching Experiment Sessions 2 and 3

Before conducting clinical interviews with prospective teachers, we planned to use a video clip about students' preferred strategies, such as using singleton units (single unit) or composite units based on the problem context for the Balloon problem and Grocery problem. Lamon's (1994) study showed that the balloon problem revealed students' preferred strategies: built-up strategy, unit rate strategy, and composite unit strategy (p.103). As expected, in our study, prospective teachers provided different strategies for the Balloon problem (e.g., unit rate/single ratio, composed unit, or using pictorial representation). However, surprisingly, none of the prospective teachers talked about how students construct and coordinate composed units. In our video database, most seventh-graders solved this problem correctly using cross-product multiplication and unit rate. Also, the student in MCV3 correctly solved this problem. However, she constructed iterable composed units (three balloons as one group-single unit) instead of applying direct cross-product multiplication like other seventh-graders. She recognized the advantage of using a composed unit. Then, she calculated how many three groups she could make using 24 balloons. After she found eight by counting threes, she multiplicated eight by 1,5 to find the total price. Finding the number of groups of three balloons she could make from 24 was a significant part of

her solution. Because she accepted three balloons as a reference unit, she reinterpreted the situation regarding that unit. Then, we gave her another solution (incorrect) and asked her to examine it as in the procedure for the Racetrack problem.

Carpenter et al. (1999) stated that students at this level mostly use build-up strategies in which a ratio is constructed using repeated addition, multiplication, or a combination of these operations when integer ratios exist. Therefore, the learning trajectory continued with the Balloon problem, with the values forming an integer ratio, to reveal students' thinking about how to make sense of the build-up strategies. We decided to use the student's video in the Session 2.

In our video database, one of the solutions to the strategy of the Grocery problem was unexpected. For the grocery problem, the student in MCV2 used the unit rate approach to dividing given numbers without considering unit rate meaning. Even though she thought she found the price for one banana in Grocery A and Grocery B, she found the number of bananas per lira. We did not say anything about whether her solution was correct or incorrect. We asked her to solve the problem using another strategy; "Hmm, I got your solution, but if you do not mind, can you solve this problem in another way to understand it better?". Then, she used pictorial representation. Her first solution was wrong, and she recognized her fault while solving the problem in another way.

We decided to use this video clip in the second-week teaching experiment session because, surprisingly, none of the prospective teachers except one anticipated this difficulty. Although one prospective teacher anticipated this difficulty, she could not provide instructional action to overcome it. Also, as Carpenter et al. (1999) stated at Level 2 and Level 3, the presence of integer or non-integer ratios is a significant indicator of students' performance to recognize their perception of the ratio. In this regard, we prepared this video clip that involves the difficulty in the unit rate approach and linked units developed by Kaput and West (1994) as noteworthy mathematical elements.

MVC2 belongs to the Balloon problem is 7.18 minutes and involves composite unit construction (see Table 3.5). Also, MVC3 belongs to the Grocery Problem is 76.52

minutes and involves the difficulty of the unit rate approach and linked composited units.

3.5.3.3. Preparation of MCV4 and MCV5 for the Teaching Experiment Sessions 3 and 4

Before conducting pre-interviews, we planned to prepare a video about students' difficulty with similarity in geometric shapes (i.e., the Photograph Problem). The pre-interview results showed that using an additive strategy for geometrical similarity was an expected difficulty. For instance, prospective teachers thought that students would apply erroneous additive strategies, increasing the length by 20 (30-10=20 cm) and the height by 26 (6+20=26 cm) for this problem, which was multiplicative in nature.

However, the pre-interviews indicated that some prospective teachers did not address significant mathematical situations such as the preservation of sides' equality and internal and external ratios (Freudenthal, 1983). The reason might be that those prospective teachers considered easy type ratio problems and had routine solving procedures ($\frac{6}{10} = \frac{?}{30}$). On the other hand, geometric similarity problems are seen as the most vulnerable to additive errors (Kaput & West, 1994, p. 275) because these problems do not have a clear indication of unit measure and do not use integer ratios (non-integer ratios). In this regard, we decided to select Hilal's solution in which she solved the Photograph problem correctly by applying multiplicative reasoning, increasing the lengths by the same ratio (three times 6 to get 30, so three times six equals 18). Then, we gave her an incorrect solution (including additive reasoning) for the problem and asked her to explain why these two solutions differed.

For the Session 5, we first selected two video clips that involved students' understanding and their solving strategies for mixture problems. According to Tourniaire and Pulos (1985), mixture problems are different from ratio problems in many ways. First, the ratio quantities in a mixture problem generate a new object (e.g., red and blue paint mixed makes purple) or modified object (e.g., lemonade juice mixed with water makes a weaker lemonade). Besides, students must understand what

happens when two components are mixed (p. 183). We realized that, even though prospective teachers anticipated possible student solutions for the Lemonade problem, they did not stress the critical components about the relationships between quantities and how two quantities vary (i.e., covary). The ratio of one quantity to the other does not change (i.e., invariant). In addition, prospective teachers did not use proportional reasoning language in their explanations, such as "Mixture A represents that there are two glasses of lemon juice corresponding to every three glasses of water".

Based on the discussion data and pre-interview results, MVC4 is a 5.01 minute-video clip involving students' understanding of rescaling geometric figures such as invariant and covariant relationships and MCV5 is a 6.43-minute video clip involving student's different strategies such as part-part and part-whole relationships within each mixture or equivalence classes ($\text{i.e.}, \frac{16}{24}, \frac{12}{24}$). In these video clips, the students solved the problems correctly, but as done previously, the researcher did not give any information about the correctness or incorrectness of her solution. Then, the researcher asked her to solve the problems in other ways. MCV4 and MVC5 are both high in windows, depth, and clarity. As Carpenter and colleagues (1999) stated, students at Level 4 could recognize the proportional relationships (i.e., within-between and part-part-whole ratios). This understanding leads them to flexibility while choosing the appropriate and efficient strategies. The video clips selected from the video database are summarized in Table 3.5 below.

Table 3.5 Video clip coding across the criteria

Sessions (Ss)	MCVs	Length (min)	Video clip coding			Key understandings
			Windows	Depth	Clarity	
S1	MCV1	7.47	High	High	High	Additive and multiplicative reasoning
S2	MCV2	7.18	High	Low	Low	Unitizing and Norming
S3	MCV3	6.52	High	High	High	
S4	MCV4	5.01	High	High	High	Proportional Relationships
S 5	MCV5	6.43	High	High	High	

3.6. Procedures for Implementing the Teaching Experiment

Teaching experiment implementation procedure was illustrated in Table 3.6. After I conducted pre-interviews and prospective teachers prepared initial lesson plans, video case-based teaching experiment sessions were conducted. The weekly sessions were consisted of two 90 minutes teaching sessions and all sessions were completed into five consecutive weeks. Teaching experiment sessions were conducted in a class at Faculty of Education. Participants watched micro-case video clips from the projector in the classroom. In the following, the implementation procedures of teaching experiment were given in detail (see Table 3.6). Video-clip enhanced intervention process has two steps: (a) individual analysis and reflecting on micro-case video clips, and then (b) video-case-based group discussions. The details about these steps are given ın Table 3.6 below.

Table 3.6 Design of Teaching Experiment

Weeks	**Sessions (Ss)**	**Overview of sessions**
Week 1-2		Pre-clinicals interviews with PSTs Preparation of lesson plans by PSTs
Week 3	S1	MCV1 Additive and multiplicative reasoning • Reflection papers and group discussion-MCV1
Week 4-5	S2-S3	MCV2-MCV3 Reasoning about unitizing and norming - Single unit - Composite unit • Reflection papers and group discussion • Revised version of lesson after Session 1
Week 6-7	S4-S5	MCV4- MCV5 Reasoning about proportional relationships - within-between relationships - part-part-whole relationships - invariant and covariance • Reflection papers and group discussion-MCV4 • Revised version of lesson plan after Sessions 2 and 3
Week 8	S6	• Reflection papers and group discussion-MCV4 and MCV5 • Revised version of lesson plan after Session 4 and 5
Week 9		Post-clinical interviews with PSTs Last version of lesson plan after Session 5

3.6.1. Individual analysis and reflections on micro-case video clips

Teaching experiment sessions began with prospective teachers' analyzing and reflecting on micro-case video clips individually. First, the prospective teachers watched the micro-case video clips on the projection screen. Then, they wrote a reflection paper about video clips in order to respond to the following prompts that were similar to those used by Jacobs et al. (2010):

a) Please describe in detail what you think this student did in response to this problem *(attending)*

b) Please explain what you have learned about the student's understanding? *(interpreting)*

c) Pretend that you are the teacher of this student. What will be the following instructional strategies you provide to the student in the video clip? (*deciding how to respond)*

Before starting teaching experiment sessions, reflection papers were distributed, and the prospective teachers began watching video clips. They were free to write reflection papers while watching video clips or after. Besides, prospective teachers were allowed to watch video clips one more time when they required it. The facilitator initiated a video-clip-based group discussion when participants finished writing their reflection papers.

3.6.2. Video based group discussions

In order to investigate the development of prospective teachers' abilities to notice students' thinking, their video based discussions were analyzed. The prospective teachers watched video clips individually, and then, they wrote reflection papers. Individual video analysis and writing reflection paper took approximately 30 minutes. Then, group discussion was initiated by the facilitators (i.e., my supervisor and me).

Recent studies showed that pre-service teachers who examine classroom videos learn to attend to the noteworthy mathematical details of student understandings (e.g., Goldsmith & Seago, 2008, 2011; van Es & Sherin, 2008). Teachers whose attending

skills develop change their interpretation of students' mathematical understanding and teaching practices based on what they notice about the events and how they analyze them over time (van Es & Sherin, 2008). In addition to the importance of videos, the role of the facilitator in engaging teachers in productive discussions of students' mathematical understandings in the videos has been highlighted in previous studies (e.g., Goldsmith & Seago, 2011). As the facilitator, I used a framework to facilitate the video-based group discussion introduced by van Es, Tunney, Goldsmith, and Seago (2014). The framework includes four categories to facilitate group discussion: *orienting the group to the video analysis task*, *sustaining an inquiry stance*, *maintaining a focus on the video and the mathematics*, and *supporting group collaboration.* Then, facilitators initiated group discussion. The details of these four dimensions are given below.

Orienting the group to the video analysis task: First, the facilitators guides prospective teachers to orient the group to the video analysis task. This dimension has two moves: *contextualizing and launching.* Before watching the video clips, information about video clips is given to assist prospective teachers as an entry point **(**contextualizing**)**. Then, the facilitator provides general prompts to invite prospective teachers to share their video clip observations as a part of launching (e.g., Let us talk about the video. What did you notice in the video?)

Sustaining an inquiry stance: In our study, another concern of the discussion was to sustain prospective teachers' inquiry stance. In this regard, as the facilitators, we followed six moves. we used highlighting to draw prospective teachers' interest in the noteworthy events they did not notice. Similarly, if one of the noteworthy ideas presented by prospective teachers was ignored, we *lifted* this idea and made it the focus of group discussion (Tunney et al., 2014). Furthermore, two other ways were used to expand prospective teachers' ideas: pressing and clarifying. The prospective teachers were encouraged to elaborate on their ideas through questions. These were "Please tell me more about this student's understanding of unit rate?" or "Ok, I got your idea but can you give me more details on how you understood it?". In addition, their ideas were rephrased by the facilitator to confirm the shared points of the ideas using expressions such as "So, what you are saying is that student used one lira as a benchmark while

matching units" (Tunney et al., 2014, p.346). Finally, offering an explanation and countering moves were used to ensure productive group discussion by introducing a way of interpreting or making sense of what was happening in the video and exploring alternative points of the idea (Tunney et al., 2014, p.347).

Maintaining a focus on the video and the mathematics: To keep participants' focus on the video clips, as facilitators, we used three strategies: redirecting, pointing to evidence, and connecting ideas. When the participants had an off-topic discussion by talking about unrelated issues, we redirected them to the topic by posing questions such as "Can we get back to the video clip?" (Tunney et al., 2014, p. 346). During the group discussion, prospective teachers were asked to show evidence for proposed claims. Finally, we tried to connect different ideas raised in the group discussion by posing question such as, "Is this idea similar to Nazli's solution in the video clip?" or "What would be your strategies if your student would solve this problem like that?".

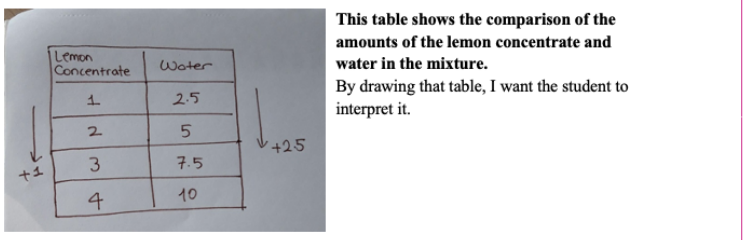
Supporting group collaboration: To support group collaboration, we used three moves: standing back, distributing participation, and validating participant ideas. During the discussion, we stood back and allowed participants to discuss the video clips without interrupting the discussion. we encouraged others who were not participating to share their ideas, for example by saying, "Tuna, tell me what you think about your friend's idea?" or "Is there anyone who thinks differently?". Finally, we supported the participants' contributions to the discussion by saying, "Hmm, this idea is interesting; I did not think about that before".

3.6.3. Post-intervention process

In order to investigate the development of prospective teachers' professional noticing of students' thinking at the end of video-clip-enhanced intervention sessions, the revised lesson plans were examined, and post-clinical interviews were conducted with the prospective teachers. The details of this process are explained below.

3.6.4. Making revisions in lesson plans

At the end of each session, prospective teachers were asked to revise their lesson plans if needed. These weekly revised lesson plans were utilized to follow their progress regarding what they noticed about students' understandings, how they interpreted this issue considering their mathematical understanding, and how they used it to respond to students' mathematical understanding. Figure 3.13 shows an example of weekly revisions of lesson plans. Making weekly revisions was not compulsory. At the end of each week, if the prospective teachers wanted to make some changes to their lesson plans, we asked them to write why they needed to make revisions based on the previous teaching experiment session. For instance, as seen in Figure 3.13, the prospective teacher made weekly revisions to her lesson plan by explaining the reason why she made these revisions. She revised her lesson plan after Session 2, providing her rationale for these changes and improvements based on comments during video-based group discussions. Additionally, she revised the next instructional actions in the lesson plan after Session 4, including specific revisions mentioned in the group discussion. The examples of prospective teachers' initial and revised lesson plans are presented in the Appendix D.



This table shows the comparison of the amounts of the lemon concentrate and water in the mixture.
By drawing that table, I want the student to interpret it.

Answer2(estimation)
We should add 8 cans of water in total. If we add more than that we have tasteless lemonade.
As the amount of water increases, the taste of the lemon decreases. (estimating)
Remediation of the misconception
First, I would ask the student how did s/he come up with this solution or what is the reason behind his/her estimation. I would ask his/her what it means to have the same taste or the same ratio of the amounts that are mixed.

[redacted] March 04, 2019
Sınıfta izlediğimiz videolardan yola çıkarak öğrencinin soruyu çözerken tahmin de edebileceğini düşündüm. Bunu da olası cevaplara eklemek istedim. Burada öğrenci tahmin ederek cevaplıyor. İşlem yapılmıyor sadece cevaba yakın, ona mantıklı gelen bir sayı söylüyor. Bunu ortadan kaldırmak için de neden böyle düşündüğünü veya neyi düşünmediğini anlamam gerekiyor. Aslında kavram yanılgısı değil de tahmini bir cevap. Peki olması gereken sonucu nasıl bulabiliriz, bunun için neleri düşünmeliyiz gibi sorular sorabilirim diye düşündüm.
Burada miktarlardan bir tanesini sabit tutup diğerini değiştirince karışımın tadı nasıl değişir, onu da sorabilirim.

[redacted] April 19, 2019
Kavram yanılgısını gidermek için şunu yapıyoruz: öğrenci önce tahminen 8 bardak su eklemeliyim demişti. Sonra limonatayı çizdik beraber. İçinde 2 bardak limon konsantresi var 5 bardak da su var. karışımımızdaki maddeler bunlar. Sonra kendi limonatası ile karşılaştırdı. Kendisininkinde 4 bardak limon konsantresi ve 8 bardak su vardı. Fark etti ki baştaki limonataya 2 bardak limon konsantresi ve 3 bardak su ekleyince bu oluyor. Ama bu ekleme bizim limonatamızın tadını bozuyor çünkü başlangıçtaki limonatada 2 bardak limon konsantresi ile 5 bardak su vardı. Demek ki aynısını eklesem sorun olmayacak diyor ve toplamda 4 bardak limon konsantrem olacaksa 10 bardak da suyumun olması gerekiyor ki tadı aynı kalsın çünkü 10 bardak yerine 8 bardak su olduğunda daha limonlu olur diyor. Burada öğrencinin ''composed unit'' kullanarak soruyu çözüyor. Unitlerimiz 2 bardak limon konsantresi ve 5 bardak su. O zaman her bir unit'i 2 katına çıkarabilirim böylece limonataya aslında kendisinden bir tane daha eklemiş oluyorum. Başka şekillerde de çözebilir fakat öğrencinin bu şekilde çözmüş olduğunu varsaydım.

Figure 3.13 An example of PST5's revised lesson plan

3.6.5. Conducting post-clinical interviews

Post-clinical interviews as the other data collection tool were used to examine the development of prospective teachers' noticing and to clarify the explanations that they wrote on lesson plans. Post-clinical interviews were conducted with prospective teachers at the end of teaching experiment sessions. To this end, we used the same five problems and similar prompting questions that were used in pre-clinical interviews; (a) Can you solve this problem please? (b) Can you solve this problem in another way(s) which students would use?, and (c) Imagine, you were a teacher of this student, what would be your next instructional action(s) based on the student's answer?. Then, we showed the participants their previous solutions in pre-clinical interviews. We asked, "Why did you make changes to your first solution strategies and your explanations? What changed for you?" to understand the development of the prospective teachers' noticing of students' understanding through video case-based teaching experiments. Finally, we asked questions to clarify the unclear issues in their revised lesson plans. Each post-interview took approximately sixty minutes.

3.7. Data analysis

In this study, pre-post clinical interviews, initial-revised lesson plans, reflection papers, and video-case-based group discussions were analyzed. Additional data includes field notes from group discussions. The data were analyzed using Jacobs et al.'s (2010) framework and coding scale (see Table 3.8). The researcher used a two-point coding scale for attending (limited evidence or evidence), a three-point scale for interpreting (lack of evidence, limited evidence, and robust evidence), and a three-point scale for deciding how to respond (general, case-specific-undetailed and case-specific-detailed). In addition, pre-post clinical interviews, and initial and weekly revised lesson plans have been used to examine the changes in individual prospective teachers' noticing as a result of micro-case video clip sessions. The data analysis process is explained in detail below.

In the pre-post clinical interview data analysis, thematic coding was utilized to analyze and report the themes. To this end, all the data were analyzed by incorporating the

processes of familiarization with data and generating initial codes related to prospective teachers' knowledge about students' mathematical thinking based on given proportional problems before the video-case enhanced intervention process (Braun & Clarke, 2006). Post-clinical interviews were examined through Jacobs et al.'s framework to project the developmental process of prospective teachers' noticing skills. Professional Noticing Framework was used to analyze prospective teachers' professional noticing skills in this study (Jacobs et al., 2010).

In the current study, I examined prospective teachers' reflection papers, group discussions, and weekly revised lesson plans for data analysis. I used qualitative data analysis methods (Strauss & Corbin, 1990). Firstly, we organized the data chronologically week by week. After organizing the data chronologically, I read all reflection papers, group discussion transcripts in each week of teaching experiment sessions and weekly revised lesson plans. Then, all data were analyzed based on professional noticing framework

In order to analyze the data, Jacobs et al.'s (2010) coding process was followed (see Table 3.7). We scored responses to the professional noticing prompts based on the extent of proposed evidence of the prospective teachers for the students' mathematical understandings. The responses were coded as seen in Table 3.8. Each response regarding *attending* and *interpreting* skills was scored as low, limited, and robust evidence considering the students' mathematical ideas. However, responses regarding *deciding how to respond* were scored as general, case-specific-undetailed, and case-specific-detailed. All data were independently coded by the researcher and faculty members familiar with the study's data analysis framework and coding schema. Inter-rater reliability was at least 83 percent, and disagreements were resolved through discussion.

Coding for attending

In order to determine whether the prospective teachers attended the noteworthy elements of the student's strategy, we followed Jacobs et al.'s (2010) coding process. Prospective teachers' responses were scored as limited evidence, and robust evidence

(see Table 3.8). The PSTs' scores show the extent of evidence teachers provided in attending to students' strategy details in micro-case video clips. Specifically, the details include noteworthy and relevant mathematical details such as how the student created composed units (see Grocery problem) or how the student described part/part relationships. Because of the nature of the current study (i.e., including one-on-one student clinical interviews), scoring was done as "limited" and "robust." All participants provided limited or robust evidence of attention to the details of students' strategies.

The responses categorized as "robust evidence" involved the most noteworthy and relevant details with explicit descriptions. Jacobs et al. (2010) described robust evidence as "demonstrating evidence of attention to children's strategies were phrased in many ways, but they all tracked the entire strategy with substantial detail about the mathematically important aspects of that strategy" (p. 183). On the other hand, responses that were categorized as "limited evidence" involved fewer details and general descriptions. At times, these descriptions were not consistent with students' strategies.

Coding for interpreting

The prospective teachers' responses were scored as lack of evidence, limited evidence, and robust evidence (see Table 3.8). Each response was scored based on the extent of evidence that prospective teachers interpreted for students' mathematical understandings. Instead of seeking the best interpretation, we concentrated on what students understood and did not understand and their reasoning related to students' mathematical strategies. In addition, we examined prospective teachers' interpretation of students' mathematical understanding and whether it was consistent with the existing studies about the related topic. If the prospective teachers interpreted the student's understanding in a consistent and detailed way with the student's strategy and this interpretation was consisted with the existing literature about proportional reasoning, this response was considered to be robust. If the response that provided limited evidence contained some understandings that consisted of student strategies without giving all related details, this response was considered to be a limited response.

On the contrary, if the response did not provide evidence and contained non-clear or too general claims about the student's strategy or claims unrelated to the student's understanding, this response was considered to provide low evidence.

Coding for deciding how to respond

Professional noticing of students' mathematical thinking demands not only attending to students' strategies and interpreting their understandings demonstrated in the strategies but also requires applying those understandings in deciding how to respond (Jacobs et al., 2010, p. 188). The prospective teachers' responses were scored on a 0-2 scale: general (0), case-specific-undetailed (1), and case-specific-detailed (2). These scores reflected the extent of evidence that the prospective teachers showed in deciding how to respond based on students' understandings. We sought the next instructional actions, such as follow-up questions that could be posed or further problems that could be asked based on students' mathematical understandings. In addition, we looked for the next instructional actions that aligned with prospective teachers' rationales and left space for students' thinking.

In this study, if the prospective teachers provided detailed instructional action(s) consistent with the student's strategy and understanding and left space for students' thinking, this response was considered as case-specific-detailed (robust evidence). Although responses with a score of case-specific-undetailed (limited evidence) focused on students' strategies and understandings with evidence, they included less specificity and clarity. Responses with scores of general (lack of evidence) included instructional actions in general, without providing any specific evidence of deciding how to respond based on students' strategies and understandings (see Table 3.8). Prospective teachers' reflection papers and lesson plans at the end of each session were examined, and each proposed instructional action was coded based on the categories. Then these codes were counted to show the progress before and after teaching experiment sessions (see Appendix E).

Table 3.7 Professional noticing framework (Adopted from Jacobs et al., 2010)

Noticing Skills	Explanation
Attending to mathematical noteworthy elements	This skill includes the description of mathematically noteworthy elements and consistencies or inconsistencies in students' mathematical actions in the context of proportional reasoning. ✓ Please describe in detail what you think this student did in response to this problem *(attending)*
Interpreting students' understanding	This skill includes both highlighting noteworthy mathematical elements and the practice of making sense of students' understanding with the reasons and evidence that underlie students' understanding in the context of proportional reasoning (Make sense of the reasons what students did understand and make sense of the reasons what students did not understand) ✓ Please explain what you have learned about the student's understanding *(interpreting)*
Deciding how to respond based on students' understanding	This skill includes next instructional actions (suggestions) based on each student's understanding in the context of proportional reasoning. The rationale of selected instructional actions based on students' understandings is provided for existing situation. ✓ Pretend that you are the teacher of this student. What will be the following instructional strategies you provide to the student in the video clip? (*next instructional action)*

Table 3.8 Professional noticing skills coding scheme used in analysis process of the present study

Noticing skills	Low evidence	Limited evidence	Robust evidence
Attending to the noteworthy mathematical elements	General description or general features of the student's strategy but omitted details of how the problem solved Inconsistent information with the video clips	Identification of some mathematically significant details about student's strategy	Detailed description of mathematically noteworthy elements about the student's strategies
Interpreting students' understanding	Too general (e.g., I was glad that the teacher allowed student to use student to use multiple ways of arriving to the correct answer	Maintain a focus on interpreting student's understanding but less depth than responses that demonstrated robust evidence Overgeneralizing but still focus on interpreting	Make sense of the details of each strategy and note these details what student did understand by providing evidence What strategies and understanding student did not demonstrate by providing evidence
	General	**Case-Specific-undetailed**	**Case-Specific-detailed**
Responding (Next instructional actions)	Too general and nonspecific suggestions based on the students' understanding (lack of evidence)	Providing specific future/next instructional actions/suggestions based on student's understanding for their but in more general way (limited evidence)	Providing specific future/next instructional actions based on students' understanding of ratio and proportion in a detailed way (robust evidence)

3.8. Trustworthiness

Validity and reliability are two crucial concepts that researchers must consider while conducting research, analyzing the data, and discussing the quality of the study (Patton, 2002; Yıldırım & Şimşek, 2006). Reliability and validity issues have been perceived differently in qualitative and quantitative research. Lincoln and Guba's (1985) identification was used for qualitative meaning in this regard. They identified credibility, transferability, dependability, and confirmability terms referring to internal validity, external validity, reliability, and objectivity as indicators of trustworthiness in qualitative studies. In the following parts, these strategies and how they are handled in the current study will be addressed in detail.

According to Lincoln and Guba (1985), *credibility* is one of the most significant factors related to internal validity in establishing the trustworthiness of qualitative research. Internal validity addresses the question, "how do research findings match reality?" (Merriam, 1998, p.213). Merriam (1998) points out strategies to improve internal validity, such as triangulation (using multiple sources), member checks, long-term observation, and multiple investigators. Most of these strategies were utilized in this study.

First, triangulation was used to increase the credibility of the study. Triangulation was described as "*Triangulation has been generally considered as a process of using multiple perceptions to clarify meaning, verify the repeatability of an observation or interpretation*" (Stake, 2000, p.443). The data were collected through multiple sources (i.e., pre-post clinical interviews, initial and weekly revised lesson plans, reflection papers, micro-case video analysis group discussions, field notes, etc.). In this vein, data triangulation was utilized in the current study. Besides, all sessions were observed with a professor in mathematics education, who shared valuable ideas about the teaching experiment process considering the methodological issues and what details she observed as significant and unexpected in prospective teachers' reasoning about the video clips. While conducting post-interviews, transcribed pre-interviews of participants were shared to ask whether they agreed with their ideas. In this regard, member-checking was used to verify credibility.

Moreover, in the current study, prolonged engagement with participants through nine weeks was used to increase the credibility. Consequently, we were allowed to establish trust with participants during this period.

Transferability refers to the external validity of qualitative studies. It is associated with the question of "How generalizable are the result of the study?" (Merriam, 1998, p.223). Yet, it is not possible to discuss the generalizability of qualitative research because, in qualitative research, a single case or small non-random sample is chosen to understand the context in depth, not to determine what was true for the entire community. In the current study, transferability was utilized by elaborating on research settings in detail. Therefore, thick descriptions of the study (e.g., the context of the study, participants' selection criteria, information about instruments, data implementation process, etc.) were provided.

Third, to build trustworthiness, *dependability* is concerned with the consistency and replicability of the findings (Lincoln & Guba, 1985). The consistency of the study can be confirmed by detailing each step of the research process (Yin, 2003). In this sense, the data collection procedure, the followed steps in the current study, and the nature of teaching experiment sessions were explained in detail. Moreover, the second coder was assisted during the coding processes to confirm the dependability of the coding categories. Then two coders worked collaboratively on the data analysis and discussed the codes until they reached a consensus on the categories. *Confirmability*, the last criterion of trustworthiness, refers to objectivity in qualitative studies (Lincoln & Guba, 1985). In order to provide confirmability and multiple data sources, details were utilized. In addition, to ensure confirmability, the analyses were supported by quoting from interview transcripts, group discussion excerpts, and illustrative examples of weekly revised lesson plans.

3.9. Researcher Role

Within a teaching experiment, researchers have various roles, such as teacher, listener, facilitator, coach, or discussion leader, that change based on interviews, observers, subjects or participants (Steffe, 1991; Lesh & Kelly, 2000). These roles allow

researchers to have firsthand experience with participants (Steffe & Thompson, 2000). The main aim of teaching sessions is to examine the development of prospective teachers' noticing of students' mathematical understanding in an alternative classroom context developed by the researcher. In this regard, the researcher's role as a facilitator was to design teaching experiment sessions Therefore, firstly, I selected the micro-case video clips from the video clips archive for the following session based on the weekly video-based group discussions and revised lesson plans. Then, I initiated the group discussion by asking questions to draw PSTs' attention to essential details throughout group discussions if they do not notice the details or lose their attention.

CHAPTER 4

FINDINGS

This study aimed to examine the nature and the developments of prospective teachers' professional noticing of students' mathematical thinking in the context of proportional reasoning through teaching experiment sessions in a video case-based learning environment. This chapter presents the findings in three sections based on the key components of proportional reasoning. Each section is related to one of the components, and sections were presented based on the components suggested by the Carpenter et al. (1999): additive-multiplicative thinking, unitizing and norming, and proportional relationships. I documented prospective teachers' professional noticing of student thinking (Jacobs et al., 2010) comprised of the three interrelated skills of "*attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings*" (p. 172) across these three components. Within each section, prospective teachers' existing noticing skills of student thinking in the context of proportional reasoning were analyzed using pre-interviews, initial lesson plans conducted before starting teaching experiment sessions, and each week's reflection papers. Then, I examined how prospective teachers' noticing of students' thinking developed throughout the video-case-based teaching experiment by using data from group discussions, weekly revised lesson plans, and post-interviews conducted at the end of teaching experiment sessions. The description of the problems was shown in Table 4.1. below.

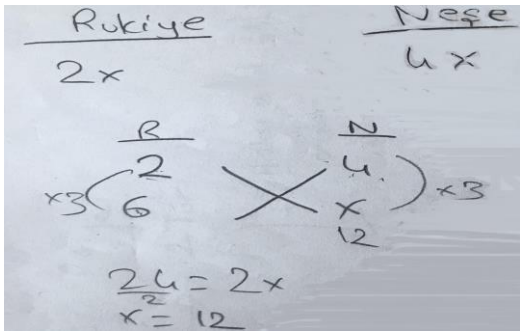
Table 4.1 Description of the problems

Micro-case video clips (MCVs)	Problems	Students	Description
Additive-multiplicative reasoning MCV1	Racetrack problem	Student 1 (S1)	In MCV1, Student 1 solved the problem multiplicatively as erroneous strategy
Reasoning with unitizing and norming MCV2, MCV3	The Grocery problem	Student 2 (S2)	In MCV2, Student 2 solved the problem incorrectly using singleton units
	The Balloon problem	Student 3 (S3)	In MCV3, Student 3 solved the problem correctly using composite units
Reasoning with proportional relationships MCV4, MCV5	The Photograph problem	Student 4 (S4)	In MCV4, Student 4 solved the problem correctly using within-between strategies
	The Mixture problem	Student 5 (S5)	In MCV5, Student 5 solved the problem correctly using part-part-whole strategies

4.1. Prospective Teachers' Professional Noticing of Students' Thinking in the Context of Additive- Multiplicative Reasoning

Based on the research questions of the current study, prospective teachers' professional noticing of student thinking in the context of additive and multiplicative reasoning was investigated through micro-case video clips of the Racetrack Problem in the following parts. First, prospective teachers' professional noticing skills (i.e., attending to, interpreting, and deciding how to respond) were examined before video analysis. Then, the contribution of participation in video-based group discussions to prospective teachers' professional noticing skills in the context of additive and multiplicative reasoning was investigated. Table 4.2 shows Student 1's solution to the Racetrack problem.

Table 4.2 Ahmet's (Student 1) strategy of Racetrack Problem, MCV1

Rukiye and Nese were running around a track and they run equally fast but Nese started before. Nese has run 4 laps, Rukiye has run 2 laps. When Rukiye has run 6 laps, how many has Nese run? 

By addressing research questions, Table 4.3 shows the prospective teachers' responses about attending to Student 1's strategy and interpreting the student's understanding of MCV1 before and after video analysis. Prospective teachers watched micro-case video clip including Student 1's incorrect solution in the context of additive and multiplicative reasoning. Before video analysis, all prospective teachers provided limited evidence of attention to the details of Student 1's strategy. Their responses included a description of the student's strategy without the meanings of these operations. For instance, PST5 described Student 1's approach generally, such as "thought proportionally," but omitted the details of how this student used this approach in the solution. Their response did not provide sufficient information for another person to recreate Student 1's strategy and a window into the student's mathematical understanding of additive and multiplicative thinking. After participating in the video clip-enhanced learning environment, all participants exhibited robust evidence in attending to the student's strategy. 4out of 6 prospective teachers provided analysis with robust evidence of interpreting Student 1's understandings. Whereas half of the prospective teachers provided general responses, including too general and nonspecific suggestions based on Student 1's understanding, and half provided case-specific-undetailed responses, including specific future instructional actions based on Student 1's understanding, but they were in a more general way and mostly procedural. On the other hand, after video analysis, all proposed case-specific-undetailed instructional actions. Their instructional suggestions were relevant but vague. Unlike

the instructional actions they proposed before the video analysis, their responses included more details and proportional language. However, they still concentrated on the numbers given in the problem context without considering the meaning of the relationships. In the following sections, the prospective teachers' professional noticing of Student 1's thinking in the context of additive and multiplicative reasoning before video analysis was examined in detail.

Table 4.3 PSTs' noticing level of evidence of attending and interpreting before and after Racetrack problem vide based intervention (MCV1)

Prospective teachers		PST1	PST2	PST3	PST4	PST5	PST6
Attending	Before video analysis	Limited	Limited	Limited	Limited	Limited	Limited
	After video analysis	Robust	Robust	Robust	Robust	Robust	Robust
Interpreting	Before video analysis	Limited	Limited	Limited	Limited	Limited	Limited
	After video analysis	Robust	Robust	Robust	Limited	Robust	Limited
Responding	Before video analysis	General	General	CSU	General	CSU	CSU
	After video analysis	CSD	CSD	CSU	CSU	CSD	CSU

*CSU= Case-specific-undetailed

*CSD= Case-specific-detailed

4.1.1. Prospective Teachers' Professional Noticing before MCV1 Analysis

This section provides data on the nature of prospective teachers' knowledge and noticing of students' thinking about additive and multiplicative reasoning. I used data obtained from prospective teachers' written and verbal explanations in pre-clinical interviews before starting the teaching experiment sessions, the initial lesson plans they planned to teach, and the reflection papers they wrote before the video based group discussion. The prospective teachers' noticing skills before participating in the video clip enhanced learning environments in the context of additive and

multiplicative reasoning. The representative examples of prospective teachers' attending and interpreting responses and instructional actions before video analysis were explained in the following sections.

4.1.1.1. Attending to the Mathematical Elements of Student 1's Strategy

To analyze the PSTs' attending skills, their responses were grouped into two categories based on to what extent they attended to mathematical elements to describe students' responses: limited evidence, and robust evidence. A significant standard of evidence of attending to the details is whether the reader could describe the student's strategy based on prospective teacher's statements. All prospective teachers provided limited evidence of attention to the details of the Ahmet's (Student 1) strategy. The representative examples of limited evidence were given below.

PST1 solved this problem incorrectly during the individual pre-clinical interviews. She used cross product algorithm focusing on the scalar relationship between numbers (e.g., times two relationship between 4 and 2). The following response of PST1 in the video analysis reflection paper showed that she provided *limited evidence* of attention to the details of Ahmet's strategies in MCV1 in the Week 3 (W3).

> PST1: Student thought that there was a multiplicative relationship (R*eflection paper, W3)*

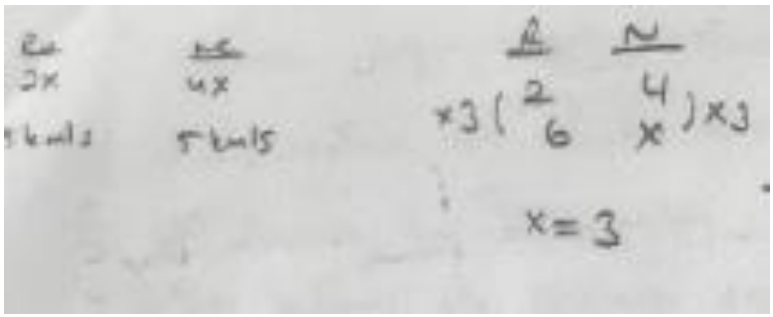


Figure 4.1 Attending to Ahmet's strategy by PST1

PST1's response includes a description of Ahmet's strategy (see Figure 4.1). However, the prospective teacher did not describe the meanings of these operations, such as applying within or between ratios. For instance, she only stated that this student thought there was a multiplicative relationship but did not state any specific evidence

of the student's strategy. Besides, this prospective teacher did not describe how the student recognized the 'times two relationship' as evidence that his strategy was correct. PST1 did not interpret what this student could not understand about this topic (i.e., additive and multiplicative thinking). Overall, this prospective teacher received a score of *limited evidence*.

Another illustrative example of *limited evidence* of attending skills is presented below. PST5 solved this problem correctly in the clinical interviews. She first focused on the words, such as a 'track', in this problem. For instance, she questioned what kind of track; round, or plane.

> PST5: *The student thought proportionally. He said that it (Nese's run) was two times more than Rukiye's run* (Reflection paper, W3)

PST5's response above demonstrated limited evidence of attending to the details of the student's strategy in the reflection paper. She described the student's approach generally, such as "thought proportionally", but omitted the details of how this student used this approach in the solution. Besides, even though PST5 paraphrased the student's statement by saying Nese's run was two times more than Rukiye's run, the references about where this "two times" relationship came from and the information of what "two times" represented there were missing. Thus, this response did not provide sufficient information for another person to recreate Student 1's strategy and a window into the student's mathematical understanding of additive and multiplicative thinking. The absence of these noteworthy mathematical details led us to categorize this response as *limited evidence of attending* to the S1's strategy.

Another response sample belongs to PST2, who solved this problem incorrectly in the clinical pre-interview, and she thought there was a multiplicative relationship between the given quantities.

> PST2: *This student applied the cross-product algorithm using direct ratios. He found 12 by multiplying six by two and found a multiplicative relationship. He recognized the two times relationship between 4 and 2. Student's answer is wrong* (Reflection paper, W3)

PST2 provided limited evidence of attending to some of the details of the student's strategy. For example, she mentioned that Ahmet applied the cross-product algorithm using direct ratios (6*4=2*x), and this student found 12 by multiplying six by two considering the multiplicative relationship. However, this prospective teacher did not describe other noteworthy mathematical details, such as what 2x and 4x represented in the student's solution and why this student's solution was wrong. The absence of these noteworthy mathematical details led us to categorize this response as *limited evidence of attending* to the student's strategy.

4.1.1.2. Interpreting Student 1's Understanding

Before detailed analysis, a pre-analysis was done by the researcher. PSTs' interpretations were categorized into three categories by Jacobs and her colleagues (2010). In this study, prospective teachers' interpretations showed that they analyzed students' understandings in both aspects: what students understood and did not understand. Therefore, PSTs' interpretations of students' mathematical understandings were examined as "what the student did understand" and "what the student did not understand". These interpretations were coded into three groups: low evidence, limited evidence, and robust evidence (see Table 3.8).

All prospective teachers provided limited evidence of interpreting student's thinking based on not only what this student understood but also what this student did not understand because the nature of the task led prospective teachers to make an explanation about student's incorrect strategies, such as "she could not understand the meaning of 'same speed' for this problem."

The following response sample was taken from the video analysis reflection paper that provided *limited evidence* to the details of Ahmet's strategy in the MCV1:

> PST3: *He thinks multiplicatively (and) he could not think that the distance will not change* (Reflection paper, W3)

PST3 made two general claims about this student's mathematical thinking. For instance, she said, "he thinks multiplicatively (and) he could not think that the distance (between Nese and Rukiye) will not change". However, she did not provide any evidence of how these details reflected what this student understood. Even though PST3 gave specific connections to the student's strategy, they were still limited. For instance, PST3 did not explain why this student could not think the distance would remain same. Hence her response was categorized as *limited*.

A representative response that provided *limited evidence* of interpreting the student's understanding in MCV1 is as follows:

> PST4: *He thought that the number of laps is direct (proportional), so there is a multiplicative relationship. Then, he found a wrong answer because of thinking multiplicatively. He tries to set up a proportion so he makes mistakes* (Reflection paper, W3).

PST4 focused on describing what Ahmet did and provided a specific connection to Ahmet's strategy, but lacked in-depth analysis. For example, he inferred that the student reached an incorrect answer because he thought multiplicatively. However, PST4 did not give information about how he reached such a conclusion from the student's solution. He also claimed that this student knew the multiplicative relations between the number of laps (i.e., scalar relationship between 2 and 4 laps). However, he did not provide any evidence of how he made this interpretation. Therefore, his response was categorized as *limited*.

Overall, although all responses were still focused on the student's understanding and provided some evidence of interpreting the student's understanding, they did not contain robust evidence of interpreting the student's understanding in MCV1. Therefore, all responses were categorized as limited.

4.1.1.3. Deciding How to Respond on the Basis of Student 1's Understanding

We used three categories that emerged from the data: "general," "case-specific-undetailed," and "case-specific-detailed." Table 4.4 illustrates the nature of PSTs'

alternative instructional actions in individual reflection papers. PSTs proposed 9 instructional actions. They mostly provided next instructional actions in *case-specific-undetailed* categories (limited evidence). 2 out of 9 instructional actions were in *general* categories (lack of evidence). The details of prospective teachers' responses were given below.

Table 4.4 PSTs alternative instructional actions before MCV1 analysis

PSTs	Suggested instructional actions			Total frequency (%)
	General (Lack of evidence)	Case-specific-undetailed (Limited evidence)	Case-specific-detailed (Robust evidence)	
PST5	0	1	0	0
PST4	0	1	0	1
PST3	2	0	0	2
PST1	0	1	0	1
PST2	0	2	0	2
PST6	0	2	0	2
Total (%)	2 (22)	7 (78)	0 (0)	9 (100)

A representative response that provided *case-specific-undetailed* instructional action(s) on the basis of Ahmet's understandings in MCV1 is as follows:

> PST6: *I would make this student model the rounds: Rukiye and Nese run, respectively, and then we can find the total round of each one.*
> *I would give him another problem in which they run at different speed, and I would ask him to compare the two solutions. Then I would make him question why these two solutions were different* (Reflection paper, W3).

Even though PST6 clearly considered this student's understanding in deciding how to respond, her response was not detailed and not specified. For instance, she proposed using models that might be useful to find the number of rounds for Nese and Rukiye. However, she did not give details about why she would use models or how she would

use the models to represent the problem. She did not provide any evidence to discuss the reasoning underlying the using models.

She also proposed another problem to gain an insight into the student's existing strategy: "I would give him another problem in which they run at different speed, and I would ask him to compare the two solutions". PST6 did not give specific details about this problem, which are not enough to link the student's existing understandings. Unlike responses that provided robust evidence, this prospective teacher did not clearly state the rationale for the following instructional actions for that student; it is unclear why she selected these actions and what this student would learn from them. In this vein, PST6's response was categorized as case-specific-undetailed.

Below is a sample response involving *general* instructional actions on the basis of Ahmet's mathematical understandings in MCV1:

> PST3: *If I were the teacher of this student, I would use technology, table and activities* (Reflection paper, W3).

PST3 says that she would use technology, table, and activities, but she did not provide any reference to Ahmet's understanding. She did not give the details about how technology or table can be used for this student and what kind of activities should be used for what purpose. Her statement was too general and did not contain case-specific mathematical details based on Ahmet's understanding in the MCV1. Providing too general instructional actions indicates that this prospective teacher did not consider the student's understanding while providing further instructional action(s). Overall, before participating Session 1, all participants attended to student's strategy and interpreted student mathematical understanding at a limited level. They responded to student thinking in MCV1 as general and case-specific-undetailed.

4.1.2. Prospective Teachers' Professional Noticing After MCV1 Analysis

After video analysis, PSTs' professional noticing of the student's understanding were developed. In the following, this process was examined.

4.1.2.1. Attending to the Mathematical Elements of Student 1's Strategy

The following illustrative examples of discussion excerpt shows PSTs' progress in attending to the student understanding through group discussion. As mentioned in the Method section, after watching the video clip, the discussion as a part of the session was initiated.

Facilitator	*So, what did you notice about this student's understanding when you watched this video? What stood out for you?*
PST6	*While solving the problem, he wrote the numbers side by side, so he thought of the relationship proportionally. He states that if there is 'x,' this should be multiplicative, not additive, so I should use cross-product.*
PST4	*At first, he stated that their speed was the same. Then he wrote the numbers side by side; he saw the scalar operator. Therefore, he applied the scalar relationship.*
PST6	*He thinks that if he used 'x', he would apply ratio, because he got used to it.*
PST4	*He called the laps as 2x and 4x, so he thought that Rukiye is faster than Nese*
Facilitator	*Hmmm, you mean that the student thought multiplicatively, right? What did this student miss?*
PST3	*The distance remains the same.*
PST4	*They have the same speed.*
PST6	*Actually, he recognized that their speed was the same, but interestingly, while solving the problem, he ignored this fact (as a reply to PST4).*
PST4	*Nese starts earlier than Rukiye and their total.*
Facilitator	*So, what kind of a relationship is there in terms of the number of laps they run?*

PST1 *Additive relationship between the given quantities. As we talked about just now, this student might not make sense of additive and multiplicative meanings.*

PST4 and PST6 focused on the "x" and how this directed the student to apply multiplication in the group discussion. When the facilitator asked what they noticed about this student's understanding, they immediately focused on "x" and indicated the writing "x" as the reason for applying the cross-product algorithm. Then as the facilitator, I wanted to guide their thinking on mathematical elements which they missed, paying more explicit attention to students' understanding of additive and multiplicative situations. Referring to the session of the week, PST4 and PST1 evaluated their own progress (see discussion excerpt above) providing *robust evidence* in the revised lesson plan as follows:

> PST4: *When I watched the video clip first, I focused on 'x' as the reason for the error that the student made. However, after the discussion, I realized that this student did not transition from additive to multiplicative thinking. Probably, he learned only cross-product multiplication at the beginning of this topic, and he did not know the reason behind the multiplicative situation. In this regard, he considered that the two variables in this problem are linked multiplicatively (two times relationship between 2x and 4x), so he solved this additive problem multiplicatively* (Revised lesson plan, W3).
>
> PST1: *During group discussions, they focused on "x" as the reason for this mistake. However, I recognized that we should talk about potential student understanding in multiplicative and additive thinking because this problem was an additive one, and he solved it multiplicatively. Even though he explained the given quantities, such as the same speed and starting earlier, Ahmet could not think additively. PST6 and PST4 should have discussed possible reasons behind this incorrect multiplicative thinking* (Revised lesson plan, W3).

4.1.2.2. Interpreting Student 1's Understandings

Unlike the individual analysis of prospective teachers, during the group discussion, they interpreted student understanding with limited and robust evidence. This diversity of interpretations led to shaping the discussion environment. This way, prospective teachers began to detail their interpretations of student understandings. There were salient differences between their interpretations in the individual analysis and the group discussion. While one of the PST reflected on their interpretation in the group discussion, others raised issues and questions about the provided interpretation. Specifically, group discussion prompted PSTs to look for different kinds of interpretations and to develop particular ways of viewing issues that were discussed in detail and at length. On the other hand, some of the prospective teachers did not listen and respond to each other. Mostly, each of them offered a different explanation that was not related to previous statements discussed by others.

The representative excerpts below show what and how prospective teachers interpret the student's understandings in MCV1. The facilitator's question shifted the discussion from attending to the mathematical details to interpreting details of the student's understanding in MCV1.

Facilitator	*So, tell me what you learned about Ahmet's understanding(s)?*
PST5	*I think the student always solves this kind of problems in the class, I mean including multiplicative algorithm. He did not solve additive reasoning problems. Therefore, he did not know how to solve this type of problem in another way. He is not aware of additive reasoning.*
Facilitator	*How do you know? So, what do you know about the objectives in our curriculum related to proportional reasoning?*
PST4	*I think there might not be anything about additive reasoning at the 7th grade, but I am not sure*
PST3	*Students learn 'rate' in the 6th grade and 'ratio' in the 7th grade, but I don't know exactly what they are.*
PST6	*I checked the objectives right now, but 6th grade objectives do not address additive reasoning either.*

PST5 *They already know how to solve multiplicative problems. As I said, they might not differentiate between additive situations and multiplicative situations because students will not think about additive and multiplicative problems together in order to discriminate their relationship.*

PST6 *As I said writing side by side and "x" direct him to apply multiplication.*

PST4 *I don't think so. He realized the scalar relationship, so he applied multiplication.*

PST6 *Do you mean the relationship between 2 and 4? He noticed that the runners' speed was the same, but he tended to find the value of "x." Because he set up proportion equations and then applied multiplication, he divided 24 by two and found 12. He applied this algorithm, and he never checked whether the answer was correct or not.*

Facilitator *Did he realize the relationship as PST4 stated?*

PST3 *I think he could not realize the scalar relationship between 4 and 2. First, he directly applied the cross-product algorithm. Then, while making the computation, he realized that two times two equals 4. Therefore, at the end of the computation, he stated that "I could find 12 by multiplying six by 2". He described the scalar relationship between numbers operationally. As we said, he could not discriminate between additive and multiplicative relationships. He could not describe this difference additively as two laps more. By the way, I am making this assumption based on the video clip.*

As seen in the excerpt above, PST5 directly focused on why he applied the multiplicative algorithm without giving any evidence from the clip. As the facilitator, I asked a question to understand whether the prospective teachers know about the related objectives in the Turkish Middle School Mathematics Curriculum. Even though they generally knew that these objectives were in the sixth and seventh grades, they were not knowledgeable about additive and multiplicative reasoning, which was

not included in the curriculum. Referring to this week's group discussion, PST2 by providing *robust evidence* wrote the following comments:

> PST2: *In the group discussion, I realized that we did not know the objectives related to proportional reasoning. Before teaching cross-product algorithms directly, we should talk about proportional and nonproportional situations. I recognized that students use multiplicative strategies for even additive problems after learning ratio and proportion because this type of problem, I mean the problems in which the fourth quantity is not given, might be seen as multiplicative problems, and students are familiar with the same type of problems after the 6th grade. Therefore, using daily examples, I started discussing proportional and nonproportional in my lesson plan. And also, I discussed the potential students' answers in detail for all my questions in the lesson plan, and I tried to explain these answers using more mathematical language* (Revised lesson plan, W4).

Then, PST6 focused on "x" as a contributing factor to applying multiplication. She stated that this student wrote "x" to set up a proportion equation, and this equation seems to invite a multiplicative rather than additive comparison between the given quantities. As a response, PST4 claimed that Ahmet first recognized the scalar relationship. Then, he applied the cross-product algorithm without giving reference to the student's mathematical understanding. At this point, PST3 elaborated on her interpretations by making specific connections to the student's strategies, and these interpretations were consistent with what Ahmet did. Even though PST3 stated that this student recognizes the scalar relationship between quantities, she did not provide evidence of whether the student realizes this scalar factor purposefully or not. After the group discussion, PST6 commented on why she would change her lesson plan, and PST3 explained her progress in noticing student strategies. Their provided *limited* and *robust* evidence respectively in the following:

PST6: *Before the group discussion, I did not recognize that our curriculum does not focus on additive and multiplicative reasoning or proportional and non-proportional situations. Students start learning ratio and proportion in the sixth grade. Then probably, students might show the tendency to apply the cross-product algorithm for all situations. In this regard, I decided to change my lesson plan, and I would like to focus on additive and multiplicative situations and the relationship between quantities in different contexts* (Revised lesson plan, W3).

PST3: *Before the group discussion, I did not think of the reasons for erroneous procedures of multiplicative methods. My friends' interpretations made me consider Ahmet's conceptions of additive and multiplicative situations. Ahmet could not describe the strategies, such as* within or between strategies *in this problem. Also, I recognized that other significant factors influence additive strategies, such as the numbers given in the context. If one number is a scalar multiple of another, students might directly apply multiplicative methods* (Revised lesson plan, W3).

4.1.2.3. Deciding How to Respond on the Basis of Student 1's Understanding

Table 4.5 shows the number of PSTs' instructional actions on the basis of Student 1's understanding after individual video analysis. The PSTs provided more specific instructional actions after video analysis, as seen below. According to the following table, PSTs proposed 19 instructional actions after video analysis. 53% of them were case-specific-detailed and provided detailed information on proposed instructional actions based on the student's understanding, including the purpose of the action and how to use it. In addition, 42% of the statements were case-specific-undetailed and did not provide general explanations. Finally, one of the PSTs provided general instructional actions that included a non-specific and undetailed explanation about the ways of instruction.

Table 4.5 Distribution of PSTs deciding how to respond PSTs' instructional actions after MCV1 analysis

PSTs	Next instructional actions			Total frequency (%)
	General (Lack of evidence)	Case-specific-undetailed (Limited evidence)	Case-specific-detailed (Robust evidence)	
PST1	0	1	2	3
PST2	0	1	2	3
PST3	0	3	2	5
PST4	0	2	1	3
PST5	0	1	2	3
PST6	1	2	1	4
Total frequency (%)	1 (5)	8 (42)	10 (53)	19 (100)

In group discussions, the PSTs mostly focused on alternative instructional actions to handle the difficulties of the student's understanding. Although the instructional actions that the PSTs provided in their individual reflections were mostly case-specific and undetailed, in the group discussion, the PSTs began to provide case-specific and detailed instructional actions/suggestions regarding Student 1's understanding(s).

The discussion excerpts below and the examples of instructional actions that the PSTs provided in revised lesson plans are given together.

Facilitator	Pretend that you are the teacher of this student. What instructional actions might you pose next?
PST1	I wonder if we use *different numbers such as 3 and 5 instead of 2 and 4.*
PST6	But he found the "x" and then he realized this relationship.
PST2	Why did you change the numbers?
PST1	Because, the relationship is so clear between given 2 and 4. In the video clip he told that "hmmm, there are two 2s in 4", so I would change the numbers as 3 and 5 in which there is no direct relationship.
Facilitator	What do we call this relationship in multiplicate methods? You remember?
PST2	*Scalar relationship or scalar factor…* I could not remember. (The group agreed by nodding their head)
PST4	Yeah, scalar relationship. Actually, he first applied cross product and then realized 2- and 3-times relationship. I think it won't let the situation change.

PST3	I agree with you. He would still make cross product multiplication.
Facilitator	Do you think he realized this relationship?
PST4	I don't think so. Even if we provide 3 and 5 instead of 2 and 4, he will apply cross product multiplication. He could not realize the additive relationship.

In the group discussion, PST1 questioned the potential effect of numerical content. PST1 proposed different numbers, such as 3 and 5. She thought that this student recognized the scalar relationship between 2 and 4 or 2 and 6 (one number is a scalar multiple of another number), so he applied the cross-product algorithm. In this regard, she assigned different numbers that were not evenly divisible to see whether he concentrated on the additive strategy. PST6 and PST4 immediately implied that the student realized the multiplicative relationship after applying cross product algorithm by explaining the student's understanding of MCV1.

After the Session 1, PST2 changed her lesson plan as follows, by pointing out the scalar relationship between the numbers mentioned in the discussion above. She provided all the following details about her suggested instructional actions: for whom and for what purposes. Furthermore, PST5 made revisions, and she explained the instructional strategy related to the student's understanding in the video by giving reference to the group discussion. Their comments and the illustrative example of initial and revised lesson plans were presented in the Table 4.6 below. In this regard, PST2's response was categorized as *case-specific-detailed.*

> PST2: *I completely changed this part of my (lesson) plan because this part could not reveal the student's thinking about additive and multiplicative relationships. I wrote questions about multiplicative relationships in my first lesson plan. After watching the video clip, I recognized that I did not focus on additive and multiplicative relations these and the questions I wrote cannot elicit students' understanding of how they construct these relationships. Besides, when I watched the video clip, like others, I only thought that 'this student solved the problem multiplicatively'. I could not concentrate on the reason of this difficulty. For instance, I could not realize the effect of numbers such as 2-4 or 4-8 (i.e., multiplier of others). The problem in the video contained easy*

multiplies, which would encourage the student to apply multiplication. Therefore, I would add two problems with similar context including non-divisible numbers to reveal his understanding of additive and multiplicative relationships (Revised lesson plan, W3).

PST2 pointed out that she completely changed her initial lesson plan after the video analysis. In her initial lesson plan, she proposed one problem to present the relationship between quantities. Even though she aimed to reveal the relationships between the numbers in the problem (i.e., the cupcake problem, see Table 4.7), she did not explain what kind of relationships were explored and the purpose of the focus of the expected relationships. In this regard, these instructional actions were grouped as undetailed. However, in her revised plan, she explained all the details about the following instructional actions by referring to the group discussion. Specifically, she pointed out the student's difficulty in discriminating additive and multiplicative relationships in MCV1 and how the relationship between the given values differs. She used a similar problem context to reveal the student's understanding and to clarify additive and multiplicative relationships for the student in the video. In this sense, she concentrated on understanding the quantitative relationships underlying the additive and multiplicative situations, their distinctions, and task-related factors such as the numbers in the videos.

Instructional actions in the individual reflections of the PSTs were mostly case-specific and undetailed. However, after video analysis, the PSTs started to provide case-specific and detailed instructional actions (see Table 4.7) during group discussions and in their weekly revised lesson plans (W3). In order to show the comparison of before and after video analysis, I assigned certain numbers to each level: 0 for general comments, 1 for case-specific-undetailed comments, and 2 for case-specific-detailed comments. Then, I calculated the average points of each PST's response for "deciding how to respond". For instance, the average point of PST1 before discussion was (0x1 + 1x1 + 2x0)/1=1 according to the calculation. Then, the average point after discussion was (0x0 + 1x1 +2x2)/3=1.7 based on Table 4.7.

Table 4.6 Examples of instructional actions presented in lesson plans

Example parts of …	
Initial lesson plan (before video analysis)	Revised lesson plan (after video analysis, W3-W4)
Question 1: Seher is a pastry cook having different recipes for cupcakes in different colors. While garnishing the cupcakes with cream, she adds food coloring in different colors and obtains many cupcakes. When garnishing is finished, these cupcakes are put to pastry shop for the sale. In the following table shows the number of sold cupcakes in different colors in one day. By using the information in the table, you are expected to solve the question. Find as many as possible relationship between the number of sold cupcakes in different colors and write your arguments in the following. e.g. The number of the green cupcakes sold is 9 more than the blue ones. The number of the green cupcakes sold is twice the number of the blue cupcakes sold.	**Question 1:** Ahmet and Özgür want to make the journey by their cars between two cities. They drive at the same speed and Ahmet started to drive before Özgür. When Ahmet drove 6 km, Özgür drove 4 km. How many km will Özgür drive when Ahmet drives 9 km? **Question 2:** Ahmet and Özgür want to make the journey by their cars between two cities. They drive at the different speed. When Ahmet drove 6 km, Özgür drove 4 km. How many km will Özgür drive when Ahmet drives 9 km?

	Yellow	Orange	Purple	Blue	Pink	Green	**Total**
Number of the sold cupcakes in the first day	15	12	8	9	10	18	**72**

The following table shows that all PSTs improved their deciding how to respond skills, which helped them to come up with more quality responses about the student's understanding after video analysis sessions. Before participating Session1, prospective teachers provided instructional actions based on Student 1's understandings in the context of additive and multiplicative reasoning, but somewhat without specificity. On the other hand, after video analysis group discussion, prospective teachers demonstrated a notable difference in case-specific-detailed responding as compared to general and case-specific-undetailed responding. Response as robust evidence included extensive consideration of Student 1's strategy in the video clip.

Table 4.7 The comparison of PSTs' average points of comments on responding in MCV1

Prospective teachers		PST1	PST2	PST3	PST4	PST5	PST6	Average of total points
Responding	Before video analysis	1	1	0	1	1	1	5
	After video analysis	1.7	1.7	1.4	1.7	1.7	1.3	9.5

Assigned the certain numbers to each level;

*0 for general comments
*1 for case-specific-undetailed comments
*2 for case-specific-detailed comments

4.2. Prospective Teachers' Professional Noticing of Students' Thinking in the Context of Unitizing and Norming

Based on the research questions of the current study, the nature of prospective teachers' professional noticing of students' thinking in the context of unitizing and norming was investigated in two parts: *unit rate construction* (i.e., Grocery Problem) and *composite unit construction* (i.e., Balloon Problem). The following part presents the prospective teachers' professional noticing of Student 2's thinking in the context of unit rate construction.

Unit Rate Construction

In this section, the micro-case video clip of the Grocery problem was used to examine the prospective teachers' professional noticing of the student's understanding in the context of unit rate construction.

Table 4.8 Asya's (Student 2) strategy of the Grocery problem (MCV2)

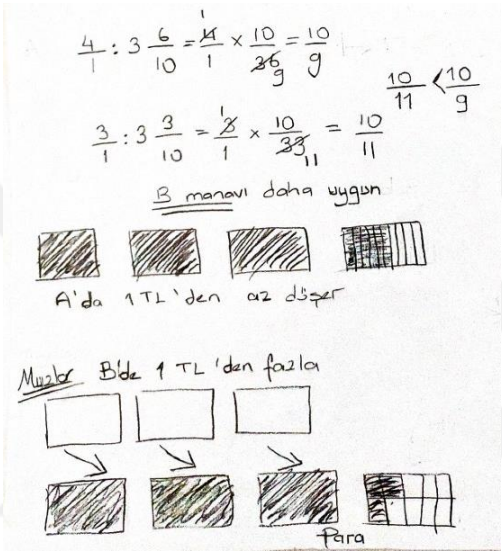
	Ayse paid 36/10 liras for 4 bananas in Grocery A. Berna paid 33/10 liras for 3 bananas in Grocery B. Which is the better buy?

Table 4.9 shows the level of prospective teachers' responses about attending to Student 2's strategy, interpreting, and responding to Student 2's understanding before and after MCV2 analysis in the context of unit rate construction. As seen in the table below, most prospective teachers (4 out of 6) attended to the student's strategy and they (5 out of 6) interpreted the student's mathematical understanding at a limited level. Even though they maintained a focus on interpreting Student 2's understanding, mostly provided limited evidence. 83% of prospective teachers provided case-specific and undetailed instructional suggestions, and one of the prospective teachers' response was general. After video analysis, 4 out of 6 PSTs' responses provided robust evidence of attention to the student's strategies and robust evidence of interpretation of the student's understanding. 50% proposed case-specific-detailed instructional actions in response to Student 2's thinking. On the other hand, half of them proposed case-specific-

undetailed instructional suggestions. Their response included limited evidence of deciding how to respond on the basis of the student's understanding.

Table 4.9 PSTs' noticing level of evidence of attending and interpreting before and after Grocery Problem video-based intervention (MCV2)

Prospective teachers		PST1	PST2	PST3	PST4	PST5	PST6
Attending	Before video analysis	Robust	Robust	Limited	Limited	Limited	Limited
	After video analysis	Robust	Robust	Robust	Limited	Robust	Limited
Interpreting	Before video analysis	Limited	Robust	Limited	Limited	Limited	Limited
	After video analysis	Robust	Robust	Robust	Limited	Robust	Limited
Responding	Before video analysis	CSU	CSU	CSU	General	CSU	CSU
	After video analysis	CSU	CSD	CSD	CSU	CSD	CSU

*CSU= Case-specific-undetailed

*CSD= Case-specific-detailed

4.2.1. Prospective Teachers' Professional Noticing Before MCV2 Anaysis

This section provides the nature of prospective teachers' knowledge and noticing of students' thinking about *unit rate construction.* Table 4.9 shows the nature of PSTs' noticing skills of student understanding in the context of unit rate construction before MCV2 analysis. The representative examples of prospective teachers' attending and interpreting responses and next instructional actions before video analysis were explained in the following sections.

4.2.1.1. Attending to the Mathematical Elements of Student 2's Strategies

PST2 solved this problem by correctly finding the price of a banana for each grocery. The following response samples of PST2 were taken from the video analysis reflection paper, which provided evidence of attention to the details of Student 2's strategy:

PST2: *In order to solve this problem, the student divided the number of bananas in Grocery A by the price and followed the same computational process for Grocery B. Starting from this point, the student found the proper fraction for Grocery A and improper fraction for Grocery B and concluded that less money is paid in Grocery B. The student used models as another solution. The models showed the student the necessity of paying more than 1 lira in Grocery B and less than 1 lira in Grocery A. Thus, the student stated that Grocery A is cheaper. Then, in the first solution, the student thought that there may be computational mistakes and tried to understand the correct computation* (Reflection paper, W4).

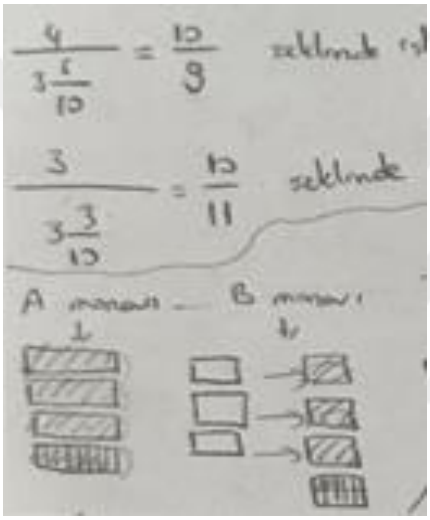


Figure 4.2 PST2's attending to Student 2's strategy

PST2 focused on more mathematical noteworthy elements by describing the student's response, and she increased her attention by providing robust evidence based on the reflections on MCV2. PST2 not only described the fractions and the correspondence relationship between two variables but also provided specific descriptions of the operations that the student made. PST2 stated that the student assumed to find the price per banana, but instead found the number of bananas per one lira. Additionally, PST2 captured other mathematically critical elements in the student's response. Specifically, she realized the linked units (linked one lira and one banana) between quantities on the model for Grocery B. PST2 recognized the matched units; (i.e., one lira and one banana) accepted by the student as the benchmark. In other words, the price of one

banana was measured by one lira (i.e., referencing unit). Overall, this prospective teacher provided robust evidence.

The following response samples were taken from the video analysis reflection paper that provided limited evidence of attending to the details of Student 2's strategies in the Grocery video clip:

> PST5: *The student first divided 4 by 3.6 liras and 3 by 3.3 liras. She found that* $\frac{10}{9}$ *and* $\frac{10}{11}$*. I think she did not know why he divided 4 by* $3\frac{6}{10}$*. Asya compared the proper fraction and improper fraction. Then, she stated that the improper fraction was greater than the proper fraction. Next, the student chose the proper fraction that represents the cheaper grocery. When the researcher used another method, the student used models to search how many three there are in 4 whole bananas and how many three there are in 3 whole bananas. Then, this student realized that the price of a banana in grocery A is less than 1 lira, and the cost of a banana in Grocery B is more than 1 lira. She realized that the first solution was wrong. However, she could not make sense of it. For instance, what does this fraction-*$\frac{10}{9}$ *represent? It represents the amount of banana per lira (for Grocery A). In the same way,* $\frac{10}{11}$ *represents the number of bananas per lira in Grocery B. She could not interpret the unit rate like that, so she made a mistake (Reflection paper, W4*).

PST5 provided limited evidence of attending to some of the details of the student's

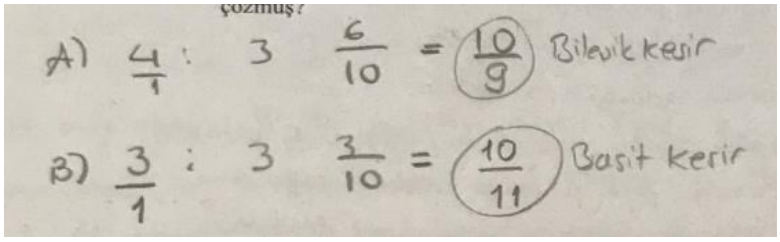


Figure 4.3 PST5's attending to Student 2's strategy

strategy. For example, PST5 described the computational process systemically in the first strategy of the student. Then, PST5 described the meaning of fractions and the amount of banana per one lira (banana/price) within the existing contextual setting.

However, PST5 did not describe the second strategy clearly and stated that Student 2 used models to find out how many 3,6 in 4 wholes (bananas) and how many 3,3 in 3 bananas (how many times 3,6 goes into four and how many times 3,3 goes into 3). However, PST5's attention to the student strategy is partially correct as she did not attend to the details in the second model, such as using linked representations of quantities, or linked units: 1 banana is linked to 1 lira. Besides, PST5 did not specify the steps and computations, and some of her descriptions were vague, lacking mathematical language. Also, any person who does not know the student's strategy will not be able to picture it fully based on this prospective teacher's response. Overall, PST5's response generated limited evidence of attending to the strategies the student used in the video clip.

Another response sample taken from the video analysis reflection paper provided *limited evidence* about attending to the details of Asya's strategies in the Grocery video clip:

> PST3: *The student tried to find the price of one banana but actually she found the number of bananas per lira. She did not recognize it. Then, the student used models (see Figure 4.4) as a second solution and found the price of one banana by making a guess. So, she realized that she needed to divide* $3\frac{6}{10}$ *by* 4. *It means that* $\frac{price}{(the\ number\ of)\ banana}$ *should be applied. She could not interpret her own solution:* $\frac{(the\ number\ of)banana}{price}$. *If she had made sense of this unit (unit amount), he would have chosen the greater fraction within three of them* (Reflection paper, W4).

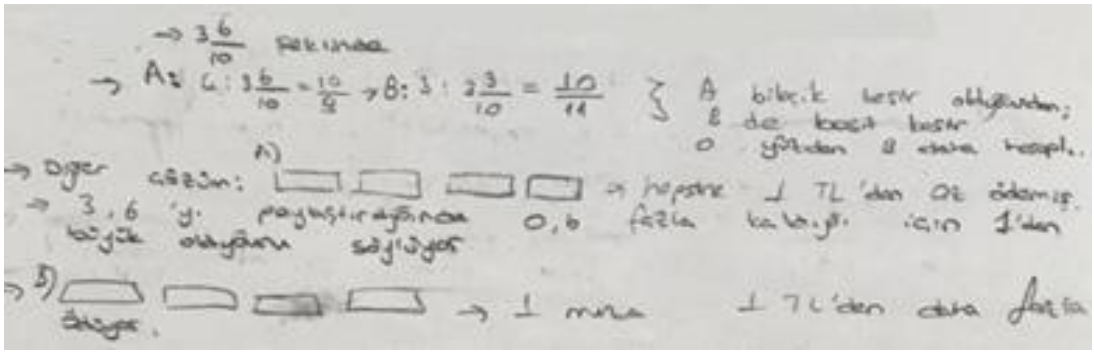


Figure 4.4 PST3's attending to Student 2's strategy

PST3's response demonstrated *limited evidence* about attending to the student's strategy as it missed out some critical mathematical details. At first, PST3 articulated the distinction between unit amount and unit price in the student's response. For example, in the first strategy of the student, PST3 described the meaning of fractions and the amount of banana per lira, instead of just stating the operation of fractions. Even though PST3 systemically described steps in the first strategy, such as drawing the models as Asya drew, she did not provide the details of how the student built a link between money and bananas for each grocery in the second strategy.

4.2.1.2. Interpreting Student 2's Understanding

As aforementioned, PSTs' interpretation skills were categorized according to how well they interpreted students' understanding. All PSTs (except PST2) provided limited evidence of interpreting the Student 2's understanding in the video.

The following response sample was taken from the video analysis reflection paper of PST5 who provided *limited evidence* about interpreting the details of Asya's strategies in the Grocery video clip:

> PST5: *This student divided* 4 *by* $3\frac{6}{10}$ *, but I think she did not know why she was dividing because when she solved the problem using models, she was able to realize that her first answer was wrong as 4/*$3\frac{6}{10}$ *shows how much banana is in 1 lira. The computation is correct, but her interpretation was different. That's why, this student could not make sense of it or interpret it. For instance,* $\frac{10}{9}$ *represents the amount of banana per one lira in Grocery A. Similarly,* $\frac{10}{11}$ *represents the amount of banana per one lira in Grocery B. It seems to be wrong because she could not interpret the quantities in that way* (Reflection paper-W4).

PST5 described the student's understanding with fewer specific and more general claims. As a specific claim, PST5 said that Asya represented the number of banana per 1 lira in Grocery A, but it seems wrong because she could not interpret the quantities in that way. This prospective teacher provided some evidence and interpreted the

student's understanding but was unable to make sense of unit rate. Even though the teacher's statements had specific connections to the student's strategies, they were limited to characterize the distinction of unit types, unit price (price/unit), and unit rate (unit/price). PST5 also made a general claim by stating that the student did not know why she was dividing. In this way, the teacher provided some evidence, but this was a broad conclusion. Although PST5 focused on the concept of division, she did not highlight a potential source of confusion for this student who sought a unit but only through one unit factor approach. Also, this prospective teacher did not interpret the student's second strategy, which used models to link money and bananas. Even though the PST's response focused on interpreting the student's understanding(s), she partially interpreted the student's understanding with *limited evidence* and general claims.

The following is a sample response with *robust evidence* on the interpretation of Asya's understanding in the Grocery video clip:

> PST2: *She supposed that she found the price of one banana, but the student found the amount of banana per one lira. Therefore, she thinks that Grocery B is the cheaper one. She found the amount of banana per lira but could not realize it. She could not interpret 10/9 and 10/11 and did not discriminate between 10/9 and 9/10 within the context* (what student did not understand). *In her second solution, she used fair sharing on models. She linked one banana and one lira to make a comparison. I mean, the student seemed to work with the linked units by considering one lira as an anchor/benchmark, but I made this assumption based on her work* (what student did understand) (Reflection paper, W4).

As seen above, PST2 made substantiated claims. Contrasting the response with the response that provided limited evidence stated above, we can notice specific claims. For instance, PST2 pointed out that the student did not know the unit rate types, and she concentrated only on finding one type of unit, i.e., unit price. Hence, according to her inferences, the student could not interpret what she found. This comment pointed out that PST2 made a specific interpretation of what the student did not understand with robust evidence. Besides, PST2 explained how the student used models and made

inferences about the student's application of linked single units to compare groceries. Therefore, PST2's comments provided *robust evid*ence of the student's understanding of applying linked units to models.

4.2.1.3. Deciding How to Respond on the Basis of Student 2's Understanding

Table 4.10 shows that PSTs had difficulties responding to the student's understanding of video clips. Prospective teachers proposed 8 instructional actions and 88% of participants with an overall deciding how to respond score of case-specific-undetailed included minimal specificity in their reasoning. Only 12% of participants (PST4) with overall score of general included any specific details based on student's understanding in MCV2. Even though all prospective teachers proposed case-specific instructional actions, most of them were undetailed (limited evidence).

Table 4.10 PSTs' proposed next instructional actions before video analysis (MCV2)

PSTs	Next instructional actions			Total frequency
	General (Lack of evidence)	Case-specific-undetailed (Limited evidence)	Case-specific-detailed (Robust evidence)	
PST1	0	1	0	1
PST2	0	2	0	2
PST3	0	1	0	1
PST4	1	0	0	1
PST5	0	2	0	2
PST6	0	1	0	1
Total (%)	1 (12)	7 (88)	0 (0)	8 (100)

The following is a sample response regarding *case-specific-undetailed* instruction actions (robust evidence) on the basis of Asya's understanding in MCV2.

PST1's comments reflect not only her detailed interpretation of the student's understanding but also the reasons why she decided to use some specific instructional actions to improve the student's understanding. PST1's instructional action was consistent with her interpretation. PST1 wanted to use the scalar multiplicative

relationship between banana and price considering the student's difficulty in MCV2. The follow-up question provided evidence that PST1 wanted to pose this question based on the student's mathematical understanding

> PST2: *If I were a teacher of this student, I would ask, "let us buy 20 bananas for each grocery. So how much do we need to pay?" (follow-up question). I would start my instructional action by choosing this question because I would like to understand whether Asya realized the scalar relationship (scalar multiplicative strategy) between the price and the banana. In the video clip we watched, she used fractional knowledge at first, and then she used fair sharing by linking one banana and one lira. This question has two quantities. Therefore, I would like to ask her a question about how quantities vary in relation to each other* (Reflection paper, W4).

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PST6 provided *case-specific-undetailed* instructional actions/suggestions on the basis of Asya's understanding in the MCV2a as follows:

> PST6: *If I were a teacher of this student, I would ask an additional question to support finding unit rate. I would ask questions involving fair sharing and unit rate. Therefore, this student could grasp the process of division* (the divisor and the divided) (Reflection paper, W4).

Even though PST6 provided case-specific instructional actions, this comment cannot be considered as detailed because it did not explain the essence of the student's understanding in the light of clear and now vague intent to reference the operation of division.

The following is the response of PST4, who provided *general* instructional actions on the basis of Asya's understanding in the video clip:

PST4: *If I were a teacher of this student, I would ask her similar problems and ask her to use different strategies such as modeling, unit rate, etc. I would understand the student's understanding in the light of similar type of questions, and then I would provide gear-up problems (R*eflection paper, W4).

Responses in PST4's video analysis reflection paper tended to move away from customization and were generally written with minimal specificity. For instance, PST4 offered more than one suggestion, such as asking similar questions and solving them using different strategies. However, there was no clear action about how to use models or unit rate.

4.2.2. Prospective Teachers' Professional Noticing after MCV2 analysis

The prospective teachers' development of professional noticing of Student 2's thinking after participating Grocery Problem video-case (MCV2) discussion was examined below. The discussion excerpts and prospective teachers' comments in reflection papers and revised lesson plans are given below.

4.2.2.1. Attending to the Mathematical Elements in Student 2's Strategy

After writing reflection papers, the discussion phase was initiated. The participants started to discuss Asya's strategy in MCV2 as follows:

Facilitator	*What did you notice about the student's understanding in the second strategy? What are you thinking about the model she drew?*
PST1	*I think in the first model representing Grocery A, the student was wrong.*
PST4	*I think the student did well in both representations.*
Facilitator	*Could you explain that?*
PST1	*If I did not hear wrong, the students said that in the first representation for Grocery A, the models represented both banana and money. However, I think the student did not make sense of why she drew like that.*

PST4 *I don't think so, because the student drew four bananas and then divided the last banana into ten pieces to find the price (3.6 liras). The student illustrated that four bananas were 3.6 liras and she understood that one banana cost less than one lira. Therefore, the first presentation and also the second one looked correct to me. What about the second representation for Grocery B?*

PST1 *In the second one, the teacher (who did a clinical interview with the student) asked a question to probe the student to represent money and banana separately. I think the student could not think of this kind of representation before the researcher's guidance. Then, I think the student decided to make fair sharing. She distributed the money to bananas by linking a banana and one lira. When she realized that for Grocery B, each banana was more than one lira, she checked the first representation for Grocery A.*

PST5 *Yeah, bananas and money were distributed/linked as one banana is one lira. Then, the student realized that if she pays 1 lira for each banana, there will be 0.3 liras left. In this case, she would pay more than 1 lira and thus, she needed to divide 0.3 liras by 3 (bananas).*

PST5 *Uh, I see. Now I get it, okay. In that case, the student could not make sense of the first drawing before drawing the second representation, and I could see that the student linked one banana and one lira to make a comparison as you said.*

In the group discussion, PST4's statement showed that she did not notice the representation in the student's solution that corresponds to linked units, underlying the unit factor approach. After PST1's response, PST4 stated her disagreement with the representation of quantities (bananas and money) as in the student model because she just claimed that the first representation was correct without providing specific and noteworthy details. When PST4 wondered the PST1's explanation for the representation of Grocery B, first, PST1 and then PST5 immediately emphasized linking the units on the models that put the student in the position of predicting appropriate values on the basis of what was seen in the noticeable representations. By referring to the advantages of the group discussion, PST4 focused on the details of

mathematical elements in the student's understanding after the video clip analysis. PST4 evaluated her previous approach as follows:

> PST4*: In the group discussion, I realized the link between the objects, i.e., money and banana. For Grocery A, the representation showed that the student thought one banana forms a unit and one lira forms another unit. The student drew four bananas and tried to place 3.6 liras inside those four bananas without drawing other models. Three liras were linked with three bananas one by one, respectively. I think the student used one lira and one banana as a benchmark. The rest of the money (0.6 liras) was linked to the fourth banana. Therefore, the student realized that each banana would be less than one lira in Grocery A. For Grocery B, first, the student drew three bananas and another model representing the 3.3 liras. After linking one banana and one lira, she realized that the rest of the money (0.3 liras) was left unpaired, so she realized that one banana would be more than one lira. As a result, the student concluded that Grocery A was cheaper* (Revised lesson plan, W4).

PST4's response reflected *robust evidence* of attention to the noteworthy details of the Student 2's strategy. PST4 realized the noteworthy details of the student's understanding related to the linked units that she did not notice before the group discussion. PST4 noticed that the student chose the units as one lira and one banana and then linked these units to describe the particular relationship between them, such as less than one lira or more than one lira.

Another illustrative example was presented to show PST6's approach, which showed limited evidence in the individual analysis of MCV2:

Facilitator	*What did you notice about student's understanding when you watched the video clips?*
PST3	*The student tried to find the amount of price per one banana but she found the number of bananas per lira.*

PST6 *Hmm, I did not think like that before. How did you understand this?*

PST3 *Because the student divided the number of bananas by the price, and then she thought that she found the unit price of one banana, so she thought Grocery B was cheaper. The student misinterpreted the answer and she did not consider choosing the grocery that provides more banana per lira.*

PST6 *You mean that she found the unit but for one lira. Am I right?*

PST4 *Yeah, you are right. Additionally, the student chose 10/9 because she thought about how many bananas she could get for one lira. However, as my friend explained, the student believed that she found the price per banana. Therefore, she interpreted 10/11 as being smaller.*

In the group discussion, PST6's questions about the PTS3's explanation indicated that she did not notice types of units. PST3's explanation led her to question the unit amount approach in the student's understanding. After confirming PST6's inference, PST4 reconfirmed his explanation by implying the choosing of the smaller one as the reason. PST6 noticed the missed noteworthy mathematical details in the student's response in MCV2. After the group discussion, PST6 made the following explanation:

> PST6: *Before the group discussion, I just focused on the division expression that the student wrote (dividing four by 3.6 and 3 by 3.3). I thought she had problems with division. After the discussion, I realized that she did not make sense of the unit rate and did not apply it to other situations. Also, in the discussion, my friends addressed an issue I did not realize: I chose 10/11 instead of 10/9 to decide on the cheaper grocery. I did not realize that 10/9 represents the number of bananas per lira* (Revised lesson plan, W4).

PST6 provided *robust evidence* of attention to the student's noteworthy mathematical details after participating in the group discussion as a part of the teaching experiment

session. Her last statement indicated that PST6 noticed the details of the student's understanding of the unit concept which she did not notice before the group discussion. Specifically, PST6 focused on the unit types, the number of bananas per lira, and the price per banana rather than evaluating the answer as an incorrect computation.

4.2.2.2. Interpretations of Student 2's Mathematical Understanding

Some illustrative examples of discussion excerpts were presented to show how prospective teachers made interpretations of Asya's understanding in MCV2.

Facilitator	*So, tell me what you learned about this student's understanding based on the strategy she used?*
PST3	*I learned that she expected to find the price of one banana (unit price), so she did not correctly interpret what she found*
PST4	*I think so. She thought that she found the price of one banana. So, she chose the smaller one (price). She made this computation unconsciously*
PST6	*In terms of giving quantities as whole number and fraction, she seems to consider the right step to divide the whole number (integer) by fraction or decimal. I mean she had this kind of a misconception*
PST3	*I was going to focus on precisely the same point. As you said, the student seems to use the whole number (the number of bananas) as a dividend as it was easy for her.* *Also, I think she was not aware of the meaning of division, and she did not know the difference between changing the place of divisor and dividend. This is one of the reasons why she chose Grocery B as the cheaper one.*
PST1	*I agree. As you said, the student wanted to use the whole number (the number of bananas) as a dividend as she found it easier. Also, I think she was not aware of the meaning of division, and she did not know the difference between changing the place of divisor and dividend. This is one of the reasons why she thought Grocery B was cheaper.*
PST3	*Yeah, I agree. She did not distinguish between the banana/the money and the money/the banana.*

As seen above, PST3 first focused on the student's solution and claimed that the student incorrectly interpreted the unit (the price of one banana). Then, PST6 claimed that the student did not interpret the answer correctly, which is related to the tendency to accept whole numbers as dividends and to divide whole numbers more easily than other numbers (e.g., fractions and decimals). As a response, PST3 supported PST6's claims. At this point. PST1 elaborated on his interpretation of the student's understanding, as reflected in individual analysis. The prospective teacher connected the role of quantity and lack of knowledge of unit types. After a group discussion, PST1 and PST6 explained their development in noticing Asya's understanding of the unit rate approach. Their responses were respectively categorized as *robust* and *limited*. The relevant interpretations are as follows:

> PST1: *In the discussion, my friends stated that the student's difficulty would be related to the numbers given in the context. As my friends explained, fractions (*$3\frac{6}{10}$ *and* $3\frac{3}{10}$*) would interfere with her understanding of finding single units. Maybe she thought the dividend must be the whole number, so she experienced a conceptual lead when she worked with fractions. Then, I became more aware that this student did not know how quantities were related to each other to determine appropriate operations. Therefore, she did not interpret the meaning of operations (the number of bananas divided by the money and the money divided by the number of bananas). If she knew the meaning of operations, the sort of number (whole number or fraction) would not be a problem to find the unit types and interpret them correctly, and she could make the basic comparison by using unit rates as a benchmark* (Revised lesson plan, W4-W5).

> PST6: *I thought that her incorrect strategy was related to an operation like a misconception; the whole number would be a dividend if the other quantity were a fraction or decimal. On the other hand, some of my friends had different ideas. They claimed that the student did not know how to interpret what she found and did not know the unit factor types* (Revised lesson plan, W4-W5)

These comments show that both PSTs noticed the interpretation of the student's unit rate understandings through group discussion. PST1's comments pointed to the relevant details of the student's interpretation of the unit rate approach. The prospective teacher had a better interpretation of the student's understanding after group discussion while the interpretations were more limited in the individual analysis paper. PST1 provided *robust evidence*. Besides, PST6 provided different viewpoints about the reason for the student's interpretation of what she found, considering the student's misconceptions about numbers. Then, however, PST6 only repeated what PST1 and PST3 stated, which led to a *limited evidence* response.

Another example is that the PSTs discussed different stances about making a comparison by using sharing models, corresponding to linked single units. In the group discussion, the facilitator invited prospective teachers to explore the reasons for using single units as an anchor or benchmark while making the comparison.

Facilitator	*What is your interpretation about Asya's drawing?*
PST4	*I think she made the modeling very well.*
PST1	*I don't think so. Firstly, I realized that representations (for Grocery A and B) differed. For instance, in the first representation, if I did not hear wrong, she stated that the shapes (models) represent both banana and money, but I could not understand how she did it. Then, she drew the second model and represented bananas and money individually. Her teacher (the facilitator) guided her to represent the quantities on separate models. The student applied fair sharing by distributing the money to the bananas- one lira per banana. She realized that one banana costs more than one lira in Grocery B. I think she fselt the need to control the first one and realized that she had four models for bananas and three for money. Hence, she realized that one banana costs less than one lira in Grocery A.*
PST4	*I did not understand first, but when I saw her distributing the money across bananas, I got it. If she drew four rectangles representing bananas and divided each into ten parts and then distribute 3.6 liras to those parts, it would be more accurate.*

PST3 *For Grocery B, she made a distribution by matching one lira to each banana and there was no banana left. I think she was aware of applying fair sharing to make a comparison. However, there is one thing that I couldn't understand in the video clips. What did she mean when she said, "I am looking how many 3,6 in 4"?*

PST2 *It seems that she thought four bananas and 3.6 liras as groups. Then, I think she was confused to explore "how many 4s would go into 3.6 liras" or "how many 3.6 liras would go into 4 bananas". I realized the same difficulty as in the first strategy that the student was not able to reason with unit rate types.*

PST3 *I agree, I think she was not able to make sense of unit or internalize units as a part of proportional thinking.*

PST4 *For Grocery B, the student could directly realize the relationship on the second representation. For instance, she linked quantities (linked single units) with arrows, and the rest of the money was not linked to bananas. Therefore, the student stated that one banana costs more than 1 lira in Grocery B, but she did not find the exact price of one banana. It seems that she used one lira as a reference point (anchor/benchmark), and I think this strategy was an indicator of the student's proportional thinking.*

PST1 *I think the student's approach seems unrelated to proportional reasoning and fractions. I mean she seems to focus on fair sharing and is stuck in fractions instead of producing single units. I am not sure whether she recognized the scalar relationship (proportional relationship) between quantities (bananas and the price).*

In the discussion, the PTSs first focused on the representation of distribution on models and how the student used quantities to find the cheaper one regarding the construction of unit rate. Then, they discussed the potential reason(s) for the student's understanding underlying the possible difficulties of distribution such as linking single units to models. For instance, PST1 provided robust evidence for the details of the student's sharing models, portraying linked quantities by keeping track of the linked quantities. PST3 questioned the student's reasoning with the question ("How many 3.6

liras would go into 4 bananas?"). PTS2 reinterpreted the situation by accepting 3.6 liras and 4 bananas as one group and elaborated on PST3's question of "how many 4s would go into 3.6?" or "how many 3.6 would go into 4". PST4 focused on the representation for Grocery B and claimed that using one lira as a reference (anchor/benchmark) and linking single units (one banana and one lira) was evidence of students' proportional thinking. However, PST1 expressed hesitations about the student's understanding of proportional reasoning. This indicates that PST1 recognized a need for more evidence to confirm PST4's interpretation of linked single units on sharing models as evidence of proportional reasoning.

After the group discussion, the PSTs explained their development in noticing (evaluating) the student's proportional reasoning regarding the distribution of money across bananas as a part of the unit rate concept. As an example, PST4's written statement is given below:

> PST4: *This week, I realized the different perspectives of the students on the representations. After the discussion, I realized that this process seems to involve grouping the quantities (4 and 3.6). The student tried to find how many 4s there are in 3.6, as PTS2 stated. On the other hand, the student changed her approach and matched up one lira and one banana. She used one lira as a reference point and decided which grocery was cheaper based on the leftover money. I thought those would be evidence of the student's proportional reasoning, but PST1's interpretation made me confused* (Revised lesson plan, W4-W5).

This explanation revealed that PST4 provided *robust evidence* of the student's strategies (her ways of reasoning with shared models) that represent linking quantities on shared models while she interpreted them without giving any specific noteworthy mathematical details in the individual video analysis. Even though PST4 initially thought this student had proportional reasoning skills considering her use of the linked single units for making the comparison, PTS1's explanation encouraged PST4 to look at the details from a different perspective. Moreover, the PSTs put forward different stances about the student's proportional reasoning as a striking point. PST3, who did

not make any comments about the student's proportional reasoning in the individual video analysis, appeared to realize that point as seen below:

> PST3: *Before this week's discussion, I did not think whether this student solved this problem through proportional reasoning. The discussion served as an anchor for me to consider significant details. It led me to consider the understanding of the student in a different way. For example, distributing single units across other units (as PTS4 said) and the scalar relationship between quantities (as PST1 said) were the essential components of proportional reasoning. However, I realized that video clips have limitations in showing evidence of proportionality, whether this student applied proportionality or not (the construction of proportional reasoning)* (Revised lesson plan, W4).

PST3 directly concentrated on the construction of proportional reasoning. She extended the interpretation by describing and providing specific details of the student's strategy. This statement indicates that PST3 deepened her interpretation, focusing on mathematical details related to student strategy and essential understandings of proportional reasoning. Therefore, her response demonstrated *robust evidence*. Another striking example was given to show the developments in PST5's interpretations of the unit rate approach of the student's understanding in MCV2.

> PST5: *The student divided the number of bananas by price: 4/3.6 and 3/3.3. Then, she found that Grocery A is cheaper than Grocery B. In the group discussion, I realized that her strategy looks procedural rather than conceptual since she is supposed to find the unit price. Still, she just found unit amounts by dividing numbers in a memorized manner instead of making sense of the unit rate approach. However, she found the number of bananas per lira by dividing 4 by 3.6 and 3 by 3.3. The student seems to have unit-rate thinking, but I am making an assumption based on her solution. Even though this student knew the requirement of one banana price to make the comparison, she was not able to make sense of a single unit, and she was not able to think*

about other strategies, such as the build-up strategy (Revised lesson plan, W4-5).

Contrary to the previous response before the video analysis, PST5 described the student's difficulty in detail providing *robust evidence* of Student 2' understanding. While explaining the student's understanding, PST5 characterized the distinction of single unit types, unit price, and unit amount within this context. Besides, PST5 hesitated by saying "The student seems to have unit-rate thinking, but I am making an assumption based on her solution". This statement indicates that PST5 realized more evidence to prove or disprove his interpretation. PST5 did not focus on the meaning of division, as stated before. Additionally, PST5 specifically pointed out what this student knew and what this student did not know by saying, "Even though this student knew the requirement of one banana price to make the comparison, she was unable to make sense of a single unit, and she was unable to think about other strategies such as build-up strategy".

Besides, the prospective teachers stated the significance of group discussion as an anchor. The discussion helped prospective teachers reframe their attention and interpretation of the student's mathematical understanding in the video clips. One noteworthy feature of the group discussion mentioned by PST3 was to provide different foci and additional evidence to support the prospective teachers in developing noticing expertise. Therefore, the discussion excerpts were given to present the professional noticing development progress.

4.2.2.3. Deciding How to Respond on the Basis of Student 2's Understanding

Table 4.11 shows that the PSTs provided a detailed response on how to respond to Asya's understanding of MCV2. PSTs proposed 15 instructional actions after the video analysis. 53% of them were case-specific-detailed and included detailed information on proposed instructional actions based on the student's understanding of MCV2, including the purpose of proposed instructional actions and how to use that instructional actions. Also, 47% of the statements considering the student's

understanding of MCV2 were case-specific-undetailed but non-specific and undetailed about the ways of instruction.

In the group discussion, PSTs focused on instructional actions to handle the difficulties in Student 2's understanding. The instructional actions that the PSTs offered were based on the student's understanding and had a connection with specific instructional actions. Even though the instructional actions in the individual video-based reflection papers that the PSTs provided were mostly case-specific and undetailed, in the group discussion, the PSTs began to provide case-specific and detailed instructional actions regarding the student's understanding(s). Below are the discussion excerpts and the examples of instructional actions that the PSTs provided in the revised lesson plans.

Table 4.11 Instructional actions after individual video analysis

PSTs	Suggested instructional actions			Total frequency
	General	Case-specific-undetailed	Case-specific-detailed	
PST1	0	2	0	2
PST2	0	0	2	2
PST3	0	1	2	3
PST4	0	2	0	2
PST5	0	0	3	3
PST6	0	2	0	2
Total (%)	0 (0)	7 (47)	8 (53)	15 (100)

Below are the discussion excerpts and the examples of instructional actions that the PSTs provided in the revised lesson plans.

Facilitator	*Pretend that you are the teacher of this student. What instructional actions might you pose next?*
PST1	*I would ask Asya to think about the price of 12 bananas for each grocery. Then, the student probably could realize that Grocery A is cheaper than Grocery B. I would ask the reason why the previous result and this were different.*

PST6	*Why did you choose twelve bananas? I wonder if using 12 bananas makes sense in the unit rate approach? Because she had difficulty in conceptualizing unit rate approach, using 12 would be problematic and not useful.* *(PST2 and PST5 showed their agreement by nodding their head up and down)*
PST4	*In order to compare the price after equalizing the number of bananas (as a reply to PST6).*
PST1	*Um, maybe. So, we would ask her "let's buy the equal number of bananas for each grocery". I would like to make her realize that she can use 12 as an alternative strategy.*
Facilitator	*What does the number 12 represent here?*
PST1	*I used 12 considering one group/one thing (composed unit) and 12 would be more efficient for students because 12 is the smallest common multiplies of 3 and 4.*
PST4	*You can use 6 instead of 12, right?*
PST1	*Yeah, but I think, using 6 would be less effective to see the scalar relationship for Asya.*

In this part of the group discussion, PST1 provided a response using 12 bananas as common multiplies to encourage students to compare the results. Although her response was related to the observed student's understanding showing the lack of scalar multiplication relationship, this suggestion was criticized by another PST (PST6) addressing the student's understanding, and she stated her disagreement with PST1's suggestion with her elaboration. According to PST6, this student was struggling with the unit concept. Even though she understood that his solution was wrong via this comparison, she still would have the same difficulty. PST2 and PST5 showed their agreement with PST6's statement by nodding their head up and down. These gestures might show us joint attention to the point and their co-construction of a shared understanding of the meaning of the unit rate concept. This comment revealed PST6's noticing of the student's understanding by prompting other participants to consider PST1's response. Even though PST6's comment showed that she diagnosed the students' difficulty during the group discussions, she did not provide instructional actions. Although she recommended encouraging students to evaluate incorrect

answers by making repeated addition or drawing shapes in her lesson plan, she did not give specific information about why she preferred using these instructional actions. She did not articulate her concern which she mentioned in the group discussion about the construction of the meaning of unit rate and unit types. In this regard, PST6's comments were grouped as case-specific-undetailed.

PST1 continued to explain how to use 12 bananas as an example of composite units, which significantly contributed to the PSTs' interpretations and alternative instructional actions. At this point, PST1 directed the group to the issue of constructing a composite unit. In the discussion, PST1 did not provide details about how quantities vary. For instance, she did not refer to a composite unit containing one that can be iterated twelve times or four that can be iterated three times based on the student's understanding. In the revised lesson plan, PST1 did not provide specific details on constructing the meaning of unit rate addressing the student's difficulty in the video clip. She used the visual presentation to show the coordination of two counts (e.g., logs and burning time) in terms of only one type of unit rate (e.g., 3/2 logs are needed for one hour).

Overall, PST1's response was categorized as *case-specific-undetailed* as well. PST2' comments and. the illustrative examples of initial and revised lesson plans (Table 4.12 and Table 4.13) are presented below.

PST2: *In the video, even though the student found the unit rate, she did not elaborate on what she found. As we discussed, she focused on single type of unit rate. I understand that the construction of unit rate concept is important to understand the relationship between quantities. In my first lesson plan, as the student thought, I focused on only one type. So, I changed my plan and I used one of the questions. I assume that the student cannot make sense of the meaning of the unit price and unit amount of electricity used in this question. Students may be familiar with comparing the quantities by finding unit price (i.e., the amount of electric per lira) but unfamiliar with comparing them by finding the unit electricity use (kW/TL). Also, we can say that they could not interpret the amount of electric use (unit kW) when we pay 1 lira. Thus, it can be said that students may know how to find the unit rate, but they may not know the different representations of the unit rate like unit price* (Revised lesson plan, W4).

Even though PST2 did not suggest instructional actions during the group discussion, in the revised lesson plan, her selective attention, determined by the student's difficulty in MCV2, can be recognized easily. In the initial lesson plan, PST2 presented the unit concept (see Table 4.14 and Table 4.15). She found the relationship between the number of cupcakes with different colors. She did not clearly provide the multiplicative relationship between those quantities (15 yellow cupcakes and ten pink cupcakes), and she did not use notational language to assist students in viewing problems from various unit-type perspectives. However, she used notational language in the revised lesson plan to represent how unit types can be formed by coordinating varying quantities. PST2 provided a comment by referencing the group discussion that week. PST2's explanation clearly showed that the video analysis enabled her to recognize the difficulty of circularity in the number-operation relationship to produce units of quantity. In this regard, PST2 implemented an activity of unit coordination to show reversibility as being interchangeable (e.g., x kilowatt per lira and x lira per kilowatt) based on the student's understanding of the video.

PTS2 highlighted using ratio tables for students to understand and interpret how given quantities vary. To sum up, PST2 provided all the following details about using the suggested instructional actions: for whom (the student in MCV2), for what purpose (to

develop unit construction), and how it is used (by recognizing the relationship between quantities regarding unit). Therefore, her response was scaled as *case-specific-detailed*.

Overall, instructional actions in the individual reflections of the PSTs were mostly case-specific and undetailed. However, after video analysis, the PSTs started to provide case-specific and detailed instructional actions during group discussions and in their weekly revised lesson plans. Moreover, Table 4.14 indicates the development of each PST's statement on responding. The table shows that most PSTs developed deciding how to respond skills, which helped them to come up with more quality responses about the student's understanding after video analysis.

Table 4.12 Examples of instructional actions presented in lesson plans

Example parts of …	
Initial lesson plan (before video analysis)	Revised lesson plan (after video analysis)
Seher decides to mix two colors to reach to one shade of the green for cupcakes. She tries to obtain the shade of the green she wants by mixing blue and yellow in different quantity, and finally she reaches to the recipe. According to the recipe for green cupcakes, she mixes 2 drops of blue and 5 drops of yellow food coloring. However, she wants to increase the blue food coloring about 4 more drops for making enough cupcakes. How many drops of yellow food coloring should be needed to use to keep the shade of the green same? Open the GeoGebra File and fill in the table with the total numbers of yellow and blue food coloring which you used to keep the shade of the green same. Then, teacher wants students to think about these situations; ⇒ As you see from the table, for 9 blue cupcakes sold, 18 green cupcakes are sold. Well, what do you think about 1 blue cupcake which was sold? or what do you th about 1 pink cupcake sold when it is compared with yellow cupcakes sold? **Expected Answers:** - The ratio of the blue cupcakes to the green cupcakes is 1:2, so 2 green cupcakes a needed for 1 blue cupcake or ½ blue cupcake is needed for the 1 green cupcake. - The ratio of the pink cupcakes to the yellow cupcakes is 2:3, so 3/2 yellow cupcak needed for 1 pink cupcake or 2/3 pink cupcake is needed for 1 yellow cupcake. With these questions, teachers aims to reveal students' thoughts about the unit rate and emphasizes what would compare two quantities when one of the quantities is 1.	A and B electricity distribution company in a city distributes electricity with different charges. While company A charges 3 kw electricity use with 1,8 TL, company B charges 8 kw electricity use with 6,4 TL. Which company is more economic? **Expected Answer 1:** Company A: $\frac{1,8}{3} = 0,6\ TL\ for\ 1\ kw$ Company B: $\frac{6,4}{8} = 0,8\ TL\ for\ 1\ kw$ Students can find the unit price of each company in order to compare these companies. In here, students divide price by kw like in the left. In other words, they find the ratio of price to the amount of electricity use. By doing this, they use the within ratio in this question and find the **unit price per 1 kw**. If we compare the prices for each company, we can see that while Company A charges the electricity use with 0,6 TL per kw, Company B charges it with 0,8 TL per kw. From this solution, we can say that Company A is more economic than Company B due to 0,6 TL < 0,8 TL. **Expected Answer 2:** Company A: $\frac{3}{1,8} = \frac{30}{18} = \frac{5}{3}\ kw\ for\ 1\ TL$ Company B: $\frac{8}{6,4} = \frac{80}{64} = \frac{5}{4}\ kw\ for\ 1\ TL$ Students can find the unit kw of each company in order to compare these companies. In here, students divide kw by price like in the left. In other words, they find the ratio of amount of electricity use to the price. By doing this, they use the within ratio in this question and find **the unit kw per 1 TL**. If we compare the kw amount for each company, we can see that while $\frac{5}{3}\ kw$ is used for 1 TL in company A, $\frac{5}{4}\ kw$ is used for 1 TL in company B. From this solution, we can say that Company A is more economic than Company B due to $\frac{5}{3} kw > \frac{5}{4} kw$ for 1 TL.

Table 4.13 Examples of instructional actions presented in lesson plans

Example parts of …	Revised lesson plan (after video analysis)

Expected Answer 5:

For Company A		For Company B	
3 kw	1,8 TL	8 kw	6,4 TL
6 kw	3,6 TL	4 kw	3,2 TL
9 kw	5,4 TL	**12 kw**	**9,6 TL**
12 kw	**7,2 TL**		

Students can use the covariation or composed unit by solving this question. Thanks to multiplicative reasoning of students, they can think these two quantities (kw and TL) vary together. With this reasoning, they try to find common point of electricity use or price like above for each company. While reaching to the common point, they use the iteration of quantities. For example, when they **iterate 3 kw and 1,8 TL in Company A four times**, they reach to the 12 kw and 7,2 TL in the solution. In here, we can say that they use the scaling up in proportion by increasing the quantities in the same ratio and iterating. Also, they use **scaling down** to reach to the common point. For example, in Company B, they use the **scaling down (scale factor 1:2)** dividing 8 kw and 6,4 TL by 2 and get 4 kw and 3,2 TL. According to these reasonings, students can achieve to equalize the electricity use in 12 kw. If the electricity use is 12 kw for each company, the price will be 7,2 TL for company A and 9,6 TL for company B due to covariation and multiplicative relationship. If we compare the prices, we can see that company A is more economic for electricity use due to 7,2 TL < 9,6 TL.

Misconception 1:

Company A:

$\frac{3}{1,8} = \frac{30}{18} = \frac{5}{3}\ kw\ for\ 1\ TL$

Company B:

$\frac{8}{6,4} = \frac{80}{64} = \frac{5}{4}\ kw\ for\ 1\ TL$

Students can find the unit kw of each company in order to compare these companies. In here, students divide kw by price like in the left. By doing this, they find the unit kw per 1 TL. If we compare the kw amount for each company, we can see that while $\frac{5}{3}\ kw$ is used for 1 TL in company A, $\frac{5}{4}\ kw$ is used for 1 TL in company B.

However, students can think the value of kw/TL as unit price per kw. Because of $\frac{5}{3} > \frac{5}{4}$ for 1 kw according to students' reasoning, company B is more economic than Company A.

"In this misconception, students cannot make sense the meaning of unit price and unit electricity use in this question. In here, they may be familiar to compare quantities by finding unit price, but unfamiliar to compare them by finding unit electricity use (kw/TL). Also, we can say that they could not interpret the amount of electricity use (unit kw) when we pay 1 TL. From this solution, it can be said that students know to find the unit rate, but they may not know different representations of unit rate like unit price (TL/kw) and electricity use (kw/TL). Therefore, teacher can use in the following methods to overcome this misconception."

Table 4.14 The comparison of PSTs' average points of comments on responding

Prospective teachers		PST1	PST2	PST3	PST4	PST5	PST6	Average of total points
Responding	Before video analysis	1	1	1	0	1.5	1	0.92
	After video analysis	1	2	1.7	1	2	1	1.45

Assigned the certain numbers to each level;
*0 for general comments
*1 for case-specific-undetailed comments
*2 for case-specific-detailed comments

Composite Unit Construction

As aforementioned, unitizing and norming were examined in two parts as unit rate construction and composite unit construction. In this part, the prospective teachers' professional noticing of Nazli's (Student 3) thinking in the Micro-case video clip 3 (MCV3) was examined in the context of composite unit construction in the Table 4.15.

Table 4.15 Nazli's (Student 3) strategy in the Balloon problem (MCV3)

Derya, Ahmet, and Kagan bought three balloons and paid 1.5 liras. How much do they need to pay for balloons for their class of 24?

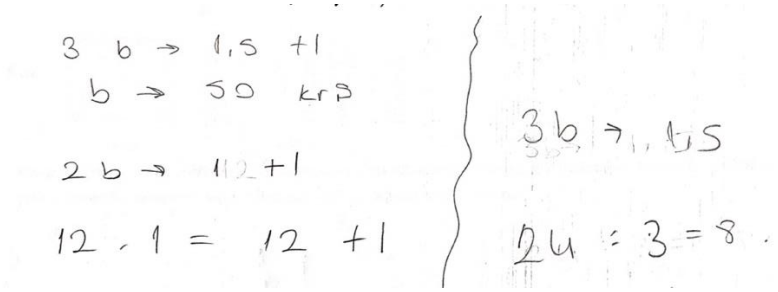


Table 4.16 shows the level of prospective teachers' responses about attending to Student 3's strategy and interpreting Student 3's understanding, and responding to Student 3's understanding before and after MCV3 analysis. All the participants (except PST4) provided robust evidence of attention to the student's strategy. Their descriptions included noteworthy mathematical details of the student strategies in the MCV3. Besides, the participants started using more proportionality language in their explanations. Three of the participant's responses showed robust evidence of interpretation of Student 3's understanding. Even though they focused on the student's

understanding, their responses were less detailed and did not include evidence. Three participants provided robust evidence of interpreting the student's mathematical understanding. They made sense of the details of each strategy (i.e., made sense of constructing composite units). They explained how the details that they attended reflected what the student did understand and did not understand. All the participants (except PST4) responded to the student thinking as case-specific-undetailed. Their responses focused on the student's understanding but included fewer details of future strategies and, in general, were somewhat vague.

Further, video-based intervention assisted in developing prospective teachers' noticing skills. All prospective teachers provided robust evidence in attending to Student 3's strategy, and 5 out of 6 provided robust evidence of the student's mathematical understanding in the MCV3. 4 out of 6 prospective teachers provided case-specific-detailed instructional actions, including specific future instructional actions based on students' understanding of composite unit construction in a detailed way. In this vein, the following table demonstrates the development of PSTs' ability to notice in the context of composite unit construction.

Table 4.16 PSTs' noticing level of evidence of attending and interpreting before and after Balloon Problem video-based intervention (MCV3)

Prospective teachers		PST1	PST2	PST3	PST4	PST5	PST6
Attending	Before video analysis	Robust	Robust	Robust	Limited	Robust	Robust
	After video analysis	Robust	Robust	Robust	Robust	Robust	Robust
Interpreting	Before video analysis	Robust	Robust	Robust	Limited	Limited	Limited
	After video analysis	Robust	Robust	Robust	Limited	Robust	Limited
Responding	Before video analysis	CSU	CSU	CSU	General	CSU	CSU
	After video analysis	CSD	CSU	CSU	CSU	CSD	CSU

*CSU= Case-specific-undetailed

*CSD= Case-specific-detailed

4.2.3. Prospective Teachers' Professional Noticing Before MCV3 Analysis

This section provides the nature of prospective teachers' knowledge and noticing of Nazlı's (Student 3) thinking about composite unit construction before video analysis.

4.2.3.1. Attending to the Mathematical Elements of Student 3's Strategies

The following response (Figure 4.5) belongs to PST1 and was taken from the video analysis reflection paper. It provides *robust evidence* of attention to the mathematical details of Nazli's strategies in MCV3:

PST1: 24/3=8 groups



Figure 4.5 PST1's attending response in reflection paper

*She stated that she would find the answer by using groups. 8*1.5=12 liras, and found 12 liras. In order to be sure, she made repeated addition: 8 times 1.5. She stated that 'Oh yes, it was 12.' Also, she said that she could find one balloon's price and then multiply it by 24. In her first solution, she divided 24 by three. She examined groups of three. Then, she multiplied eight by 1.5. She did it mentally and found 14. However, in her first strategy, she found 12. In order to check the answer of 8*1.5, she added eight times 1.5 using her fingers. She found 12 again by coordinating 1.5 liras and three balloons.*

As another solution, she stated that she could find one balloon's price, and then she multiplied it by 24. She recognized that two balloons' price is one lira, so for 24 students, she multiplied one lira by 12. She could construct different composite units and coordinate them (Reflection paper, W4).

PST1's response captured the noteworthy mathematical elements of the strategy. Specifically, she articulated the student's ability to use different strategies flexibly and to coordinate unit types (i.e., composite units). For instance, PST1 stated that this student noticed the correspondence between the number of balloons and the cost of these balloons (groups of three). Therefore, she multiplied eight times 1.5 liras. PST1 explained how this student conceptualized a composite unit in her solution. Then,

PST1 pointed out that the student found the price of one quantity or the unit in the problem as another student strategy. Also, this prospective teacher employed context-specific language when describing the elements (e.g., composite units, unit rate strategy, and iterating). Thus, her response was categorized as robust.

> PST4: 3 b→ 1,5 TL
> 24/3=8 8*1.5=12 TL
> *Student made those operations and then deleted it. She found 14. Then, she calculated 8 times 1.5 using her fingers and found 12* (Reflection paper, W4).

PST4 attended to only a few mathematical elements of the student's solution. The response shows that PST4 missed out on some critical mathematical details. Despite describing all the student's computations to solve the problem identically, the prospective teacher could not recognize the details of strategies (e.g., sets of three singleton units become composite units). For instance, the student created multiplicand, in this case, the composite units of three and the multiplier, the number of groups (8 groups or eight sets). However, PST4 did not explain the roles of these numbers that the student used in her strategy. In this regard, PST4's responses were categorized as *limited*..

4.2.3.2. Interpreting Student 3's Understanding

The following response sample provided limited evidence on interpreting the details of Nazli's strategies in the Balloon video clip:

> PST6: *The student did not have difficulty finding units and showing the relationship between numbers (quantities). She found 0.5 as a unit and tried to build a relationship between 0.5 and 2. Then, she stated that 4 balloons cost 2 liras. The student correctly found the (number) groups of three or the (number) groups of four that could make 24. Making correct multiplication and division shows that this student understood the topic very well* (Reflection paper, W5).

PST6 described the student's understanding by providing more general claims and less specific statements. The prospective teacher made two claims with *limited evidence*. For instance, she claimed that this student was good at finding a unit and showing the relationship between numbers. However, this comment was limited and less specific. She did not refer to any evidence about the underpinning relationships that the student considered when making those operations. Her claims were not grounded in the particular parts she observed in the video. Besides, PST6 wrote, "The student correctly found the (number) groups of three or the (number) groups of four that could make 24. This statement shows that the student understood the topic very well." This broad conclusion goes beyond the evidence based on the student's solution to only one problem. Although PST6's response focused on interpreting the student's understanding(s), she, unfortunately, described the student's answer underlying the procedure with general comments.

The following is a sample response that provided *robust evidence* on the interpretation of Nazli's understanding in the Balloon video clip:

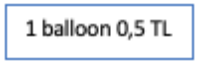
> PST1: *First, the student took "3 balloons=1.5 liras" as composed units. Therefore, she divided 24 by three and then found how many groups of three were in 24. There were eight groups of three. Then she multiplied eight by 1.5. Then, the student made a mistake in the multiplication. She added eight times 1.5 repeatedly (iterating composite unit-3 balloons). This situation made me think that the student would have difficulty with multiplication with decimals. But I need to see more evidence about it. Moreover, she found the price of one balloon and then multiplied it by 24. Forming groups as a reference (referent units) shows that this student understood how to construct composite units and how to coordinate them. She recognized that she could decide on the unit by herself (flexibility in unitizing) because she was able to treat a group as follows (Reflection paper- W4).*
>
> 
>
> *one unit is three balloons*
>
> 
>
> *one unit is one balloon*

Figure 4.6 PST1's interpreting response in reflection paper

PST1 interpreted the student's understanding in different ways and made substantiated claims on the student's understanding (Figure 4.6). First, she made sense of the noteworthy details of Nazlı's strategy and how these noteworthy details reflected what the student did understand and did not understand. For instance, PST1 pointed out that this student could think flexibly about determining units (using unitizing, singleton unit, or composite unit) given in the problem context. Also, PST1 was able to use the language of ratio and proportion. She evidenced this by calling related elements "composed unit" and "linking" in her response. Besides, PST1 recognized what understanding the student did not present. For example, PST1 expressed her hesitation at the student's answer about multiplying decimals by saying, "The student made a mistake about multiplication. But this situation made me think that student had difficulty in multiplication with decimals". This claim was not directly related to a multiplicative relationship. She highlighted that she needed more evidence to make this interpretation. Overall, All these claims were relevant and consisted of details reflecting Nazli's understanding. In this regard, PST1's response was categorized as *robust evidence* of interpretation.

4.2.3.3. Deciding How to Respond on the Basis of Student 3's Understanding

Table 4.17 shows that the PSTs had difficulties deciding how to respond to the student's understanding, as exhibited in the student's correct strategies in MCV3. The PSTs proposed ten instructional actions in total, and their representative responses of each level are given below. While 31% of them were case-specific-detailed, 15% of them were general. The representative examples of instructional actions were given in Table 4.17 below. The following is a sample response that provides *general* and case-specific-undetailed instructional actions on the basis of Nazli's understanding in MCV3:

> PST4: *If I were the teacher of this student, I would show the incorrect solution which was solved by modeling to ask the error (general). Then, I would ask some probing questions (related to unit rate) in order to understand whether she could make sense of unit rate or not* (Reflection paper, W4).

Table 4.17 PSTs' next instructional actions

PSTs	Next instructional actions			Total frequency (%)
	General (Lack of evidence)	Case-specific-undetailed (Limited evidence)	Case-specific-detailed (Robust evidence)	
PST1	0	1	2	3
PST2	1	1	0	2
PST3	0	2	0	2
PST4	1	1	0	2
PST5	0	0	2	2
PST6	0	2	0	2
Total	2 (15)	7 (54)	4 (31)	13 (100)

PST4 suggested two instructional actions in his reflection paper. The first instructional action was unrelated to the student's understanding of the video. Even though he suggested the next strategy (e.g., using an incorrect solution), she did not specify how this incorrect solution could be linked to the student's existing understanding. In this regard, this instructional action was *general* (lack of evidence). Also, PST4 shared another instructional action ("I would ask some probing questions to understand whether she could make sense of unit rate or not") with a rationale considering the student's existing understanding. However, this comment was vague and included minimal specificity in the reasoning. Even though PST4 considered the student's existing strategy (e.g., finding one balloon price-unit rate) in deciding how to respond, he did not provide details about what kind of gear-up question can be asked in the teaching process. In this vein, this response was characterized as *case-specific-undetailed.*

As another sample response, PST1 provided *case-specific-detailed* instructional action on the basis of the student's understanding in MCV3 as follow:

> PST1: *If I were the teacher of this student, I would provide different quantities. In this task, the numbers were nice numbers. Therefore, she could solve the problem by creating composite units. If I were the teacher, I would use a problem that provides no easily divisible quantities to see how she would solve the problem. I mean, which strategy would she use? For instance, if she pays 1.4 TL for three balloons, how much she needs to pay for one balloon? My*

expectation would be to see whether she could say 14/30 TL for one balloon or not. How much would she pay for four balloons? I would like to see whether this student recognized the unit rate: the price/the number of balloons-14/30 or the number of balloons/price-30/14.

Also, I liked the student's solution: dividing 24 by three and then making eight groups. I did not think of this potential solution before. This strategy reminds me that we can create unit types (composing units). Therefore, I would like to add her strategy as an expected answer in my lesson plan, and then I would like to ask the student to create different composite units from 16 objects. For instance;

This activity might lead the student to explore the creation of different types of composite units (Reflection paper, W4).

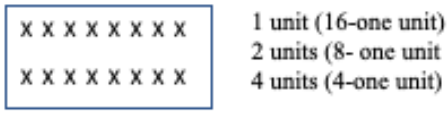


Figure 4.7 PST1's next instructional action (deciding how to respond)

PST1's comments involved more details about why she decided to use the next instructional actions for the student in the video clip. This response above explicitly shows that PST1 considered the student's existing strategy. For instance, in the Grocery problem, Nazli showed evidence of forming composite units. She divided 24 balloons to find the number of groups of three balloons in 24. There were eight groups of three balloons, so there should be eight groups of 1.5 liras. Then, PST1 provided a follow-up instructional activity. Making groups of three into 24 balloons led her to form the goal of finding how many composite units the student could make from 16 objects. Although she did not illustrate the detailed figure regarding how she constructed different composite units, she clearly explained why she suggested the possible next strategy. Providing an opportunity to ask about making composite units that include a different number of objects might encourage her to work with composite units. Besides, PST1 expressed her curiosity in seeing the student's approach if non-divisible numbers (e.g., three balloons were 1.4 liras) were given. She wondered about the student's preference for singleton units or composite units. Making groups of three into 24 balloons led her to form her goal of finding how many composite units the

student could make from 16 objects. Even though PST1 did not illustrate the detailed figure regarding how she constructed different composite units, she clearly explained why she suggested the possible next strategy.

4.2.4. Prospective Teachers' Professional Noticing After MCV3 Analysis

This section provides the development of prospective teachers professional noticing of Nazli's (Student 3) understanding in the context of composite unit construction in the MCV3.

4.2.4.1. Attending to the Mathematical Elements in Student 3's Understanding

Some illustrative examples of discussion excerpt display PST's attention to Nazli's strategies through group discussions. After writing individual reflection papers, the group discussion was initiated.

Facilitator	*What did you notice about the student's understanding in the video?*
PST5	*She found the number of groups of three.*
PST1	*The student stated that she took 3 balloons as one group. Then, she found how many groups of three there were in 24. She created groups of three to make a single group or composite unit: 3 balloons-1.5 liras. She thought that if she is required to buy 24 balloons, she will buy 8 groups of three balloons. She multiplied 1.5 by 8 and found 14. She realized that she made a mistake.*
PST4	*Yeah, she found 14 and she was not sure about it. Therefore, she decided to calculate by fingers.*
PST6	*She wrote 12 directly without making computation, because in her previous solution, she used unit rate and found 12.*
PST1	*Yeah, after her hesitation, she iterated eight times of 1.5 using her fingers.*

Facilitator	*Hmm, you used composite unit and iteration. First, what do you mean by composite unit and iteration?*
PST1	*I mean, she took 3 balloons as one group or new unit, called composite unit. Also, she sequentially put up her fingers eight times-this is iteration- by saying "1.5-3-4.5-…-21.5-24"*
Facilitator	*What about you PST2? Is there anything you noticed?*
PST2	*One more thing I would like to add is she stated that she used four balloons to find the result. She took four balloons as the composite unit. In this regard, I think she was able to flexibly work with composite units.*
	Can we watch the video clip again? I missed something about her strategy (The group watched the video clip again).
PST2	*Here it is, she stated that we could use 1 lira because one balloon was 0.50 kr and 1 lira means 2 balloons.*

In the group discussion, prospective teachers focused on composite units. PST1 described all the actions in the student's response by giving details. She highlighted noteworthy mathematical details, including the rationale for the student's mathematical understanding. For instance, she stated that the student took three balloons as one group and then found the number of groups. Then, the student multiplied 1.5 by 8. In order to provide evidence, PST1 showed the student's explanation of how she iterated eight times 1.5 with her fingers. PST2 captured another point of her strategy about composite units with evidence of attention to the student's strategy. PST2's question led the group discussion to shift to an in-depth focus on composite units. This question guided us to talk about the previous week's topic-unit rate and also the new terminology-invariant relationship. PST2 provided further details in her reflection paper after the video clip analysis and referred to the advantages of group discussion. She evaluated her previous approach as follows:

> PST2: *The students solved the problem in more than one way, including unit rate and composite units. In the group discussion, we focused more on composite units based on the student's existing strategies in the video. Before the discussion, I did not recognize the significance of the composite*

> *unit concept and its relation to a unit rate. Nazli used composite units and iterated these composite units by preserving one balloon's price. For instance, she used two and three balloons as an example of composite units by keeping one balloon's price. Besides, as I mentioned before, we started using more proportionality language such as iteration, composite units, or invariant* (Revised lesson plan, W5-W6 and post-interview).

By referring to the benefit of the group discussion, PST2 focused on different details of the mathematical elements in the student's strategies in MCV3. The comments revealed that PST2 noticed the importance of constructing composite units for the student's understanding of the relationships between two given quantities. Moreover, she used multiple sources of evidence by attending to her strategies. For instance, in the reflection paper, PST2 stated that the student used three balloons as one group as *robust evidence* of attending to the student's strategy. After video analysis, she provided more detailed evidence of the student's strategies for taking two and three as one group.

Further, in her previous comments, she criticized using less proportional reasoning language throughout the group discussion. She stated that they started using more proportionality language in their explanations. Similarly, PST5, who provided evidence in individual video analysis, pointed to significant details about the invariant relationship, which was interpreted as a unit rate based on the student's strategy in MCV3. She wrote the following comments:

> PST5: *After the group discussion, I realized different details which I did not previously attend. For instance, this student used different composite units, such as three and two balloons. Specifically, she could make composite units using two balloons. As we discussed, she also realized the invariant relationship because she said, "we can use two balloons instead of three balloons. Since one balloon is still 0.5 lira"* (Revised Lesson plan, W5-6).

PST5 provided *robust evidence* of increased attention to the student's mathematical thinking. Her comments could be enlightening to notice evidence of what details the student attended to and how she attended to it. She realized the invariant relationship between composite units by recalling the student's specific statements in the video clip.

Besides, PST6 and PST4 did not provide evidence of increased attention to the student's thinking after the group discussion. They described the student's procedural steps with general statements. For instance, PST6 states that the student wrote 12 without calculating because she found 12 before. They described some of the details, but they could not fully get the gist of the student's strategy. Also, the data indicated that while PST6 recognized the unit rate in the student's strategy, she had difficulty recognizing the *composite units*. In that week's lesson plans, they did not make any revisions on the basis of the student's strategy in MCV3, which can be additional evidence for the claim that they provided limited evidence of attending to the student's strategy.

4.2.4.2. Interpretations of Student 3's Mathematical Understandings

Some illustrative examples of discussion excerpts were presented to show how prospective teachers interpreted Nazlı's understanding in MCV3.

Facilitator	*So, tell me what you learned about this student's understanding based on her strategy?*
PST4	*She understood the proportional relationships very well. Therefore, she easily divided 24 by 3 and then multiplied the result by 8.*
PST3	*Yeah, I think this student is good at multiplicative relationships because she was able to work with composite units, such as three balloons, and then she was able to create another composite unit, such as two balloons. Therefore, she could reinterpret the situation regarding those units. For example, she found eight groups for three balloons and 12 groups for two balloons. Then, she multiplied 1.5 by eight and wrote 12. It seems that she was not sure about the result.*

	So, she started iterating 1.5 eight times using her fingers to find the total price, and she seemed to know when to stop. However, I am still unsure whether she might be able to find a way to work with appropriate units. This video did not show me this construction explicitly.
Facilitator	*You said she iterated 1.5 using her fingers. Why did she use her fingers?*
PST3	*I believe she was trying to keep track of how many times she needed to iterate the composite unit. As PST1 stated, she stopped counting without saying anything. In this regard, I believe that she had a concept of 1.5 that she tried to coordinate with the concept of eight, and the student was able to determine how many 1.5 she made. Therefore, I think she conceptualized the iteration.*

Despite multiple inputs about attending, PST4's interpretations in the Balloon task remained limited to what the student understood or did not understand. For example, he made interpretations about what Nazli understood without grounding interpretations in evidence. He worked to interpret the student's understanding, focusing on correct or incorrect responses with evaluative assumptions.

PST1 explained the terms "composite units" and "iteration" in the discussion. For instance, she said this student took three balloons as one group and then found how many groups of three there were in 24. PST1 provided robust evidence for the student's composite unit generation details. Then, she responded with the phrase, "But still, I am not sure…". This terminology showed the teacher's hesitancy or uncertainty about the student's composite unit construction. She explained this hesitation with specific and robust evidence in the video clip.

When the facilitator wondered why the student used her fingers, PST3 immediately said that the student used her fingers to keep track of counting till she reached the desired quantity. PST3 evaluated what she attended to and then explained how she arrived at that interpretation with robust evidence. After the group discussion, PST5

and PST2 explained their progress in noticing of Nazli's understanding of composite units and iterating composite units by providing robust evidence as follows:

> PST5: *This week we focused on composite units, including creating new composite units and iterating those composite units. At first, I thought this student understood working with composite units. However, PST1's comment made me think that we cannot say this student conceptualized composite units very well. I mean, the correct solution did not mean that she understood the composite unit. As she stated, there was not enough evidence about how she formed composite units and worked with composite units in different contexts. Therefore, I could explain the student's work asking probing questions while solving the problem* (Revised lesson plan, W5).

> PST2: *I focused on different terms, iteration, and linking composite units. Although we already used composite units and iteration on ratio tables, we should know the language to describe what students do while communicating about students' conceptions of ratio and proportion* (Post-interview, W8-W9*)*.

These comments were aligned with the increase in the dept of the PSTs' interpretations of the student's understanding of ratio and proportion. PST5 stated that she needed more evidence to assist her interpretation of the student's understanding of composite unit construction. She tries to be careful about not to over-attribute to the student's understanding in this context. Besides, PST2 stated that she tended to focus on proportionality language as a key part of attending to and interpreting the student's mathematical understandings.

Another interesting example was the developments in the PSTs' interpretations of noteworthy mathematical elements in the student's strategy in MCV3. The facilitator posited a question to examine the PSTs' reasoning about the relationship between given quantities.

Facilitator	*What do you think about the student's understanding about composite unit and unit rate?*
PST2	*I think she realized that the unit price never changed. In the video, the student said that she could use two balloons as well because one balloon's price was still 0.5.*
Facilitator	*I see, you claim that while iterating two balloons, she realized that she needed to keep one balloon's price the same or invariant. Did I understand correctly?*
PST2	*Yeah… (Some students nodded their heads).*
Facilitator	*Yeah, we keep one balloon's price invariant. This invariant relationship between those linked composite units was described as unit rate.*
PST5	*Hmmm, I wonder both 1.5 and 3 were linked composite units, then?*
PST1	*I think so.*

In the discussion, the facilitator guided the PSTs to think about the relationship between composite units and unit rates. According to PST2, this student recognized the relationship between the composite unit and unit rate by providing evidence from the video. At this point, PST5 elaborated on her interpretation after the group discussion. PST5's comments are given below:

> PST5: *So far, I have realized that content knowledge helped me notice the details of students' understanding and offered extended next step. For instance, we have learned the relationship between composite units and unit rates. Therefore, I could be more aware of the student's strategy and interpret the student's understanding when she used two balloons 12 times. She realized that while iterating two balloons, she kept one balloon's price the same* (Revised lesson plan-W5).

Notice that PST5 connected her observations to her interpretations of the student's understanding. She stated that ratio and proportional content knowledge is necessary to fully interpret and offer the next instructional suggestion on the basis of the student's

understanding. In this regard, she provided *robust evidence* of interpreting students' understanding with a specific claim.

4.2.4.3. Deciding How to Respond on the Basis of Student 3's Understanding

Table 4.18 shows the number of next instructional actions the PSTs provided for deciding how to respond to the student's understanding in MCV3. The PSTs proposed 11 instruction actions in total, and 46% of them were case-specific-detailed. 18% of proposed instructional actions were general and did not reference Student 3's understanding in the video clip. The discussion excepts, and representative responses of each level are given below.

Table 4.18 Next instructional actions after micro case video analysis

PSTs	Next instructional actions			Total frequency
	General	Case-specific-undetailed	Case-specific-detailed	
PST1	0	0	2	2
PST2	0	2	0	2
PST3	0	0	0	0
PST4	1	1	0	2
PST5	0	0	3	3
PST6	1	1	0	2
Total frequency (%)	2 (18)	4 (36)	5 (46)	11 (100)

In the group discussion, the PSTs focused on alternative instructional actions to make some vague points in the student's understanding clear. Therefore, instructional actions that the PSTs offered were based on the student's understanding and had a connection with specific actions. Before video analysis, although all the PSTs provided case-specific-undetailed instructional actions, most PSTs suggested case-specific-detailed instructional suggestions.

Facilitator	*Pretend that you are the teacher of this student. What instructional actions might you pose next?*
PST1	*She was able to realize the relationship between three balloons and 24 balloons. This student realized the advantages of using composite*

units- three balloons. So, I would like to ask whether this student considers the most appropriate unit because in the video, her approach about how to choose a more appropriate unit by envisioning the whole was not obvious.

Facilitator *How?*

PST1 *For instance, I would give her more difficult numbers to see how she generates and uses different composite units in context. I would give activities to internalize the composite units. Six balloons are 5 liras. What about the price for 15 balloons? Here, the numbers are not divisible, and I wonder which unit she might choose.*

PST4 *I would ask her to create models.*

PST3 *I agree with PST1's idea. Besides, I wonder how she constructs unit rate. In her solution, she used different strategies, such as finding one balloon's price and using composite units. However, I am unsure about how she made sense of the unit rate because there was not enough information on the student's understanding of the unit rate in the video. As we discussed last week, two unit rates correspond to the given quantities in the context. We talked about each of them as reciprocal to the other one. In this regard, I would like to use models to see how she identifies the meaning of 3/1.5 and 1.5/3 as unit rates, which representation (for balloon or money) she draws first and what will be her reason for drawing this first. Then, I would ask her how she distributes one to another and if we could do a vice versa.*

Facilitator *Is there any different suggestion?*

PST4 *I wonder about the same issue, so I would use cognitive conflict to catch her understanding of the unit rate. For instance, I would give an incorrect solution, such as 3/1.5=2 without saying anything about whether it is correct or not. I would ask her the difference between the solutions. Therefore, she can compare the results.*

Also, I would like to encourage the student to develop different strategies, such as build-up strategies (even though the facilitator

asked him to explain this response, he did not give specific details about it).

In this example, PST1 suggested using different instructional actions to analyze whether this student conceptualized making composite units. Especially for non-divisible numbers, if the student cannot divide 15 by 6 (6 is a composite number) directly, PST1 wonders how she alters her composite unit strategy. She would like to ask a question for clarification. Then, PST3 shifted the attention toward another point. She stated that she did not have enough evidence about the student's construction of unit rate based on evidence in the video. In this regard, she suggested a particular instructional action to reveal the student's thinking about types of unit rates (i.e., the number of balloons/the amount of money or the amount of money/the number of balloons). By exporting the student's reasoning in detail, PST3 demonstrated that she tried to make correct and robust interpretations based on the student's understanding. PST4 provided a response using models without explaining any rationale. Then, he anticipated another instructional action with the rationale of using the incorrect solution to compare her answer, which accounted for the student's existing performance but with minimal specificity.

Furthermore, he offered another instructional action. He said, "I would like to encourage the student to develop different strategies such as build-up strategies," which indicates openness to the student's ideas. However, this statement did not describe the rationale for selecting this instructional action. Once the foci of their noticing shift to unclear points of the student's mathematical understanding, there is evidence that PST1, PST3, and PST4's shifted their responses to elicit more ideas from this student rather than to overcome potential incorrect answers. Even though PST1 and PST3 showed this progression explicitly in their lesson plans and post-interviews by providing case-specific-detailed instructional actions, PST4 shared case-specific-undetailed instructional actions considering the student's understanding of MCV3. PST4's comments and illustrative examples of initial and revised lesson plans (See Table 4.19) are presented below.

> PST4: *This week I would like to pay attention to within and between ratios as a part of unit rate. First, I wanted to add one more problem to lead students to use different strategies. I added market problem as an additional activity and I added "use more than one strategy to solve the problem" to push students to apply within and between ratios for this problem. The teacher could ask them questions about their strategies to understand whether students were aware of using within and between ratios. If they were not aware of them, the teacher could teach them* (Revised lesson plan, W5).

PST4 could not consider himself a teacher of the student who solved the problem correctly using more than one strategy (i.e., unit rate and composing units). He only provided the problem above (see Table 4.19) without taking the student's understanding of MCV3 as a reference. Although he superficially stated why he preferred this problem, he did not give specific information about the meaning of the given quantities in the problem context. For instance, he did not explain the meaning of 5 litter or why he chose 15 to determine the cheaper market. Also, his explanation was general and somewhat vague. For example, he stated, "The teacher will ask them some questions about their strategies and see whether they are aware of they are using within and between ratios or not." He did not define what questions he might ask in the next step. Thus, his response was categorized as *case-specific-undetailed*.

Table 4.19 Examples of instructional actions presented in lesson plans

Example parts of …	
Initial lesson plan (before video analysis)	Revised lesson plan (after video analysis, W3-W4)
"Doğan and Utku are two friends. Doğan have a car model A and Utku has model B. In Doğan's car storage there are 24 liters fuel and in Utku's car storage there are 30 liters fuel. Doğan's car is driven at a distance of 120 km by 4 liters and Utku's car is driven at a distance of 100 km by 5 liters of fuel consumption. Can you find how many kilometers they can go by 1 liter of fuel consumption?" **Expected solution:** Model A: 120/4=30 km for 1 liter of fuel consumption and Model B: 100/5=20 km for 1 liter of fuel consumption This is the first question. In this question, the teacher will activate students' previous knowledge about **M.7.1.4.1.** this objective. The teacher can bring two toy cars to the class to use them visually for this problem. Then, the teacher will move on to second part of the problem.	There are two markets which are A and B in Zuhal's hometown. You can buy a 2.5 liters cola for 6 TL in market A and 3 liters cola for 7.5 TL. If Zuhal wants to buy a cheaper cola, which market should she choose? (Use more than one strategy to solve this problem)" **Expected Solution(s):** The students can use within ratio and between ratio for this problem. Therefore, they can divide liters by money or vice versa. 2.5/6=0.41-liter cola per TL for market A 3/7.5=0.4-liter cola per TL for market B. Since we buy more cola with 1 TL, Zuhal should choose market A for a cheaper cola. 6/2.5=2.4 TL per liter cola for market A 7.5/3=2.5 TL per liter cola for market B. Since we pay less money for 1 liter cola, Zuhal should choose Market A for a cheaper cola. These strategies are built on within ratio. In this problem, within ratio is more effective solution. The students can use build up strategy. For market A, if we pay 6 TL for 2.5 liters, we will pay 36 TL for 15 liters. For market B, if we pay 7.5 TL for 3 liters, we will pay 37.5 TL for 15 liters. We can understand that market A is cheaper than market B.

To sum up, Table 4.20 indicates the development prospective teachers' professional noticing of the student's understanding in the context of composite unit construction. Four of them provided robust evidence of attending to the student's strategy and interpreting the student's understanding. PST6 provided robust evidence of attending to the student's strategy but limited evidence of interpreting the student's understanding. However, PST4 provided limited evidence for both skills (i.e., attending to and interpreting). All proposed instructional actions were case-specific-undetailed. They proposed case-specific-undetailed instructional actions on the basis of the student's thinking.

Instructional actions in the reflection papers of the PSTs were mostly case-specific and undetailed (see Table 4.17). However, last week after MCV2 analysis, majority of PSTs provided case-specific and detailed instructional actions (see Table 4.10) during group discussions and in their weekly revised lesson plans. Table 4.20 shows that most of PSTs improved their responding skill, which helped them to propose more qualified and detailed responses considering the student's understandings after MCV3 analysis.

Table 4.20 Comparison of PSTs' average points of comments on responding (MCV3)

	Prospective teachers						Average of total points
	PST1	PST2	PST3	PST4	PST5	PST6	
Before video analysis	1	1	1	0	1.3	1	0.88
After video analysis	1	2	1.7	1	2	1	1.45

Assigned the certain numbers to each level;

*0 for general comments

*1 for case-specific-undetailed comments

*2 for case-specific-detailed comments

*including initial-revised lesson plans, pre-post interviews and group discussions

4.3. Prospective Teachers' Professional Noticing of Students' Thinking in the Context of Proportional Relationships

Based on the research questions of the current study, the nature of prospective teachers' professional noticing of students' thinking in the context of proportional relationships was investigated into two parts; invariance and covariance and part-part-whole relationships. There are two tasks: photograph and mixture problems related to proportional relationships. The following part presents the prospective teachers' professional noticing skills of Student 4's thinking in the context of invariance and covariance in micro-case video clip (MCV4).

Invariance and Covariance

Table 4.21 Hilal's (Student 4) strategy in the Photograph Problem (MCV4)

Aynur would like to enlarge the photograph of his brother as seen below. How many times the enlarged photograph is bigger than the original photograph?

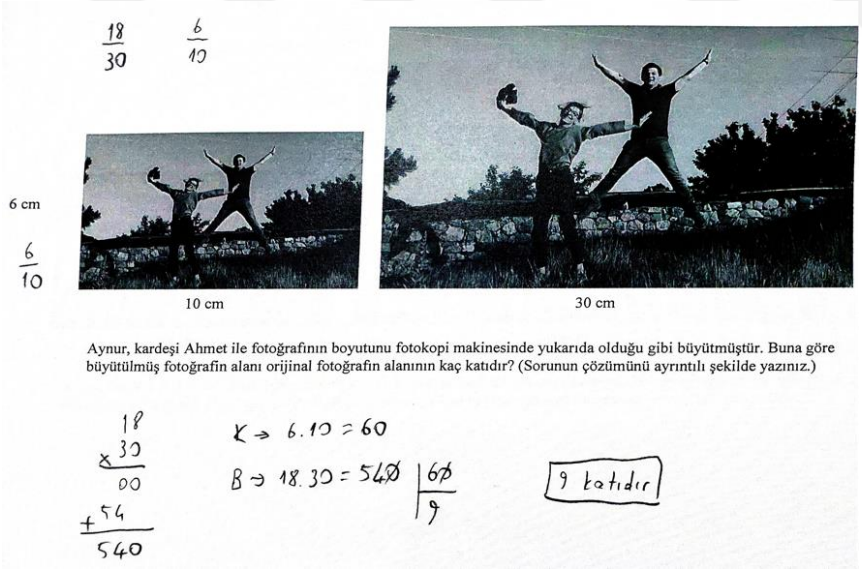


Table 4.22 shows the levels of prospective teachers' responses about attending to the student's strategy and interpreting the student's understanding in MCV4 before and after video analysis. Even though some prospective teachers' before and after attending

and interpreting levels' were the same, their professional noticing skills were developed. For instance, this progress involved attending to some pertinent features and disattending to others. Prospective teachers proposed multiple valid interpretations of the student's understanding, including "what the student understood" and "what the student did not understand." In this vein, the group discussions became more interpretive to make sense of what attended to the student's strategies. Besides, in group discussions and revised lesson plans, they provided more extended, detailed, and case-specific instructional actions on the basis of Student 4's understanding of MCV4 rather than other instructional actions before video analysis.

As seen below, before the MCV4 analysis, 5 out of 6 participants attended to the student's strategy, and 4 out of 6 interpreted the student's mathematical understanding at a robust level. They focused on interpreting Student 4's understanding, and all provided robust evidence. While 2 out of 6 prospective teachers provided case-specific and undetailed instructional suggestions, 4 out of 6 provided case-specific and detailed ones. After video analysis, all PSTs' responses provided robust evidence of attention to the student's strategies, and 4 out of 6 included robust evidence of interpretation of the student's understanding. 4 out of 6 proposed case-specific-detailed instructional actions in response to Student 4's thinking. On the other hand, two prospective teachers' responses (PST4 and PST6) included case-specific-undetailed instructional suggestions. Their response included limited evidence of deciding how to respond based on the student's understanding.

4.3.1. Prospective Teachers' Professional Noticing Before MCV4 analysis

This section provides the nature of prospective teachers' knowledge and noticing of students' thinking about *invariance and covariance.* Table 4.22 shows the development of PSTs' noticing skills of Student 4' understanding in the context of invariance and covariance after participating in Session 4. The representative examples of prospective teachers' attending and interpreting responses and instructional actions in the video analysis were explained in the following sections.

Table 4.22 PSTs' noticing level of evidence of attending and interpreting before and after Photograph Problem video-based intervention (MCV4)

Prospective teachers		PST1	PST2	PST3	PST4	PST5	PST6
Attending	Before video analysis	Robust	Robust	Robust	Limited	Robust	Robust
	After video analysis	Robust	Robust	Robust	Robust	Robust	Robust
Interpreting	Before video analysis	Robust	Robust	Robust	Limited	Robust	Limited
	After video analysis	Robust	Robust	Robust	Limited	Robust	Limited
Responding	Before video analysis	CSD	CSD	CSD	CSU	CSD	CSU
	After video analysis	CSD	CSD	CSD	CSU	CSD	CSU

*CSU= Case-specific-undetailed

*CSD= Case-specific-detailed

4.3.1.1. Attending to the Mathematical Elements of Student 4's Strategies

The following response of PST5 was taken from video analysis reflection paper, which provided evidence of attention to the mathematical details based on the student's strategies in MCV4:

PST5 focused on noteworthy and related mathematical elements of the student's solution in the video clip, and she provided *robust evidence* based on the reflection of MCV4. First, she stated that the student applied the same scale factor (i.e., 3) to the length of the rectangle so that she could find the missing length of the enlarged one. Then, she realized that the corresponding lengths in the written response were not multiplied by the same scale factor, so the rectangles were not similar (i.e., 6cm*10 cm and 26cm*30 cm). Furthermore, she pointed out that she preserved the ratio change (i.e., 6/10) between the sides of the original photograph to avoid distortion. Even though she did not use the term constant ratio, she stressed that the faces in the written response looked taller and thinner than the original photograph because it was enlarged

without keeping the ratio the same. This response was categorized as robust evidence of attending the student's strategy.

> PST5: *First, this student found the area of the original photo. The (rectangular) area was short side lengths x long side lengths, so she found the area of the original photos as 6x10=60. She stated that the short side lengths (the height) of the second photo were not given. In order to avoid distortion of the second photo, she enlarged the first photograph with the same ratio (constant ratio). Therefore, she provided the relationship between the long side lengths of the two photos. The length (base) of the original photo was multiplied by 3, so the height of the original photo should be multiplied by 3 (scale factor). So, she found 6x3=18 cm. Then, she calculated the area of the enlarged-photo; 18x30=540. The area of the enlarged-photo was nine times bigger than the area of the original one. When another solution in which the student thought additively was shown, she pointed out that it would be multiplied by three instead of 20. The reason for this was to preserve 6/10 (constant ratio-the ratio of height to width) in order to avoid distortion. She explained that the ratio of 26/30 was not the same as the ratio of 6/10* (Reflection paper, W6).

The following response of PST4 was taken from video analysis reflection paper, which provided limited evidence of attention to the mathematical details based on the student's strategies in MCV4:

> PST4: *(short side length) x (long side length)*
>
> 6x10=60
>
> *In order to avoid distortion, the photograph was enlarged by the same ratio (scale factor).*
>
> *Therefore, if I multiplied 10 by three, I multiplied 6 by three and found 18. Then, she found the areas of the photos and then found 18 by correctly dividing the enlarged one by the small photos* (Reflection paper, W6).

PST4 provided limited evidence of attending to some of the details of the student's strategy in the video clip. She stated, "To avoid distortion, the original photograph should be in the same ratio." However, PST4 did not describe other details, such as how the student obtained three and why the student multiplied the short side by three. Also, he did not use proportional language in his explanation when comparing other responses, which provided robust evidence. In this regard, PST4's response was categorized as *limited evidence* of attending to the student's strategy.

4.3.1.2. Interpreting Student 4's Understanding

The following response sample was taken from a video analysis reflection paper that provided *robust evidence* of interpreting the details of Hilal's understanding in MCV4:

> PST1: *The student was not able to understand the topic. In order to preserve the structure of the photograph (the people's face, mouth, and nose), the ratio should be preserved (constant ratio). When another solution with addition was shown, she stated that the ratio would be preserved. This explanation shows that the student could make sense of the topic.*
> *The student's statement that there should be a ratio of 6/10 between the small and large photographs indicates that she has made sense of multiplicative reasoning.*
> *In the end, the student said, "Even the area grows, the ratio is still 6/10", which shows that the ratio (constant ratio) should be preserved to obtain a similar photograph. This explanation indicates that the student was aware of the meaning (conceptually) of her steps and knew what she had done so far* (Reflection paper, W6).

As seen above, PST1 described the student's understanding with substantiated claims by providing robust evidence. For instance, she pointed out that the student could conceptualize the invariant relationship within the sides of the original photograph (internal proportion). PST1 claims this student could intentionally explain why additive reasoning (the constant difference strategy) was inappropriate to correctly describe the missing length between similar figures. She justified her reasoning by

saying that even though this task was solved using a constant difference strategy, there would be a constant ratio between corresponding lengths. In this regard, PST1 provided a reference to the student's statement that "the area (of the original photograph) is enlarged, but the ratio of 6/10 (its width to its length) is still same" as evidence of implicit understanding of preserving the ratio within a single object. Also, PST1 used proportional reasoning language to explain the student's understanding. In this regard, PST1's response was considered as robust evidence of interpreting the student's understanding.

The following is another sample response that provided robust evidence of interpretation of Hilal's understanding in MCV4:

> PST2: *The student thinks that there should be a constant ratio between the short side length and the long side length to avoid distortion. She realized that she needed to enlarge the photograph by a ratio (scale factor). Therefore, she found the ratio of the long side lengths of both rectangles (three times relationship). The student would know that this ratio between long sides (three times relationship) should be the same between the short sides, so the student multiplied the short side by 3. Then, a wrong answer was shown to the student without saying it was incorrect. Even though she noticed that it was incorrect to increase the short side by adding 20, she could not explain why we could not focus on the difference between given sides rather than the proportionality illustrated in the problem context. The incorrect response challenged her to further her reasoning. Then, she stated that this ratio (i.e.,* constant ratio*) should be preserved to avoid distortion. She stressed the invariant relationship (*within ratio*) between the short and long sides of the photograph. She said, "even though the area becomes larger, the ratio will be the same." Also, she recognized that to obtain similarity, the scalar relationship* (i.e., scalar factor) *between the corresponding sides (short side-short side and long side-long side) of two photographs would be the same. She explained this relationship as three times relationship* (between ratios). *Her explanations showed that this student made sense of the within and between relationships between*

> *quantities and realized that "when one side increases by three times, the corresponding side also increases by three times" and "the ratio between short side and long side for each shape would be preserved"* (Reflection paper, W6).

PST2 interpreted the student's understanding in different ways and made substantiated claims on the student's understanding. She made sense of the noteworthy details of the student's strategy and how these details reflected what the student did understand and did not understand. For instance, PST2 pointed out that this student considered the relations between the short and long lengths of the original photograph, and she evidenced this claim by referring to the specific and related parts of the student's work. She stated that the student realized the original photograph should be enlarged so that its lengths preserve the ratio. Therefore, she expressed the ratio of long side lengths between the original photograph and the enlarged one: three times relationship. This "three times relationship" should be applied to the missing short side length of the second photograph. PST2 reasoned this relationship as between ratios. In this context, she revisited the concept of ratio as referring to the number to scale (scale factor) of the long side lengths in the original photograph to get a similar enlarged photograph. She made sense of invariant relationship because she told that “the ratio between short and long side would be preserved”.

Second, PST2 also realized what understanding the student did not demonstrate. She stated that this student did not explain why she first enlarged additively (i.e., 30-10=20 and 6+20=26 cm). In the wrong solution, another student solved the task by increasing the short side of the length by 20 cm to find the missing lengths of the second photograph. According to PST2, this student reasoned about similarity and distortion using within and between ratios. Then, PST2 wrote, "She found the invariance (constant ratio) by using within ratio between short side lengths and long side lengths of the original photograph. She realized that this ratio keeps constant to avoid photograph distortion". These statements and particular linking interpretations about the student's understanding were used as evidence from the video to advance her interpretation. Overall, this explanation indicates that PST2 made sense of the student's

understanding based on the student's knowledge and the proportional content. PST2's interpretation was categorized as *robust evidence of interpretation* in this regard.

Below is a sample response that provided *limited evidence* of interpretation of the student's understandings:

> PST4: *First, the student stated that it would be distorted if the same ratio did not enlarge the photograph. She realized the multiplicative relationship, but the student could not state another reason when the incorrect answer which included the additive strategy was given. She did not make sense of that solution and told me that the side would not increase by 20 (cm). The student may not be familiar with additive reasoning because she did not understand it at first. She directly applied multiplicative strategy because she applied the memorized algorithm. When the student saw the different solution, she showed her hesitation and could explain why the additive solution was incorrect. Her initial explanation showed why an additive strategy could not be applied, but she could not recognize it* (Reflection paper, W6).

PST4 interpreted the student's understanding by providing more general claims with limited evidence responses using limited proportional language. For example, to reflect the student's understanding, PST4 selected particular and noteworthy events. More specifically, he explained, "The student stated that if the same ratio enlarges the (original) photograph, it will be distorted. It means that she realized the multiplicative relationship. Nevertheless, when the incorrect solution including addition was shown, the student could make another reasoning. Even though he stressed that the side of the lengths had to have the same ratio, he did not mention how the student found the constant ratio and how to apply it to the second photograph. Besides, PST4 claimed that this student directly made multiplicative operations by heart, but she did not prove how he reached this point. Even though PST4 recognized what understanding the student did not demonstrate, he supported this statement with superficial evidence. He interpreted that this student would not be familiar with the additive relationship because, at first glance, she could not explain why the second solution was incorrect.

In this regard, PST4's response was categorized as *limited evidence of interpretation* of the understanding of the student in the video clip.

4.3.1.3. Deciding How to Respond on the Basis of Student 4's Understanding

Table 4.23 shows that the PSTs provided a detailed response on how to respond to Hilal's understanding in MCV4. Prospective teachers proposed case-specific-undetailed and case-specific-detailed instructional actions. None of the PSTs proposed instructional actions that were categorized as general.

Table 4.23 PSTs' instructional actions before MCV4 analysis

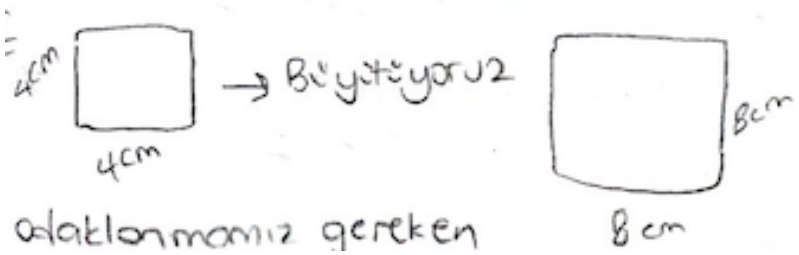
PSTs	Next instructional actions			Total frequency
	General (Lack of evidence)	Case-specific-undetailed (Limited evidence)	Case-specific-detailed (Robust evidence)	
PST1	0	2	0	2
PST2	0	1	1	2
PST3	0	2	1	3
PST4	0	1	0	1
PST5	0	0	3	3
PST6	0	1	0	1
Total (%)	0	7 (58)	5 (42)	12 (100)

Below is a sample response pertaining to *case-specific-undetailed* instruction actions (limited evidence) on the basis of Hilal's understanding in MCV4:

> PST4: *Regardless of this question, I would ask two questions: additive and multiplicative relationships. Then, I would learn the student's reasoning based on her solution. I would ask questions to explore the difference between additive and multiplicative relationships. Because the student's proportional reasoning was not developed yet, she did not know the difference between additive and multiplicative thinking* (Reflection paper-W6).

PST4 proposed the next instructional actions as to how he would respond based on the student's understanding in the video clip. He stated that he would use two problems to show the difference between additive and multiplicative relationships. Even though he tended to focus on the student's difficulty and then developed his instructional action, he did not provide the details of the instructional action. For instance, he claimed that "the student's proportional reasoning was not developed yet." However, he did not provide specific connections to the student's strategy and understanding. Also, the rationale for his instructional choice in response to what he noticed was unclear. Although he considered the student's strategies and understanding, he did not provide the details of the next instructional actions and her rationale. In this regard, PST4's instructional actions were grouped as *case-specific-undetailed* (limited evidence).

PST5 provided *case-specific-detailed* instructional actions on the basis of the student's understanding in MCV4 as follows:

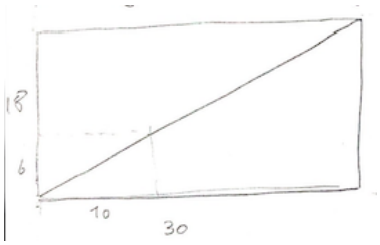
> *If I were a teacher, I would provide one rectangle in which two sides (vertically and horizontally) were equal; the ratio of corresponding lengths (vertical/horizontal or horizontal/vertical) is 1. As seen above, the increase of both side lengths is equal. I would use this example intentionally to reveal how the student reason this change multiplicatively and additively, or why we can think this relationship both additively and multiplicatively. She will be able to realize that the increase in both lengths is equal. The shape is square, so the shape could be enlarged or stretched in both ways proportionally and additively.*
>
> *Example:*
>
> 
>
>
> *In this example, both side lengths are increased by adding 4 cm. Then, I would ask her: "Is this one side length (long side of first rectangular) has increased by 4 cm, so another side length (shot side length of first rectangular) has increased 4 cm as well?". Alternatively, I would ask her to explain the change between two shapes. She can say*

that both sides of the length were increased two times, or both sides of the length were increased by adding 4 cm.

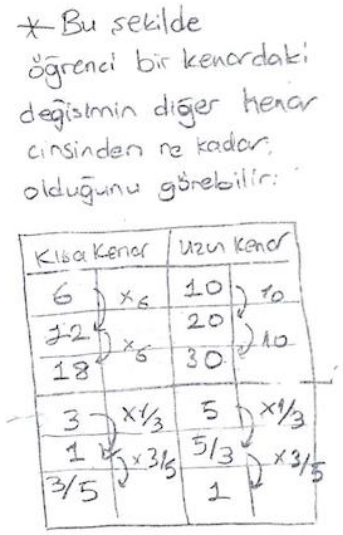
Here, we need to concentrate on the increase in the same amount (i.e., 4 cm or two times) both vertically and horizontally for this case. For instance, in this case, there is an increase of 1 cm unit of horizontal length for the increase of 1 cm unit of vertical length. In another way, we can explore how much each side length increased (the long side length of the first rectangular- the long side length of the enlarged rectangular). For instance, the long side of the first rectangular was enlarged two times (increased by 100%). I would subsequently give these rectangles (cut out of the rectangles) below. I would ask which ones look alike (similar).



For similar rectangles, I would ask the student to stack these rectangles with the largest on the bottom, so that they are aligned at one corner. Here, I would make her recognize that the diagonal edge of the similar rectangles lined up (the first rectangle and the second rectangle). I would make her observe that the rate of change between the lengths (short side and long side) was invariant (constant ratio).



Then, I would ask her why the third rectangle was not similar (I would compare the first and the third one in the same way). I would ask her to explain why they were not similar although the difference in the side lengths is the same. Here my expectation is that the student could recognize that the change in side lengths is not independent of each other within the shape. The change in the side lengths affects other side lengths. In this regard, I would create a table to show these relationships.



In this way (using the table), the student can recognize the change in the lengths of one side considering the change in another one. The ratio table shows that the lengths of the rectangle increase multiplicatively (using the same scale factor), not additively. For instance, the increase of 6 cm in short side length results in 10 cm in long side length. The change of 1/3 times (scale factor) in the short side length results in 1/3 times in the long side length.

Figure 4.8 PST5's responses on the basis of Hilal's understanding in MCV4 (Revised lesson plan)

PST5's response showed case-specific-detailed evidence of deciding how to respond on the basis of the student's understanding in the video clip. Her comments involve detailed and specific explanations about how and why she chose these instructional actions above to develop the student's understanding of similarity in the video clip. In her response, she proposed two squares that help the student progress from additive to multiplicative reasoning in a similar context. She highlighted the difference of additive and multiplicative approaches in similarity context and then focused on the invariant and covariant relationship between quantities (within-between ratio). For instance, she interpreted that the student could not explain why additive reasoning was not valid for this task. In this regard, she provided two rectangles in which side lengths were 4 cm and 8 cm. It means that those rectangles were square, so their side lengths are in the ratio of 1 to 1. The second rectangular (square) length was increased by two times or adding the same amount of difference-4 cm. She encouraged the student to examine the relationship between both shapes additively and multiplicatively.

Then, she provided three different rectangles. The second rectangle was obtained by increasing the first rectangle's lengths by two times. On the other hand, the third rectangle was obtained by adding 6 cm to each length. She would ask the student to stack the rectangles with the largest on the bottom. As seen above (see Figure 4.8),

they can be coordinated at one corner. When the student placed the straight edge across the diagonals, she could notice that the opposite corners of the first and the second rectangles were lined up. Even though PST5 did not mention the slope, she pointed to this relationship as the ratio of the sides (invariance).

In order to show more clearly, she utilized a ratio table to make sense of vertical and horizontal relationships between variables. She structured the ratio table vertically, as presented above. Using ratio tables, she thought the ratio table would support the student as a tool to notice the relationship between quantities (i.e., invariation and covariation relationships). PST5 multiplied the side lengths with the same scale factor (i.e., two times and three) to obtain similar rectangles. Then, she multiplied the side lengths with the same scale factor (i.e., 1/3 and 3/5) to obtain unit rates (i.e., 1 cm unit of vertical length for every 5/3 cm unit of horizontal length). The ratio table clearly shows that both sides of lengths increase/decrease with the same scale factor by preserving the ratio (unit rate/constant ratio). Hence, using a vertical ratio table, the student would be able to reason with the concepts of invariation and covariation relationship between values (i.e., within-between shapes).

PST5 paid attention to the student's uncertainty in MCV4 and provided case-specific and detailed instructional action. It can be concluded that PST5 made sense of the details of the student's strategy and then how these details reflected what the student did understand and did not understand, demonstrating robust evidence of attending to the student's strategy and the student's understanding. She strategically suggested a *case-specific and detailed instructional action* considering the student's understanding in MCV4.

4.3.2. Prospective Teachers' Professional Noticing After MCV4 Analysis

This section provides the development of prospective teachers professional noticing of Student 4's understanding in the context of invariance and covariance after video analysis.

4.3.2.1. Attending to and Interpreting Student 4's Understandings

In this part, the progress of prospective teachers' attending to the student's strategy and interpreting the student's understanding is given together. The same intervention procedure was followed; the group discussion was initiated after writing individual reflection papers. Then, prospective teachers were asked to write in response to the attending prompt. Even though the specific question (attending prompt) intended to reveal what they noticed about the student's strategy, they also moved to multiple interpretations of the student's understanding. During the group discussion, while they were attending to the student's strategy in MCV4, they simultaneously interpreted the student's mathematical understanding with evidence.

Unlike the previous sessions, the discussion excerpts (Session 3 and Session 4) were analyzed based on attending to the student's strategy and interpreting the student's understanding. The discussion excerpts were given to show the process of professional development. The group discussion was initiated after writing an individual reflection paper.

Facilitator	*What did you notice about the student's strategy in the video?*
PST6	*First, the student multiplied the short side length by the long side length and found the area of the original photograph. Then, she stated that the long side length of the enlarged one should be 30 cm because, to avoid distortion, the ratio will be preserved. It was enlarged by three times, so 6 cm should be increased by three times, too, and it became 18. Then, she found the area of the enlarged photograph, and the difference between the two photographs was 18 times. This student realized that the numerical relationship of (between) shapes should be the same: 10 cm became 30 cm, which means it was enlarged three times. Then, 6 cm should be increased by three times and become 18 cm.* *When she examined the incorrect solution, she said that although the solution was correct, she would not add 20 to the short side length (i.e., 30 -10=20 cm and then 6+20= 26 cm). She said there was no relationship between 6 cm and 26 cm. The student pointed out that there*

would be three times relationships between the sides. At this point, I have a question. She said that there was no relationship between 6 and 26. What would be her explanation if the side length is 24 (four times as large as the 6 cm)? I wonder whether she accepted that response as correct regarding the four times and three times relationship.

PST3 *I don't think so PST6, because she was looking at the ratio (between).*

PST1 *Yeah, she considers that there should be a scale factor. She never thought that the side lengths grew additively.*

(PST2 approved PST1's explanation by nodding her head)

PST6 *She already realized it (constant ratio/within ratio), stating that 6/10 should be preserved. While enlarging or shrinking the shapes, the student highlighted the preserved ratio of 6/10. I just got stuck in her sentence.*

PST4 *I agree with them (PST3 and PST1). Even though the side length was 24 (four times 6), she would not accept the four times relationship to determine the similarity. I am not sure on another point. In the beginning, the student explained her approach very well, "It should be enlarged with the same ratio. Otherwise, the faces on the photo will be distorted". She explained the reason why there was a multiplicative relationship between quantities. However, I think she was unaware of it. When we showed another solution, she was surprised and could not understand. She did not accept 20 cm (adding 20 to 6), and she pointed out the three times relationship but in a memorized manner. Therefore, I think it was the first time she saw this type of a question, and she was not be familiar with additive situations.*

PST6 *I don't agree. Even though she did not use proportional terms, she constructed a scalar relationship. I mean it was not just a drill.*

PST1 *Yeah, even though she did not use proportional language, she focused on within-between relationships to explain her reasoning (PST3 nodded her head as agreeing). For instance, she said, "Of course, the number of areas increases, but 6/10 must be the same (constant ratio)". She recognized that the ratio of the side lengths within the shape remains constant as the values of corresponding side lengths change.*

As PST4 stated, even if we gave 24 as a scalar of 6 cm, she would probably oppose multiplying by the different scale factors to maintain the multiplicative relationship. In this regard, I think that she interiorized the difference between additive and multiplicative situations.

PST4 *In the end, if she said "if we increase the side lengths an as 20 cm, the picture will be distorted", I would agree with you.*

PST3 *She already told me that. She could explain the meaning of 6/10, and then she stressed that there was no covariational relationship between the two quantities: 6 cm and 26 cm did not vary in relation to each other. All corresponding lengths vary according to the same scale factor, and she constructed the invariant and variant relationship between values.*

As stated before, group discussion indicated the process of professional noticing of prospective teachers. In the group discussion, prospective teachers focused on scale factor and constant ratio *(within-between ratio)*. As seen above, prospective teachers used significantly new language that includes such terms as covariation, similarity, invariance, and constant ratio. However, in the first weeks of the teaching experiment, they did not commonly use proportional language in the group discussions. First, PST6 used noteworthy mathematical details, which included a description of the student's strategy. They then interpreted what the student did understand and did not understand in conceptualizing the covariational relationships. For instance, she stated that the student could link the values in the first measure space (the original photograph) and the second measure space (the enlarged photograph) with the same scaler operator. She enlarged the original photograph's short side length by using the scale operator's sameness (three times) to find the missing value. Then, she explained her hesitation related to the student's strategy in MCV4. According to PST6, the student could not explain why two quantities could not combine additively.

Some prospective teachers did not agree with PST6's observation. For instance, as a response, PST4 suggested that the student was unaware of proportional relationship. Also, she explained the scalar relationship (three times relationship) as a memorized

action by drawing attention to the student's explanation in MVC3. PST4 simultaneously attended to the student's actions and evaluated those actions to make inferences about the student's understanding. While PST4 provided mathematical details to report that the student "used three times relationship as memorized manner," he ignored to provide robust evidence about how the student could not construct a 'three times relationship' (covariance) in similarity task context.

Similarly, PST1 explained why she did not share the same thoughts with PST6. She stated that the student concentrated on the scalar relationship between quantities and the constant ratio of this multiplicative situation. At this point, she elaborated on her interpretations of the student's mathematical understanding by providing specific student statements as robust evidence.

Then, PST4 re-explained why he thought that the student memorized the procedure for solving this scaling problem. Although he justified his claim that the student did not clarify why 26 cm was not appropriate to obtain a similar shape, PST3 showed her disagreement with his claim by proving the evidence from the video. PST3 went on to give details about how the student explained the meaning of 6/10 and the covariational relationship between corresponding side lengths by examining the incorrect solution. She utilized these details as evidence to make inferences about the student's understanding of invariance and covariance within the context. After the group discussion, PST2 and PST3 evaluated their progress by providing *robust evidence* in noticing the student's mathematical understanding in MCV4.

> PST2: *I thought the student in the video was aware of the relationship between the corresponding lengths of rectangles. After the discussion, I realized that even if one person knows the within-between ratio or unit factor and the composed unit, she/he will be unable to transfer them in every context. For instance, invariance and covariance structures should be worded or graphed in similarity tasks so that the relationship between quantities can become explicit. Also, I realized that it was difficult to express the relationship in natural rather than proportional language. As stated before, I noticed that we have developed using the language of*

proportionality to express concepts and relationships in more detail in the student's solution (Post-interview, W8-W9).

PST3: *Even though I used the similarity task in my initial lesson plan, I realized that I could not show the relationship between quantities. I solved the task by pointing to 'within-between' relationships directly. For instance, the figure implies that for each unit (10 cm-unit) that increases on the long side, the short side increases by 6 cm unit. This student noticed that any unit difference in quantities is always related to the numerical difference in other quantities. As we discussed, this difference should be multiplicative. The student was able to coordinate the quantities simultaneously by applying the same scale factor and preserving the constant ratio* (Post interview, W8-W9).

Facilitator *You said the student stated "It should be enlarged with the same ratio, otherwise the faces on the photo will be distorted" and some of you claimed that this student was able to make sense of multiplicative relationship. How did you understand it?*

PST2 *In her response, she coordinated the quantities and controlled them simultaneously (within and between shapes), and she was able to explain her reasoning visually and verbally. For instance, she used hand gestures to show the all-directional growth of the original photograph by applying the same scale factor and the distortion if she enlarged it by the different scale factor. Besides, I realized that she used her hands to show vertical and horizontal relationships. She realized that the sides covary together.*
(PST2 used her hands to show how the student used hand gesture to show the distortion of the photograph)

PST6 *For instance, she stated that their face will grow vertically (non-coordination of vertical scale factors) if the missing side is longer than 18 cm.*

Facilitator *Why did she say like that?*

PST4 *Because she knows that the ratio should be same (the constant ratio).*

PST1 *May I draw a graph on the board to show this change more?*
(While she was drawing a graph to show the rate of change, PST2 started to talk again).

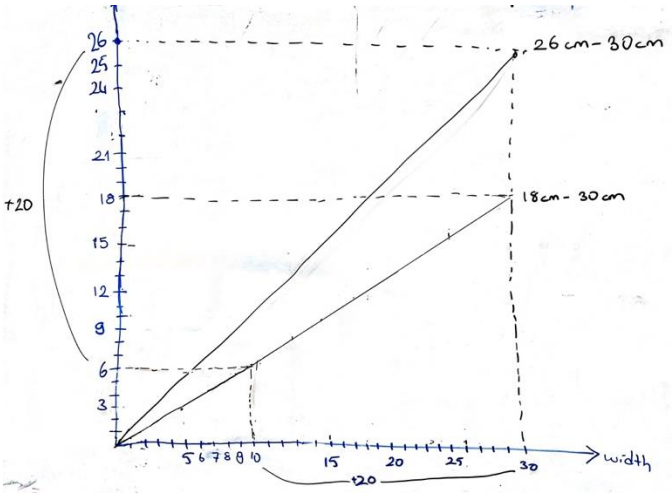


Figure 4.9 PST1's graph shows ratios of sides in similar rectangles

PST2 The short side length change corresponds to the long side length. Therefore, to preserve the similarity, we have to change 3 cm for the short side length corresponding to the change of 5 cm for the long side length. When we make a 4 cm change in the short side length and a 5 cm change in the long side length, the faces in the photo will be distorted (the rate of change). In other words, *the slope* will change. When we look at the second photograph (26cm-30 cm), it became more square compared to the original one (makes hand gestures to illustrate the distortion).

PST1 Yeah, you are right. The second photograph looks more square than the original one, and the original one looks more rectangular. I would not think before that the student could intuitively reason about the distortion of the shapes when the original photograph was enlarged additively. Adding 20 to 6 increased the height of the original photograph by 10/3, but adding 20 cm to 10 cm increased it by 2. When we look at the graph, the steepness of the lines is not the same. Therefore, I thought this student built an understanding of rate of change and scale factor.

In this part, the prospective teachers focused on providing evidence for their claims, "the student understood within and between multiplicative relationships". PST2 stated that the student coordinated the quantities and simultaneously operated the same scale factor on all sides of the rectangle. Moreover, if the constant ratio is not preserved, the figures will be distorted vertically or horizontally. She made some hand gestures (i.e., "she used her hands to show vertical and horizontal relationships. She recognized that

the sides covary together) to provide evidence that the student used her hands to express her reasoning about the distortion. She noticed that she made those hand gestures to show the vertical and horizontal relationship and enlarge the original photograph horizontally and vertically. Then PST6 supported PST2's explanation by providing the student's explanation about vertical distortion when different scale factors were used. According to the student's understanding, the facilitator directed the discussion to the reason for distortion. PST4 claimed that the student knew the ratio would remain the same without providing specific evidence of the student's understanding. PST1 preferred showing the difference between additive and multiplicative approaches on the graph.

While drawing, PST2 made some inferences by using new terminology, 'slope', that was not discussed before. She concentrated on the rate of change where the short side length and long side length covaried. PST2 stated how much the short side increases when the long side increases by ten or vice versa. For instance, for 10 cm units increase in the long side length, the short side length increases by 6 cm units. In PST1's graph, the x-apsis represents the vertical side (height), and the y-axis represents the long side (width). In this situation, the slope is the rate of change of the vertical side relative to the horizontal side (see Figure). Neither PST2 nor PST1 identified the slope formula directly. They concentrated on constructing ratios as a foundation for understanding the slope. Interestingly, without guiding the group, the discussion evolved from interpreting the student's proportional understanding as reflected in her response to their instructional suggestions based on the student's understanding in MCV4. After the group discussion, PST1's response to the revised lesson plan and during the interview highlights her attending and interpreting of the student's understanding:

> PST1: *I had never thought that I would support the student's reasoning about how the shape of enlarged rectangles would change. I mean, the shape becomes "more square" (looks like square) when the sides increase additively. I realized that we would construct the rate of change without measuring the side lengths for the formula and then forming the ratio directly. Even though the student did not verbalize 'the slope', she could generate similar rectangles: 6cm-10 cm and 18 cm-30 cm. I noticed that she could consider the quantities as two different*

> *numbers (e.g., 10cm-width and 6cm-height) because this student stated that the quantities covary together and the ratio between the sides was the same* (Revised lesson plan, W5-W6).
>
> *Interestingly, the student developed the rate of change by mentally comparing the widths and heights of three rectangles to check their similarity instead of applying the memorized formula. I wonder how she constructs the relationship of change between the height and the width, especially with a height of 1 cm and with a width of 1 cm (e.g., 10 cm for every 6 cm., which is equivalent to 5/3 cm each 1 cm short side length or 3/5 cm each 1 cm long side length). I realized that I was questioning the student's response to increase and shape my knowledge about the student's understanding through the video clips. In discussions, sometimes, we did not talk about the student's understanding of the video. However, this kind of discussion helped me make additional sense of Nazli's understanding (Post interview, W8-W9).*

PST1 interpreted the student's understanding in several points. First, she explained the details of her strategy and how those details reflected what Nazli did understand and did not understand in similarity context. PST1 drew a graph (see Figure 18) to explain why additive reasoning incorrectly identified missing values. She stated that the original photograph seemed more square when the additive difference was added to find the missing one, and she had never thought about it before. She claimed that the student was able to recognize the rate of change (the slope) between the sides of each rectangle (i.e., 3/5) and invariant relationships (scale factor) between corresponding sides of the rectangle (i.e., 3). Second, PST1 voiced some hesitations about her understanding. For instance, she wondered how she thought the relationship of change between the width and height, specifically the thought that for every increase of 1 cm in width (the long side), there will be a corresponding increase of 3/5 cm in height (the short side). The responses that demonstrated *robust evidence* of attention and interpretations of the student's understanding concentrated on making sense of strategy details in various ways.

4.3.2.2. Deciding How to Respond on the Basis of Student 4's Understanding

Table 4.24 shows that the PSTs provided detailed instructional actions on how to respond to Hilal's understanding in MCV4 before video analysis. 5 out of 6 prospective teachers proposed case-specific-detailed instructional actions. Only PST4 and PST6 provided case-specific-undetailed responses (13%) based on the student's understanding in the video clip.

In this week's group discussion, the PSTs focused on alternative instructional actions, including seeking more information on the student's understanding and extending the student's understanding of invariance and covariance relationships. Even though the student in MCV4 did solve the task correctly, PSTs did not evaluate the solution. They just labelled it as correct. They described and analyzed the evidence to show the student's proportional understanding that was embedded in the evidence. Besides, they connected the evidence with specific instructional actions to support the student's progress in proportional reasoning.

Table 4.24 Next instructional actions after MCV4 analysis

PSTs	Next instructional actions			Total frequency
	General (Lack of evidence)	Case-specific-undetailed (Limited evidence)	Case-specific-detailed (Robust evidence)	
PST1	0	0	3	3
PST2	0	0	4	4
PST3	0	0	1	1
PST4	0	1	0	1
PST5	0	0	4	4
PST6	0	1	1	2
Total	0	2 (13)	13 (87)	15 (100)

In group discussions and revised lesson plans, the PSTs provided more extended, detailed, and case-specific instructional actions on the basis of the student's understanding in the video clip rather than other instructional actions in the previous weeks. According to the Table 31, prospective teachers proposed 15 instructional actions after individual video analysis. 87% were case-specific and detailed, reflecting more tiered and systematic action steps on the basis of the student's understanding.

The discussion excerpts and the examples of instructional actions that the PSTs provided in revised lesson plans and post-interviews are given together below in order to show the PSTs' comments related to the proposed instructional actions:

PST5 *Hilal readily accepted that multiplicative scalar applied to the side lengths of similar rectangles. However, I would like to encourage her to think in detail. I would like to ask "you said that when we multiply the long side length by 3, we also need to multiply the short one by 3. However, the difference (additive) between the sides is different (i.e., 30-10=20 and 18-6=12). Why are they different?".*

Facilitator *Why did you ask this question? But in the second response, their difference (additively) is the same as well. Right?*

PST5 *Yeah, the sides were changing (covariation) together. This student appeared to construct the meaning of the variant relationship and the slope as the multiplicative comparison of width and height. Therefore, I would like to ask that follow-up question to help Nazli consider the covariational relationship between quantities in two responses based on additive and multiplicative reasoning strategies.*

PST2 *She already easily detected the distortion when she examined the incorrect response. She stated, "we should preserve the ratio of 6/10 to avoid distortion. Otherwise, the photograph would be distorted". She explained her reasoning within-between ratio relationships.*

PST6 *I would use a rectangle (3cmx5cm), and I would ask her to increase the side lengths by 3 cm. Then, I would draw her attention to the changing appearance based on which one becomes more rectangular or they were similar.*

PST3 *Yeah, but I would use squares to understand her judgment. For instance, I would say that "we have a square that is 5cm by 5 cm, and we can create a new square that is 10cm by 10cm. The difference (additive) of sides is 5 cm, and the ratio of (corresponding) lengths was two times taller. Then, I would use rectangles, as you said. Therefore, I would create (cognitive) conflict to see how the student would perceive additive and multiplicative reasoning for this problem.*

However, I am still thinking that the student could explain this relationship very well because she said, "in order to avoid distortion, we should preserve the ratio 6/10".

In this group discussion, the prospective teachers focused on the student's additive and multiplicative reasoning in similarity tasks and the student's visual perception of the shape. PST5 questioned the student's understanding of additive and multiplicative relationships for similarity context in the group discussion. Previously, she explained her hesitation about the student's understanding of the additive and multiplicative differences in MCV4. Therefore, she wanted to understand how the student constructed the relationship between additive and multiplicative reasoning, so she presented her with an additional correspondence task. PST2 provided evidence as a response to the PST5's hesitation. She claimed that the student constructed the invariant and variant relationship between the side lengths of rectangles. Her explanation contains case-specific-detailed evidence in the form of transcripts records of the student's words.

Then, PST6 offered another instructional action based on PST5's explanation. She implemented a task that included a specific rectangle and increased the side lengths by 3 cm. Then, she asked the student to explain whether they were similar or if one of them was more rectangular. She would like to monitor the student's understanding of additive and multiplicative reasoning in similarity context. PST3 offered an additional instructional sequence to PST6's response to elicit the student's understanding. However, she stated that the student constructed additive and multiplicative relationships in this context. She provided evidence of the student's explanation of what she said exactly; "in order to avoid distortion, we should preserve the ratio 6/10". PST3 proposed an instructional action based on the uncertainties demonstrated by PST5. It can be seen that she made deliberate modifications (i.e., changing the sequence) to PST6's task. They provided case-specific-detailed instructional actions in accordance with the student's understanding.

After Session 4, PST2 focused on visual recognition of the similarity of shapes in her post-interview and revised lesson plan. In the initial lesson plan, she did not pose any

instructional actions related to similarity. She mixed two colors (blue and yellow) to obtain green. Then, she provided a table to fill in regarding the given amounts of colors to have the same green. She proposed these actions to show the relationships between quantities, but the actions were general and undetailed. For instance, she stated, "What do you think about these values?" without explaining the specific purpose of this question. On the other hand, after this session PST2 revised her lesson plan and added similarity and distortion tasks. After the group discussion, she explained that this task would lead the student to recognize the relationship between similarity and proportionality. The next lines were drawn parallel to the other side for each side length. This action would facilitate the student's ability to construct a connection between the measure of multiplication and the use of this multiplication to find the area. PST2's interpretation is seen below:

> PST2: *This week, we discussed the concept of slope in similarity and distortion. I could not think of relationships for similarity problems in my first lesson plan. For instance, the slope concept is introduced in the 8th grade, and the relationship with proportional reasoning is not given. So, I did not think of slope as a part of proportional reasoning. This week's discussion led me to shift my reasoning. For instance, I realized that similarity and distortion lead to developing the student's proportional reasoning skills, specifically for covariant and invariant relationships. The student's hand gesture for enlarging (to show distortion) indicated that the student used informal proportional language (the student used hand gesture in MCV4), and I could not imagine it before. Therefore, I changed my lesson plan and added a similarity task. Besides, understanding the slope within similarity context might allow the student to recognize the connection between multiplicative reasoning and proportional reasoning. As we discussed, I gave the non-divisible numbers, so the student might apply an additive strategy to finding the missing side. Based on the discussion, I tried to visualize the rate of change by using unit squares in my lesson plan. Because this student had a sense of covariation and invariation, I would help her concentrate on the rate of change on the graph without giving further details of the slope* (Revised lesson plan, W5-W6).

PST2's comments involved more details about why and how she decided to use the instructional action in the revised lesson plan to develop the student's understanding of MCV4. She concentrated on the relationship between similarity-distortion and proportionality and how the lengths would covary together by preserving the constant ratio. She used graphs to develop the student's understanding of the rate of change and scale factor. Her instructional actions explicitly considered the student's strategy and reflected her expertise in interpreting the student's understanding in the video clip. This group discussion led her to revise her lesson plan in a different manner. For instance, some prospective teachers had hesitations about the student's understanding in MCV4 about invariant and covariant relationships. Therefore, she proposed instructional actions to reveal the student's thinking to understand how this student made sense of covariance and invariance. Overall, PST2's instructional actions were categorized as *case-specific-detailed.*

Another discussion excerpt is given below. The prospective teachers concentrated on the relationships between the ratio of sides and ratio of areas and the ratio table's role. Then, the illustrative examples of lesson plans (initial and revised) with the PSTs' comments were presented to show the progress of prospective teachers' noticing skills (see Table 4.25).

PST5 *In order to extend the student's understanding, I would place the small rectangle (original photograph) in the bigger one and ask how many small rectangles we need to get in the enlarged rectangle. Let me draw on the board. We could use grid paper, so it would be easier.*

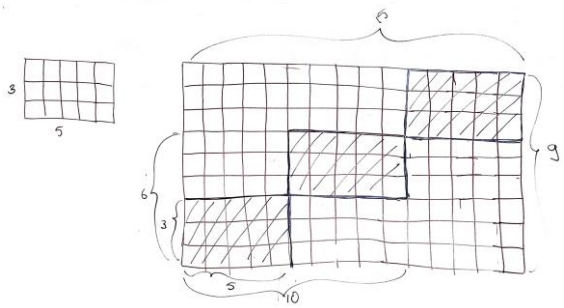


Figure 4.10 Iteration of rectangle on grid paper

*For instance, I would give two rectangles like the one seen here. The small rectangle sides are 3cm*5cm. The long side of the enlarged rectangle is 15 cm, so how could we find the short side (or vice versa)? Probably, the student will be able to find it easily. Then, I would ask her to fit the small rectangle in the enlarged rectangle and iterate it until she filled up the bigger one. She could recognize that she could fit two more pieces (the long side length) in there. Then, I would discuss how many pieces (the short side length) could fit. She would be able to notice that the enlarged rectangle contained nine original rectangles. Also, she could recognize that each side length of the enlarged rectangle was three times longer than the original rectangle. I mean, she would notice the relationship between the ratio of sides and the ratio of areas.*

PST3 *Yeah, we directly focused on the areas. I would also help the student concentrate on the relationship between the areas directly. I would use squares again. For instance, I would say, "we have a square that is 5 cm by 5 cm, and we can create a new square that is 10 cm by 10 cm". The ratio between the side lengths is 1:2, and the difference (additive) is 5 cm. One of my goals is to encourage the student to think about additive and multiplicative reasoning in a similar context as we discussed before. Then, I would ask her, as you said, how many 5cmx5cm squares we can place in 10cm x10cm. The answer is 4. She could discover the relationship between the areas by using them. After placing the rectangles or squares inside the enlarged one, I would encourage the student to draw a table (ratio table) to show the relationship between side lengths. I mean the ratio table.*

Facilitator *Ratio table? Who can explain how you would use ratio table for this task, then?*

PST3 *Ok, I will explain better if I draw it on the board. I would like to make the student notice the relationship between the quantities (drawing the table), so she would explore the relationship by herself. For instance, I would write the short and long side lengths. Then, I would ask her, "I would like to increase the long side length by two times,*

so it would be… (waiting for the student's answer) 20 cm. What about the short side length?". The student probably would say, "it should be two times more, as well." She would keep going, and then she would be able to find the missing length as 18 cm. Here, "2s" or "3s" is the scale factor, and 6/10 is *the constant ratio.*

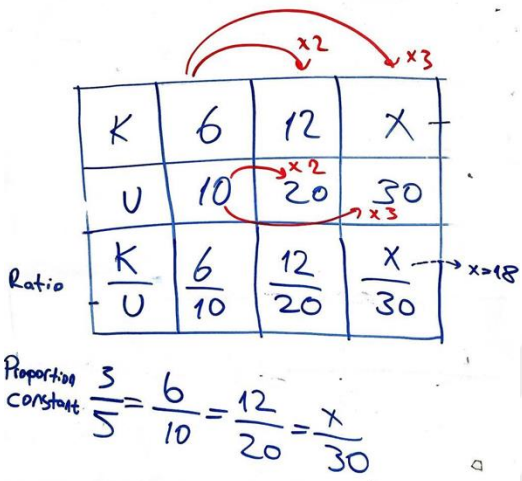


Figure 4.11 Scalar relationship between sides by PST3

PST5 focused on extending the student's understanding by using different strategies and explicitly explaining her rationale about the student's understanding in the video clip. She stated that she would use grid paper to show how many small rectangles would be placed into the bigger one. She would ask the student to iterate until she filled up the bigger one. PST5's representations above correspond to a method of determining the area of forming a grid in the rectangle. PST2 partitioned one of the sides into unit lengths (1 cm). Therefore, she would notice the relationship between the ratio of sides and areas. PST5 offered questions to probe further based on what the student understood and did not understand. PST3 supported PST5's suggestion by providing two different squares instead of rectangles. She said she would encourage the student to think about additive and multiplicative strategies in similar contexts. She explained her rationale based on the student's understanding and PST5's previous comments. Additionally, she said she would use a ratio table (see Figure 4.11) to extend the student's understanding by explicitly stressing the constant ratio and scale factor. PST5's comments and illustrative examples of lesson plans were presented in the Table 4.25 regarding the group discussion.

Table 4.25 Examples of instructional actions presented in lesson plans

Example parts of …	
Initial lesson plan (before video analysis)	Revised lesson plan (after video analysis, W3-W4)

There is a family living in a small town and they have a huge garden with a lot of trees. In a sunny day, they decide to have a barbecue party in the afternoon. They want to set their lunch table in a shady place so that the sunshine does not bother them. They have 6m long tree in their garden. They think that the shadow of this tree might be long enough to cover the sunshine and they decide to set the table under this tree. They measure and record that the shadow of the tree is 12m long at 3.00pm and the shadow of another 8m long tree is 16m. However, they need more shadier place since they are crowded family. Therefore, they decided to set the table under the biggest tree which is16m long in their garden.

1. Considering the above knowledge, what will be the shadow of the 16m long tree? (We are assuming that it is 3.00pm)

Expected answer: We can see that **the ratio** of length to shadow for the first tree is 6m/12m=2. If we look at the second tree, **the ratio** of the length of the shadow for it is also 8m/16m=2.
Since time does not change, the ratio of the length to shadow should not be change. (We are assuming that time is still 3.00pm, the sun stays its place. Therefore, the sunshine is the same. It is because of the incidence angle of the sunshine is the same.) As a result, if the tree is 16m long, its shadow should be 32m since the ratio of its length to shadow is 2.

2. What do you think about the length of the trees and their shadows?

Expected answer: The length of the tree is half of the length of its shadow. In other words, the length of the shadow is twice the length of the tree.

3. If there is, what kind of relationship they have?

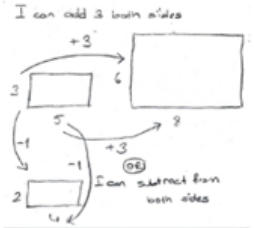
Expected answer: There is a multiplicative relationship between the length of the trees and their shadows.

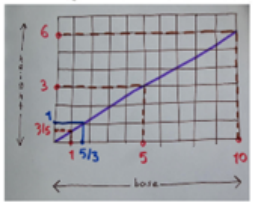
There is a rectangular picture whose height is 3 cm long and base is 5 cm long. Can you create other rectangles without distorting the image?
The ratio of the two sides should remain the same so that the image is not distorted. Let's make a table to show different rectangles whose corresponding sides are proportional to the given rectangle. These rectangles might have sides of 6-10, 9-15 etc.

	Color	Height	Base	Ratio of the height to base (purple line)
Rectangle 1	Red	3 cm	5 cm	3/5
Rectangle 2	Orange	6 cm	10 cm	6/10
Rectangle 3	Dark blue	9 cm	15 cm	9/15
Rectangle 4	Green	12 cm	20 cm	12/20

When we look at the ratio of the sides, they are all equal; **3/5=6/10=9/15=12/20**

Potential misconception: While scaling up the rectangle I can add 2 to each side. I can also subtract any number that I want but, I should do the same thing to both sides again. For instance, I can subtract 1 from each side.





Remediation of the misconception
I would give the student a few rectangles whose sides are 3-5, 6-10, 3-8, 4-8 and want the student to find the similar ones. After that, I would ask them why s/he choose the way s/he did.
Can you find the ratio of the height to base for all rectangles?
Is there a relationship between the ratios of the sides and similarity of rectangles?
I would ask his/her to draw their diagonals and compare with each other. By doing that, s/he can observe that the similar ones have diagonals that are on the same extension/line. S/he can observe this by putting one rectangle's left corner on the right corner of the previous rectangle. (as in the answer3)

PST5: *This week we concentrated on how the student reasoned about similarity. The (group) discussion led to different reasonings that the student would use. Additive strategies were one of them. The student might say that "I added the same amount to both sides". However, this strategy would be incorrect in the similarity context because it would not create similar figures. In this regard, I would like to give different rectangles to find similar shapes. When she categorized the similar rectangles, I would ask her reasoning to help the student think about the relationship of the change between the long and short side lengths. I would ask her to fill in the (ratio) tables. This activity aimed to recognize the relationship between the corresponding sides while obtaining similar figures. Therefore, I would ask the student to stack the look-alike rectangles (see the example of the revised lesson plan below) and draw a line across the diagonals. This activity would help the student make sense of the relationships between ratio tables and graphs as multiple representations. The student would recognize that opposite corners of two rectangles (3cm-5 cm and 6cm-10cm) were lined up because they were similar rectangles. Therefore, the student would construct multiple views of the rate of change and covariation. Even though it was not called a "slope," she would notice the change of one side length relative to the change of the long side length, while two quantities vary together. She would recognize that for every increase of 1 cm in long length, there would be a corresponding increase of 3/5 cm in short length, or for every increase of 1 cm in short length, there would be a corresponding increase of 5/3 cm in long* sides (Revised lesson plan, post-interview-W7-W8).

After the discussion, PST5 made changes in her lesson plan and included additional tasks and new representations. Her responses include a rationale for extending the student's understanding. Her response provided a clear connection between what the student seemed to understand and what she did not understand. For instance, with robust evidence, PST5 explained her hesitation about the student's understanding of additive and multiplicative strategies in the context of the similarity task in MCV4. First, PST5 aimed to clarify the student's understanding of the task's additive and multiplicative strategies. Then, she proposed more tasks to elicit and extend the

student's understanding of covariant and invariant relationships using different representations (i.e., ratio tables and graphs). It can be seen that she might revise her lesson plan in ways that support and extend the student's learning.

Instructional actions in the individual reflections that the PSTs provided were mostly case-specific and undetailed (see Table 4.26). However, as seen in the Table 4.26, after video analysis, the PSTs started providing case-specific and detailed instructional actions during group discussions and in their weekly revised lesson plans (W7). The table below shows that most PSTs improved their responding skill, which helped them to propose more qualified and detailed responses considering the student's existing and further understandings after MCV4 analysis. PST5's score seems to be the same, but her responses include additional tasks, new representations, and in-depth and focused questions or instructions based on the student's mathematical understanding in the video clip.

Table 4.26 Comparison of PSTs' average points of comments on responding for MCV4

	Prospective teachers						Average of total points
	PST1	PST2	PST3	PST4	PST5	PST6	
Before video analysis	1	1.5	1.3	1	2	1	1.3
After video analysis	2	2	2	1	2	1.5	1.75

Assigned the certain numbers to each level;
*0 for general comments
*1 for case-specific-undetailed comments
*2 for case-specific-detailed comments
*including initial-revised lesson plans, pre-post interviews and group discussions

Part-Part and Part-Whole Relationships

As mentioned earlier, prospective teachers' professional noticing of students' thinking in the context of proportional relationships was examined in two parts: Invariance-covariance and part-part-whole relationships. This section presents the nature of the prospective teachers' knowledge of and progress in noticing students' thinking about *part-part and part-whole relationships* in the Mixture problem before video based group discussion. Asya's (Student 5) solution in micro-case video clip (MCV5) is given below.

Table 4.27 Asya's (Student 5) strategy of the Mixture Problem (MCV5)

Esra and Sinan will taste of different mixtures (pitchers) given above. Which lemonade pitcher has stronger lemonade flavour?

Mix A: 2 glasses of lemon juice and 3 glasses of water

Mix B: 1 glass of lemon juice and 4 glasses of water

Mix C: 4 glasses of lemon juice and 8 glasses of water

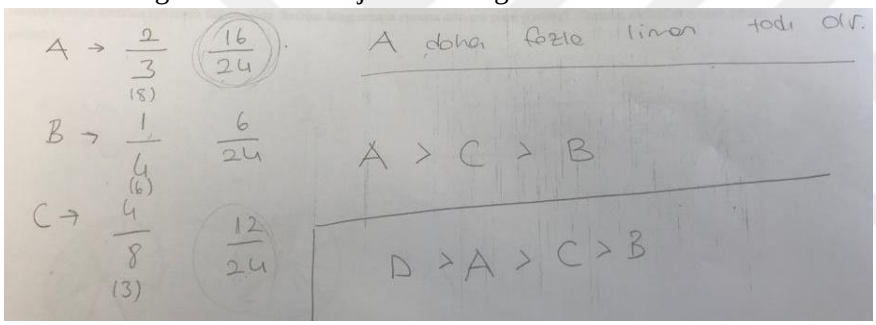


Table 4.28 shows the progress of prospective teachers' responses about attending to the student's strategy and interpreting the student's understanding in MCV5 before and after video analysis. Below is the description of the development of the PSTs' ability to notice in the context of part-part and part-whole relationships. All prospective teachers' attending and interpreting scores were the same, and they provided robust evidence for both skills. Moreover, their response nature changed. For instance, this progress involved attending to some pertinent features and disattending to others. All prospective teachers proposed multiple valid interpretations of the student's understanding, including "what the student understood" and "what the student did not understand." In this vein, the group discussions became more interpretive to make sense of what attended to the student's strategies. Besides, most PSTs improved their responding skills, which helped them to propose more qualified and detailed responses

for furthering Student 5's understanding, evidenced in the micro-case video clip (MCV5). Responses included additional tasks and various new representations to connect clearly what this student seemed to understand and did not understand. The details are given in the following sections.

As seen below, before the MCV5 analysis, all participants attended to the student's strategy, and interpreted the student's mathematical understanding at robust level. While 2 out of 6 prospective teachers provided case-specific and undetailed instructional suggestions, 4 out of 6 provided case-specific and detailed responses. After video analysis, 4 out of 6proposed case-specific-detailed instructional actions in response to Student 5's understanding. On the other hand, two prospective teachers' responses (PST4 and PST6) included case-specific-undetailed instructional suggestions. Their response included limited evidence of deciding how to respond based on the student's understanding. Representative examples of prospective teachers' responses were given in the following parts.

Table 4.28 PSTs' noticing level of evidence of attending and interpreting before and after Mixture Problem video-based intervention (MCV5)

Prospective teachers		PST1	PST2	PST3	PST4	PST5	PST6
Attending	Before video analysis	Robust	Robust	Robust	Robust	Robust	Robust
	After video analysis	Robust	Robust	Robust	Robust	Robust	Robust
Interpreting	Before video analysis	Robust	Robust	Robust	Robust	Robust	Robust
	After video analysis	Robust	Robust	Robust	Robust	Robust	Robust
Responding	Before video analysis	CSD	CSD	CSD	CSU	CSD	CSU
	After video analysis	CSD	CSD	CSD	CSU	CSD	CSU

*CSU= Case-specific-undetailed

*CSD= Case-specific-detailed

4.3.3. Prospective Teachers' Professional Noticing Before MCV5 Analysis

In the following, prospective teachers' development of professional noticing of Asya's thinking (Student 5) in the context of part-part-whole relationships before video analysis.

4.3.3.1. Attending to the Mathematical Elements of Student 5's Strategies

The following response of PST3 was taken from the video analysis reflection paper, which provided *robust evidence* of attention to the mathematical details based on the student's strategies in MCV5:

> PST3: *The student wrote the ratio of lemon and water:* $\frac{Lemon(the\ amount)}{Water\ (the\ amount)}$ *After finding the ratio between lemon juice and water for each mixture, she used equivalent fractions. She stated that there was more lemon juice in Mix A than in B and C. Therefore, Mix A has a stronger lemon flavour. When she examined the incorrect answer, she stated that "we should compare the amount of lemon juice and water in each mixture and water was not subtracted from lemon juice. This difference (between quantities) gives just a difference"* (Reflection paper, W7).

PST3 focused on the noteworthy mathematical elements by describing the student's response, and she provided *robust evidence* based on the student's solution in MCV5. She stated that the student looked at the ratio of lemon and water, (i.e., *lemon juice/water).* In order to make the comparison, the student used equivalent fractions (*equivalence ratio class*). Then, she pointed out that Mix A contains more lemon juice within those equivalence fractions. Hence, the taste was stronger than Mix B and Mix C. Also, she provided the student's specific explanations as evidence of her statements. For example, she stated that when the incorrect solution was shown, the student said, "we should compare the amount of lemon juice and water in each mixture, and water was not subtracted from lemon juice. This difference (between quantities) gives just a

difference *(additive comparison)"*. This response was categorized as *robust evidence of attending* to the student's strategy.

Another illustrative example of *robust evidence* of attending is presented below:

> PST2: *First, the student found the ratio of lemon juice to water. It was 2/3, 1/4, and 4/8, respectively in here. Then, she had the common denominator, and she respectively found 16/24, 6/24, and 12/24. When she compared these rations, she stated that Mix A has a stronger lemon flavor because the ratio of 16/24 is greater than others (i.e., comparing the fractions). When another solution which included addition was shown to the student, she realized that the solution was incorrect because she thought that instead of taking difference (via subtraction), she should consider ratio. While explaining the reason for this, she should use proportion and compare the lemon juice to water multiplicatively* (Reflection paper-W7).

PST2 provided robust evidence for the mathematical and noteworthy details of the student's strategy in the video clip. She described how the student used equivalent ratios to make the comparison for each mixture. In this regard, she demonstrated how the student created equivalent fractions and how she applied them to the mixture context. Besides, she also identified the reasons why the student explained the second response as incorrect (e.g., "While explaining the reason for this, she should use proportion and compare the lemon juice to water multiplicatively"), which provides evidence that PST2 made sense of the strategy.

4.3.3.2. Interpreting Student 5 's Understanding

The following response samples were taken from the video analysis reflection paper of PST1 that provided *robust evidence* of the understanding of Asya in MCV5:

> PST1: *The student knows that she needs to apply the ratio between lemon juice and water; hence she did not think additively. I think she recognized that she required to apply multiplicative comparisons between quantities. It means that she constructed the distinctions between multiplicative and additive comparisons. While solving (the task), she* used part-part ratios. *I think she understood the relationships that involve multiplicative reasoning and in which two quantities covary together and the relationship is 2/3, 4/8, and 1/4. However, I could not understand whether she conceptualized the relationship between the amount of lemon juice for one glass of water. There was not any information about it* (Reflection paper, W7).

PST1 made inferences about the student's understanding based on the evidence in MCV5, making statements such as, "I think she understands the relationships that involve multiplicative reasoning and in which two quantities covary together and the relationship is 2/3, 4/8, and 1/4". She focused on the part-part relationship in the video. When discussing Asya's understanding of *part-part relations*, PST1 recognized that this student conceptualizes the covariational approach, how lemon juice and water vary in relation to each other. On the other hand, PST1 explained her hesitation about how she conceptualized the meaning of ratios; the amount of lemon juice per glass of water. Overall, this response demonstrating robust evidence of interpretation of the student's understanding above indicates that PST1 made sense of the student's understanding based on the evidence captured in the video.

The following response is another sample that provided *robust evidence* of interpretation of Asya's understanding in MCV5. PST4 interpreted the student's understanding by providing evidence. First, he recognized that Asya reasoned *part-part ratios within the mixtures* (i.e., the ratios of the amount of lemon juice and water) by making connection with the unit rate (i.e., *the amount of lemon juice per glass of water)*. Then, he expressed how this student compared the denominators (e.g., 24 glasses of water). He highlighted the meaning of the part-part comparisons and used evidence from the student's practice to draw inferences about her understanding (e.g., "She compared the amount of lemon juice to 24 glasses of water").

PST4: *The student first found the ratio of lemon juice to water (lemon juice/water), a part-part relationship. She found* the amount of lemon juice per glass of water*, and to compare them, and she used the common denominator. Then, the student compared the amount of lemon juice corresponding to 24 glasses of water and chose the correct one. As a second solution, she interpreted using the* benchmark-*the more than half or less than half. Then, the student was able to compare those fractions correctly. On the other hand, I am not sure what she thinks about the part-whole comparison* (Reflection paper, W7).

Second, this participant also recognized that this student used benchmarks to make proportional judgments (e.g., greater than half and smaller than half) in order to solve the task. PST4 referred to the event as evidence (i.e., using half as a benchmark) captured in the video to support his claims about the student's understanding. Overall, this response demonstrating robust evidence of interpretation of the student's understanding above indicates that PST4 made sense of the student's understanding.

4.3.3.3. Deciding How to Respond on the Basis of Student 5's Understanding

Table 4.29 shows that the PSTs provided detailed instructional actions on how to respond to Asya's understanding in MCV5 before video analysis. Most prospective teachers proposed case-specific-detailed instructional actions. PST4 and PST6 provided case-specific-undetailed responses based on the Student 5's understanding.

Table 4.29 Instructional actions before MCV5 analysis

PSTs	Next instructional actions			Total frequency (%)
	General (Lack of evidence)	Case-specific-undetailed (Limited evidence)	Case-specific-detailed (Robust evidence)	
PST1	0	1	1	2
PST2	0	0	2	2
PST3	0	0	2	2
PST4	0	1	1	2
PST5	0	0	2	2
PST6	0	1	1	2
Total (%)	0	3 (25)	9 (75)	12 (100)

PST6 provided case-specific-undetailed instructional actions on the basis of the student's understanding in MCV5 as follows:

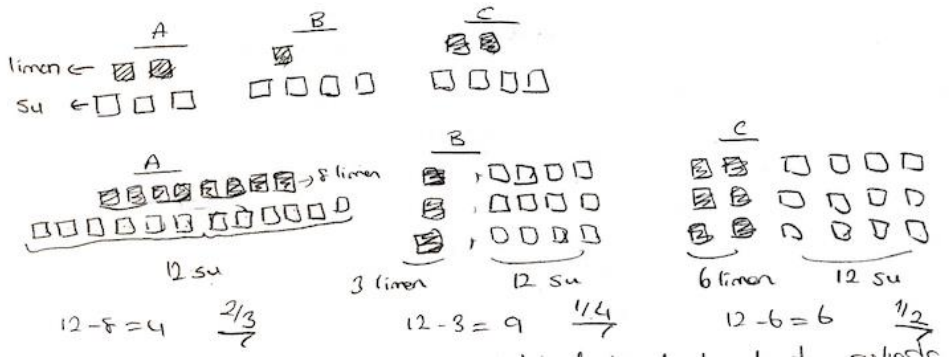


Figure 4.12 PST6's responding on the basis of Asya's understanding

> PST6: *I would give unit cubes in different colours and ask her to show the amount of water and lemon juice by using these cubes in order to make lemonade. After checking the ratios, I also show that she could make an interpretation by looking at the (additive) difference. In the video, she could not recognize that she could use the difference (subtracting lemon juice from water) to compare the quantities additively. After finding the common denominator (25 glasses of water), she could apply additive reasoning. Because the difference between water and lemon juice such as 4 glasses of water more can give stronger lemon flavor, the mixture which has smaller difference is stronger. The reason might be focusing on the proportional methods and ignoring the additive methods.*
>
> *I would ask questions to focus on the amount of water per (one glass) lemon juice or the amount of lemon juice per (one glass) water. Thus, the student would make reasoning with the unit rate or with the within-between strategy. Besides, this student used part-part ratios, so I would also ask her to explore the part-whole relationships* (Revised lesson plan, W7).

PST6 explained a problem with her rationale that accounted for the student's understanding of the MCV4 and anticipated the next instructional actions but did it in general and in less detail. For instance, she stated she would provide physical

manipulatives (unit cubes), so the student could simulate creating lemonade recipes. She built upon introducing a different method (i.e., using unit cubes as a representation of lemon juice and water) to explain ratio. However, she did not explain in detail why and how these manipulatives were used for this task.

Besides, she anticipated the next instructional action to make a comparison between three mixtures; (i.e., unit rate). For this student, she suggested that she would provide questions to invite the use of unit rate strategy: glasses of water per glass of lemon juice or glasses of lemon juice per glass of water. Although this response shows evidence of PST6's consideration of the student's understanding, her response did not include details on how she would show unit rate strategy to create and build up comparisons. Nonetheless, despite the limited details and lack of specific information about the instructional actions above, PST6's response involves building on the student's understanding and anticipating the next strategy for the proposed task. In this regard, her response was grouped as *case-specific-undetailed* (limited evidence).

The following is a sample response that provided *case-specific-detailed* instructional actions on the basis of the student's understanding in the video clip:

> PST5: *If I were the teacher of this student, I would ask questions to the student to understand how she made sense of the ratio (e.g., ⅔). For instance, she added three glasses of water to the two glasses of lemon juice. However, I do not know whether she made sense of the other meanings of ratio. For example, there is ⅔ glass of lemon juice per one glass of water or two-thirds of 3 glasses of water is two glasses of lemon juice. Because I could not catch from the video how she made sense of ratios represented as fractions, I would ask her by showing the fractions she wrote, "Can you explain the relationship between lemon juice and water?". This question might help me understand how she constructed the relative comparisons between lemon juice and water.*
>
> *As I watched the video, she wrote the fractions as part-part relationships between lemon juice and water and did not use a part-whole comparison. So, I would ask*

questions to understand whether she constructed part-whole relationships. So, I would ask her to compare the lemon juice concentrate in each Mixture (e.g., ⅔). Therefore, she would be exposed to both part-part and part-whole comparisons. I would engage her to create the part-whole ratios, such as the amount of lemon juice (part) to the lemonade (whole). For instance, Mixture A is composed of 2 glasses of lemon juice to 3 glasses of water, and we can interpret the ratio of ⅔ as two-fifths of Mixture A. So, I would ask, "can you tell me the lemon concentration in each mixture? How do you interpret the ratios?" In this way, I would not only access additional information about this student's understanding, but also-if she expressed the part-whole relationship- I would give different and difficult numbers to extend her thinking. For example, 4 glasses of lemon juice-5 glasses of water and 6 glasses of lemon juice-7 glasses of water. These numbers are not divided, so she would not make proportional judgments such as 'less than half' and 'more than half.' Also, the difference between numbers is the same (i.e., 7-6=1 and 5-4=1). I would ask the student, "The difference of the numbers is the same. There is 1 less glass of lemon juice in each mixture, so is their taste the same, as well?". Therefore, I could notice whether she compared the quantities additively or not when there was no direct scalar relationship between quantities (Reflection paper, W7).

PST5 focused on interpreting ratio meaning, representing the part-part comparison (i.e., ⅔ ratio) considering the specific student's strategy in the video. She stated that she wondered about the student's interpretation of the relationship between lemon juice and water. This point is consistent with the student's strategy, who used part-part comparison (e.g., 2 glasses of lemon juice for 3 glasses of water). It gives the student a chance to extend her understanding of part-part relationships. Moreover, PST5 would engage the student to compare the lemon juice in each mixture to show the number of ways a comparison could be made. As seen, she suggested not only showing a specific strategy but also strategically providing the next instructional actions to explore the student's understanding in the video. This follow-up question provides evidence that PST5 posed the next instructional actions in detail based on the student's mathematical understanding. In addition, there is an openness to the student's understanding. The prospective teacher aimed to create opportunities to further extend the student's

understanding. Overall, this response is categorized as *case-specific-detailed* based on the student's understanding.

4.3.4. Prospective Teachers' Professional Noticing After MCV5 Analysis

The following section illustrates the prospective teachers' development of professional noticing of Asya's thinking (Student 5) in the context of part-part-whole relationships after video analysis.

4.3.4.1. Attending to Student 5's Strategy and Interpreting Student 5's Understanding

In this part, the progress in the prospective teachers' attending and interpretation are given together. As stated before, during the group discussion, while they were attending to the student's strategy, they simultaneously interpreted the student's mathematical understanding with evidence. The following discussion excerpt displays how the PSTs' developed in terms of attending to Asya's understanding with group discussion. As mentioned earlier, the facilitator initiated the group discussion after the PSTs wrote the video analysis reflection papers.

Facilitator	*What did you notice about the student's strategy in the video?*
PST4	*First, the student found the common denominator to compare the fractions; 24 glasses of water for each mixture. Then, she compared the amount of lemon juice corresponding to 24 glasses of water. She compared the fractions by finding a common denominator to decide the stronger one (i.e., more strong taste). But it seems that she just used procedure without building any connection between the terms.*
PST6	*I don't think so, and she was aware of the relationship as she considered the denominator and numerator. For example, she used benchmarks; ⅔ is close to 1, or 4/8 is half. She was good at numbers, and also, she was able to coordinate the quantities by creating equivalent fractions.*

PST4 *I don't mean that. It seems that the fraction number sense of the student was developed, but I am not sure whether she interpreted the part-part-whole relationship between the quantities. Because she just focused on part-part relationships. However, I am not sure whether she recognized the relationship between lemon juice and water, considering the ratio meaning. For example, 2/3 or 16/24 represents different amounts, such as ⅔ glass of lemon juice per one glass of water. Her language (ratio language) did not give me enough data.*

PST1 *I agree with you (as a reply to PST4). In the video, she said she could not look at the difference (i.e., 3-2=1 or 8-4=4) to compare the stronger one. However, she could not explain why 'the difference' was not applied in this context. That is why, I thought her approach was rule-oriented and did not explain part-part-whole relationships. Therefore, in the reflection paper, I wrote that I would ask about the meaning of those fractions (i.e., ratio meaning).*

PST6 *She stated, "we would look at the ratio of lemon juice and water." This sentence shows that she understood the ratio concept even though she did not state the part-part relationship explicitly.*

PST3 *Yeah, I captured that statement and it gives clues about how she considered the concept of ratio.*

PST1 *I don't think so. It was unclear whether the student constructed the part-part-whole relationships in the video. She did not identify what each number (term) in a ratio represents here. She created the common denominator to compare the fractions. However, I could not decide whether she applied this operation just as procedurally-I mean to decide the bigger one. She knows that she should find the common denominator- or she took the same amount of water to check the lemon concentration. I need further explanation from this student to show what each term in a ratio represents.*

PST2 *Besides, I am not sure whether she constructed a unit rate in this content. She chose the biggest one (i.e., 2/3) as having stronger taste by creating the equivalent classes. Suppose she wrote 3/2 (water/lemon juice) instead of 2/3 (lemon juice/water). In that case, I wonder whether she would choose the smallest one because I did not catch anything (any statement from the student in the video) about how she reinterpreted the situation about the specified unit.*

PST6 *Specified unit?*

PST2 *I mean based on the video, it seems that lemon juice/water is the specified unit.*

PST5 *I agree with PST2. As we discussed before, I would like to add one more point; covariation and invariant relationships. While creating equivalent ratios (i/e., 2/3 equals 16/24), I could not understand whether she made sense of covariation and invariance in relation to the given quantities. While varying the quantities together, the relationship-such as 3 glasses of lemon juice corresponds to 5 glasses of water- does not change. I did not catch any evidence related to the process in the video. Therefore, I cannot claim that she reasoned with the part-part-whole ratios and connected it with covariant and invariant relationships. I mean, I need more data.*

At the beginning of the group discussion, the prospective teachers focused on the strategies of the student (i.e., part-part-whole relationship) in the video. PST4 made sense of the noteworthy mathematical details, including a detailed description of the student's strategy. Then, he interpreted the student's understanding in specific ways. PST4 pointed out how these details reflected what the student did understand and did not understand. For instance, when discussing the student's understanding of part-part-whole relationships, he noted that he was unsure whether she conceptualized the part-part-whole relationship between the quantities in the given context. He recognized that the student compared the fractions by finding the common denominator without considering relational thinking. However, he expressed his hesitation saying that "I am not sure whether she interpreted the part-part-whole relationship between the quantities because she just focused on part-part relationships. However, I am unsure whether she recognized the relationship between lemon juice and water, considering the ratio meaning". This statement indicates that she recognized more evidence of whether the student made sense of part-part-whole relationships in a ratio using comparison.

On the other hand, PST6 explained her disagreement with PST4's statement by providing evidence from the video. She claimed that the student identified fractions for the part-part relationships in the ratio of 2/3 or 1/4. As evidence, she showed the student's specific strategy- using a benchmark as an estimate of reasonableness (i.e., more than half or less than half) as evidence of her certainty about relational thinking

and wrote the exact quotes of what the student said. Unlike PST6, PST1 stated that even though this student correctly labeled part-part relationships as ratios, the video did not provide enough evidence to claim this student constructed part-part-whole relationship. This statement indicates that she realized a need for more evidence to prove or disprove her interpretations.

PST5 drew the group's attention to covariant and invariant relationships. First, she agreed with PST2's comments, and then she pointed out that this student did not have a sense of the covariant and invariant relationships based on the video. She wondered whether she interpreted the ratio concerning the invariant and covariant relationships between quantities. PST5 not only interpreted the student's understandings in several ways but also recognized what significant and related understandings the student did not show in the video.

By referring to the benefit of the group discussion, PST6 and PST2 evaluated their progress providing *robust evidence* in noticing the student's mathematical understanding in the light of MCV5.

> PST6: *After the discussion, I realized that using a benchmark for finding the stronger taste will not be evidence of part-part-whole relationships, as PST1 stated. The student concentrated on the lemon juice/water relationship in the video and did not talk about lemon juice/ lemonade or water/ lemon juice (part-whole) relationships. It could be a symbolic representation of proportion to find a stronger taste. The solution in the video may not be enough to say that "this student conceptualized the part-part-whole relationships very well* (Revised lesson plan, W7).

> PST2: *In the group discussion, PST5's interpretation made me aware of the covariant and invariant relationships in this mixture context. The student did not demonstrate the connection between the essential elements of proportional reasoning as evidence of the development of proportional reasoning. For instance, lemon juice and water are linked and must be changed together if we want to preserve the same taste. As PST5 stated, the*

student did not show enough evidence to consider the covariant and invariant relationships (Revised lesson plan, W7).

4.3.4.2. Deciding How to Respond on the Basis of Student 5's Understanding

Table 4.30 shows that the PSTs provided a detailed response on how to respond to Asya's understanding in MCV5. Most prospective teachers proposed case-specific-detailed instructional actions.

Table 4.30 Instructional actions after video-based analysis

	Next instructional actions			Total
PSTs	General	Case-specific-undetailed	Case-specific-detailed	frequency
PST1	0	0	2	2
PST2	0	0	4	4
PST3	0	0	2	2
PST4	0	2	1	3
PST5	0	0	4	4
PST6	0	2	1	3

In this week, the PSTs focused on alternative instructional actions, including seeking more information on the student's understanding and extending the student's understanding of part-part and part-whole relationships in the mixture problem. They attended to the student's strategy and interpreted the student's understanding in detail by providing robust evidence to offer more appropriate instructional actions to support and extend the student's understanding. The table above shows that most PSTs improved their deciding how to respond skills, which helped them to propose more extended, detailed, and case-specific instructional actions considering the student's understandings after MCV5 analysis. The PSTs proposed 18 instructional actions after individual video analysis. 78% were case-specific and detailed, reflecting more tiered and systematic action steps on the basis of the student's understanding. Even though some prospective teachers' scores were the same (i.e., PST1, PST5, PST3, and PST2), their responses included additional tasks, representations (i.e., ratio tables), in-depth and focused questions or instructions based on the Student 5's mathematical understanding in MCV5.

Below are the excerpts on the examples of instructional actions that the PSTs proposed throughout group discussions.

PST6 *I would encourage her to consider the part-whole relationship because she explained ⅔ as ⅔ glass of lemon juice per one glass of water. I think she understood part-part ratios. So, I would ask her, "what about the amount of lemon juice when we mix lemon juice and water for each mixture?" as another strategy. For example, Mixture A contains two glasses of lemon juice in five glasses of lemonade, and we can write it as 2/5. Then, I would ask more questions, "Which one has a stronger taste? and how did you decide?". `I would encourage her to consider both situations: lemon juice/mixture or water/mixture. Therefore, I would learn the student's reasoning strategies in detail.*

PST1 *I agree with PST6. However, as stated before, I could not understand how she coordinated the relationship between the lemon juice and water because equalizing the denominators to find the bigger one might be a routinized application. Hence, I would ask her to show the more structured part/part/whole relationships and write the corresponding part-whole ratios between lemon juice and lemonade (mixture) or the amount of water and lemonade (mixture). For instance, I would state, "I want you to compare the mixtures in different ways, such as the amount of lemon juice to the total mixtures. Can you show me the ratio (part-whole) in that way?". This way, I would learn how she made sense of the relationships by comparing the ratios.*

PST5 *Yeah, she is right. I would like to state that to decide on the stronger taste, and she created equivalent ratios (i.e., 2/3 and 16/24). However, I could not justify whether she recognized that she coordinated the linked composite units. So, I would ask her to show me the relationship on the (ratio) table. I mean invariant and covariant relationships. For example, may I draw?*

Facilitator *Sure.*

PST5

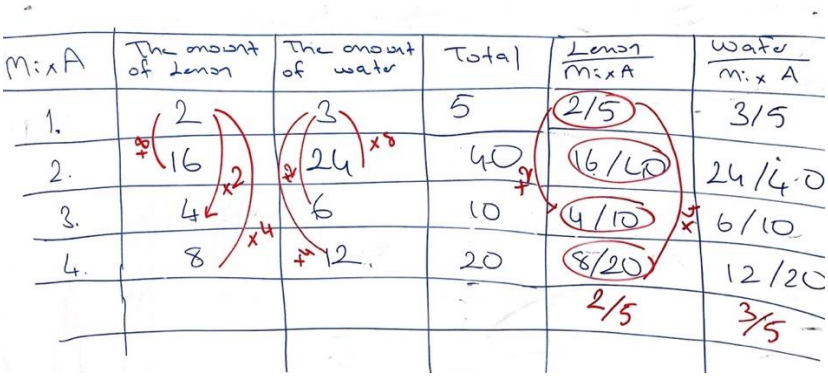


Figure 4.13 Scaling up the numbers in the ratio table

For instance, I would encourage her to write the table like that. While writing the multiplied amounts (i.e., four times two glasses of lemon juice), I would leave some sections blank. This ratio table would be helpful to elicit her reasoning about the multiplicative change of the quantities and to explore the relationships on the table. Therefore, I would understand how she conceptualized the change.

Moreover, if this student did not make sense of the multiplicative relationship, this table would help her to investigate the relationship in the table. For instance, she would construct composite units scaling by the same factor and iterate these composite units in a more organized way, such as 2-3, 4-6, 8-12, 16,24…

PST2 *Also, I would use the ratio table to show the invariant relationships between lemon and water. For instance, I would ask the amount of water for one glass of lemon juice and the amount of lemon juice for one glass of water on this table. As the student iterated these linked quantities, she would recognize the scalar relationship for one glass of lemon juice or one glass of water.*

Facilitator *How?*

PST2 *I would ask her to add another row to show the quantity for one glass to reconceptualize the constant relationship. Mix A requires 2/3 glasses of lemon juice for one glass of water or 3/2 glasses of water for one*

glass of lemon juice. The student could recognize that while covarying the quantities to have same taste, the relationship between them was kept the same. I did not recognize explicit statement of the student in the video, so I would focus on these relationships on the table.

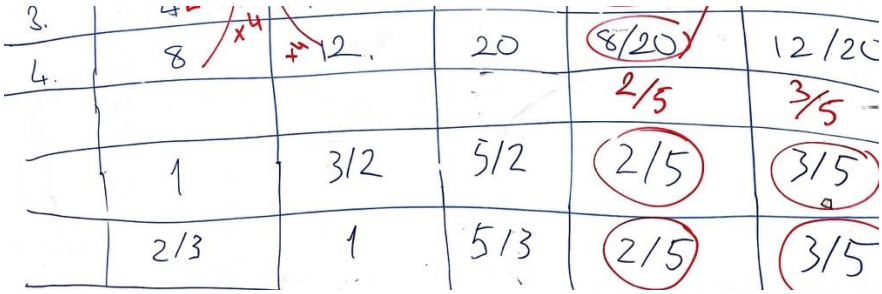


Figure 4.14 Reasoning with ratio table to find unit rate

In group discussions, prospective teachers first focused on the part-part relationship based on the student's strategy in the video. PST6 stated that the student utilized part-part ratios. Therefore, she would ask the student to compare by reconceptualizing this ratio as part-whole relationships. She gave the details of her next instructional action. On the other hand, PST1 explained her hesitation about the construction of part-part relationship by the student. She stated that she could not understand how the student conceptualized part-part relationships. She stated that "equalizing the denominators to find the bigger one might be a routinized application". Therefore, she asked specific follow-up questions to clarify the student's understanding (i.e., I want you to compare the mixtures in different ways, such as the amount of lemon juice to the total mixtures. Can you show me the ratio (part-whole) in that way?)

PST5 then responded to the student's understanding by incorporating another point. She stated her hesitation about how the student constructed linked units (linked composite units) by creating equivalent ratios. Therefore, she would ask the student to draw a ratio table to explain the relationship between quantities (i.e., scalar factor, iterating the composite units). PST5 drew a ratio table on the board to show the details of her next instructional action. Her suggestion reflects expertise not only in attending to the student's strategy (i.e., part-part ratio, equivalent ratios) and in interpreting the student's understanding but also, based on the evidence, reflected the next step to elicit the student's reasoning of proportional relationships. PST2 supported PST5 and also

pointed out that the ratio table would support the student in recognizing the invariant relationship for each mixture (i.e., the amount of water for one glass of lemon juice and the amount of lemon juice for one glass of water). She engaged in connecting her next instructional action and responding to the student's understanding by providing evidence. Overall, PST5 and PST2's responses were categorized as *case-specific-detailed*.

PST2's comments are presented in the following. The illustrative examples of initial and revised lesson plans of PST2 are given in Appendix.

> PST2: *In this week's video, the student used part-part ratios by comparing the lemon juice to water, and she chose mixture A as having a much stronger taste than mixture B and mixture C. She used a part-part ratio to solve the problem and explain her rationale. However, the solution in the video was not enough to claim that this student conceptualized the part-part and part-whole relationships. The preference for using a part-part ratio might show that the student did not conceptualize a part-whole relationship or had difficulty applying part-whole ratios. The group discussion on the ratio table made me reconsider my previous lesson plan. For instance, my friends (PST2 and PST5) used a ratio table to keep track of linked units (i.e., 2/3, 1/4, and 4/8). In my previous lesson plan, I explained the part-part-whole relationship procedurally. I realized that as a teacher I would engage the student to gain different perspectives. For instance, the ratio table could be used as a starting point to show the relationship between quantities. Besides, if the student makes sense of the part-part-whole relationship, the ratio table will explore other strategies based on the existing strategy we watched in the video. Moreover, the ratio table would lead to recognize the relationship between lemon juice and water such as the amount of water per glass of lemon juice and the amount of lemon juice per one glass of water. Therefore, I proposed some expected answers (i.e., instructional actions) on the basis of the student understanding in the video* (Revised lesson plan, W7).

PST2 revised the lesson plan based on the group discussion in Week 7 and explained her rationale why for making revisions to the initial lesson plan. For instance, she stated that she prepared a lesson plan (i.e., the initial lesson plan) without focusing on the student's strategies in detail. After group discussion, she revised her lesson plan by attending to and interpreting the student's understandings in the MCV4. In this vein, PST2 implemented a mixture problem to show the proportional relationships (i.e., part-part and part-whole) between quantities. She concentrated on extending the student's understanding; thus, she proposed various instructional actions. She pointed out that this week's group discussion encouraged her to think about how she would respond to the student mathematical thinking in the video clip. Therefore, her response was categorised as *case-specific-detailed.*

Instructional actions in the individual reflection paper showed that the PSTs made mostly case-specific and detailed interpretations and they suggested case-specific and detailed instructional actions during group discussions and in their weekly revised lesson plans (W7). Table 4.31 below shows that most PSTs improved their responding skill, which helped them to propose more qualified and detailed responses considering the student's understanding after MCV5 analysis. Even though some of the participants' scores seem to be the same, their responses include additional tasks, new representations (i.e., ratio table), and in-depth and focused questions or instructions based on the student's mathematical understanding in the video clip.

Table 4.31 Comparison of PSTs' average points of comments on responding for MCV5

	Prospective teachers						Average of
	PST1	PST2	PST3	PST4	PST5	PST6	total points
Before video analysis	1.5	2	2	1.5	2	1.5	1.75
After video analysis	2	2	2	1.5	2	1.3	1.8

Assigned the certain numbers to each level;
*0 for general comments
*1 for case-specific-undetailed comments
*2 for case-specific-detailed comments
*Including initial-revised lesson plans, pre-post interviews and group discussions

Summary

The aim of this study was to examine the development of prospective teachers' professional noticing of students' thinking in the domain of proportional reasoning through video case enhanced learning environments. The results of this study indicated that the prospective teachers' professional noticing skills improved through the video case-enhanced learning environment. By engaging in micro-case video clip analyses, they significantly improved their noticing skills in the context of proportional reasoning. Although the results of the study revealed that although prospective teachers had difficulty in deciding how to respond to students' understanding, there was improvement in the responding skills of all prospective teachers thanks to the video-clip enhanced interventions. In other words, after examining the micro-case videos and having video-based group discussions about the students' thinking, the prospective teachers substantially provided case-specific and detailed mathematical instructional actions.

On the other hand, the results of this study did not indicate a linear progress in terms of the professional noticing skills of prospective teachers. Consideration of a particular mathematical domain and its associated objectives in the current study may have improved the progress of the prospective teachers in terms of focusing on and making sense of students' mathematical understanding regarding additive-multiplicative relationships, unitizing-norming, and proportional relationships during the whole teaching experiment process. The result of the study revealed that although prospective teachers' professional noticing of students' understanding developed in general, this trend showed a variation in the light of the key understandings of proportional reasoning. This variation was mainly seen in video-based group discussions and revised lesson plans. Such finding may suggest that researchers need to study the professional noticing of students' mathematical understanding in detail for each key understanding within specific domains. The details of this situation were explained in the discussion session.

CHAPTER 5

DISCUSSION

This study aimed to contribute to the literature by investigating the development of prospective teachers' professional noticing of students' understanding in the domain of proportional reasoning in a video case enhanced learning environment. In the light of these foci, this study sought to address the following research questions:

1. How do the prospective teachers develop professional noticing of students' mathematical thinking expertise in the domain of proportion reasoning as they participate in the video case-based teaching experiment?
    a) How do the prospective teachers attend to the noteworthy mathematical details in the micro-case video clips?
    b) How do the prospective teachers interpret students' understanding in the micro-case video clips?
    c) What instructional actions/pedagogical strategies do the prospective teachers propose in response to students' understanding in the micro-case video clips?

5.1. Prospective Teachers' Professional Noticing of Students' Mathematical Thinking

This study indicated that prospective teachers' attention and interpretation skills were limited before participating in video case-enhanced learning environments. They provided case-specific-undetailed instructional actions in response to the students' understanding of the micro-case video clips (MCVs). They presented limited noteworthy mathematical details of the students' strategies in the MCVs. In this vein,

their responses provided insufficient information about the reconstruction of the students' solutions in the MCVs. The descriptions demonstrating limited evidence of attending to students' strategies typically referred the problem-solving process but excluded details about how the students solved the problem.

Furthermore, in their responses, the prospective teachers concentrated on interpreting the students' mathematical thinking using limited evidence to support their claims about the students' understanding. They made unsubstantiated or oversimplified claims about their understanding (lack of evidence). They were only able to make interpretations, but these interpretations were from a general perspective.

Moreover, a majority of the prospective teachers proposed case-specific and undetailed instructional actions. Even though they concentrated on the students' understanding, their responses included limited details and general rationale for proposed instructional actions. Also, some prospective teachers' responses were general. They proposed instructional actions without explaining why these actions were the appropriate next steps for the students. As also reported by previous studies, this study indicated that learning to notice is difficult for prospective teachers (Amador, 2016, 2021; Erickson, 2011; Jacobs et al., 2010; Star et al., 2011; Star & Strickland, 2008; van Es & Sherin, 2002).

The results of this study indicated that the prospective teachers' professional noticing skills improved through the video case-enhanced learning environment. By engaging in micro-case video clip analyses, they significantly improved their noticing skills in the context of proportional reasoning. In this vein, most prospective teachers provided robust evidence of *attending* to the students' strategies and detailed, specific, and accurate descriptions of the students' strategies in the micro-case video clips throughout the interventions. For instance, PST1 did not provide evidence of attending to the details of the student's strategy before the analysis of micro-case video clip 1 (MCV1), and her comments included more general features such as "The student thought that there was a multiplicative relationship." She did not include the details of how the student solved the problem. Thus, her response was categorized as limited. After the video-based group discussion, this prospective teacher revised her lesson

plan and provided evidence for her explanations. PST1 stated that, "After our discussion, I added tables and graphs to help the student recognize additive and multiplicative relationships. I used the same numbers in my lesson plan." Even though this response did not include every detail of the student's strategy and general impressions, PST1 attended to the mathematical essence of the strategy than the response as she had done before the video. She adopted an interpretive stance and referred to some specific moments in the video clip and the group discussion. Overall, it can be stated that at the beginning of the video case intervention sessions, the prospective teachers were not very observant of the noteworthy details in the video clips (Goldsmith & Seago, 2011; Jacobs et al., 2010; Sherin & Han, 2004; Sherin & van Es, 2009; Ulusoy & Cakiroglu, 2020; van Es & Sherin, 2008). Through video based interventions, it has been discovered that the prospective teachers' professional noticing abilities shifted from general noticing of sequences of events to a more focused noticing of noteworthy details and specific events (Sherin & Han, 2004; van Es & Sherin, 2011). Their improvement in professional noticing was demonstrated by their clear and accurate description of what the students did by focusing on the noteworthy details of the students' strategies in the micro case video clips. In parallel to the findings of this study, the studies in the literature revealed the developmental progress of prospective teachers' ability to attend to students' mathematical strategies after participating in professional development programs (e.g., Goldsmith & Seago, 2011; Jacobs et al., 2010; Kazemi & Franke, 2004; Krupa et al., 2017; Star & Strickland, 2008). This finding of the study can be considered to be an evidence that prospective teachers were more likely to attend to students' thinking at a deeper level, and they became attuned to the practice of identifying and articulating students' thinking (Teuscher et al., 2017).

In addition to the development of prospective teachers' noticing skills, the way they interpreted the students' mathematical understanding improved over time. In the initial session, the prospective teachers' interpretations were limited and lacked specificity. They did not provide any reason for their claims and just restated the students' solutions. After the second session, they began to interpret the students' understanding by providing evidence of the student's actions and words. For example, in the first session, PST4 focused on describing what the student did and provided a specific

connection to his strategy that lacked an in-depth analysis of the student's thinking. Hence, he did not provide evidence of how he interpreted the student's understanding in the MCV1. However, in the last session, PST4 interpreted the student's understanding by providing robust evidence. He realized that this student used benchmarks to make proportional judgments (e.g., greater than half and smaller than half) to solve the task. As a result, the prospective teachers' responses which demonstrate robust evidence of interpretation of the student's understanding indicates that they made sense of the student's understanding based on the micro case video clips. Even though the prospective teachers' comments were initially descriptive and evaluative, they adopted an interpretive stance and provided evidence of students' understanding to support their comments.

On the other hand, this interpretive stance led prospective teachers to examine the response in different manners. Our data analysis revealed that prospective teachers started to interpret not only what the students in the video clips understood but also what they did not understand, providing evidence of the students' understanding. They explained their hesitation about the students' understanding of the videos. They tried to make sense of the students' strategy by adopting an interpretive stance. They interpreted what the student did and did not understand, providing specific comments with evidence of the student's understanding. Likewise, Males (2017) maintained that prospective teachers concentrated more on and made sense of students' thinking about what they understood and did not understand in the following sessions. The improvement could be attributed to the analysis of one-on-one student problem-solving videos, which contained various solutions of students (i.e., correct and incorrect solutions), following a developmental trajectory of proportional reasoning and engagement in a targeted and sustained professional development program.

Moreover, the results of the study revealed that although prospective teachers had difficulty in *deciding how to respond to students'* understanding as in other studies (i.e., Jacobs et al., 2010; Krupa et al., 2017; Sánchez-Matamoros et al., 2019), there was improvement in the responding skills of all prospective teachers thanks to the video-clip enhanced interventions. In other words, after examining the micro-case videos and having video-based group discussions about the students' thinking, the

prospective teachers substantially provided case-specific and detailed mathematical instructional actions. For instance, the prospective teachers proposed undetailed mathematical future instructions in the first-week reflection paper. Their rationale for the instructional actions was vague and lacked details. However, their expertise in deciding how to respond improved through a video case-enhanced learning environment. This finding coincides with previous studies (e.g., Amador et al., 2011; Jacobs et al., 2010; Philip et al., 2011). The findings of this study can be accepted as supporting evidence for the improvement of noticing skills of prospective teachers in micro-case video clip-enhanced learning environments.

The shift in prospective teachers' interpretations of the students' understanding occurred during the intervention, and this shift caused the prospective teachers to revise and rechange their lesson plans. The results showed that the prospective teachers started to develop strategies to test the accuracy of their interpretations based on the students' solutions in the video clips after the second week of the teaching experiment. They used follow-up questions about students' mathematical understanding in each video clip or proposed new contexts to revoke their current thinking. These instructional actions were beyond just probing questions discussed in the prior research (Stein et al., 2008). The prospective teachers wanted to become more confident about students' mathematical thinking while interpreting their understanding in the video clips. In this vein, they aimed to adopt some actions to understand students' thinking, such as providing different contexts in which students would use similar reasoning. For instance, PST1 commented that the student in the MCV5 solved the mixture problem correctly by using part-part relationships and found a stronger taste. However, she stated that the student could not explain the meaning of this relationship (i.e., 2/3 glasses of orange juice to a glass of water). In this regard, by providing evidence, PST1 described the student's uncertainty about explaining the invariant and covariant relationship. Also, she claimed that this video clip was limited in terms of clearly showing the student's mathematical thinking about the part-part-whole relationships. In this context, PST1 provided the next instructional actions to test the students' hypothesis and reveal how they thought. According to Dyer and Sherin (2022), prospective teachers noticed that they also needed to focus on students' ambiguous statements as they are essential to interpret their mathematical understanding. In this

vein, the findings of this study indicated that various videotaped solutions of the students could prompt prospective teachers to interpret students' understanding in a situated and contextualized way.

The development of interpretation led the prospective teachers to identify which instructional activities promote students' mathematical understanding and how these instructional actions should be formulated and sequenced (Krupa et al., 2017). In this vein, the shift in prospective teachers' responding skills occurred based on the interpretations and attention. This study revealed that prospective teachers proposed the next instructional actions in response to students' understanding in the micro-case video clips. This situation led prospective teachers to begin their responses with the phrase, "I felt this student" or " I am not sure..." These statements were indicators of hesitancy or uncertainty in their commitment to know what the student understood or did not understand. As a result, towards the last sessions, prospective teachers preferred to use specific instructional actions to uncover the students' understanding in each context of proportional reasoning.

Jacobs and colleagues. (2010) stated that attending, interpreting, and responding are interrelated skills. Even though these three skills were examined individually in this study, throughout the last sessions, prospective teachers' comments indicated nested relationships. For instance, the prospective teachers who did not attend to the noteworthy details of the student's strategy in the MCV1 (i.e., providing limited evidence) did not provide evidence of their interpretation of the student's understanding and did not use the student's understanding in their reasoning in a detailed way. On the other hand, some prospective teachers were able to attend to the details of the student's strategies (robust evidence). However, they provided limited evidence of the student's understanding and proposed instructional actions without giving details. These results suggest that a robust level of attending and interpreting is necessary but not an indicator of a robust level of responding (i.e., case-specific-detailed). In this study, none of the prospective teachers whose attending and interpreting level was limited proposed case-specific and detailed (robust level) instructional actions. The prospective teachers who provided case-specific and detailed instructional actions on the basis of the students' understanding in the video

clips already attended to the students' strategies and interpreted their understanding. The research has indicated that students' strategies provide a window into their understanding and shape PSTs' response decisions (Carpenter et al., 1999; Jacobs et al., 2011). Many studies related to noticing claimed that attending to students' strategies and interpreting their mathematical understanding have a foundational role in deciding how to respond (Jacobs et al., 2011; Sánchez-Matamoros G. et al., 2018).

Studies illustrated some variations in how noticing is conceptualized in terms of methodological applications (i.e., studying individual teachers' noticing or looking at groups of teachers' noticing) or media used to assess teachers' noticing (i.e., students' written work, video clips of instructions and teachers' live classrooms and one-on-one student interviews) (Amador et., 2021; Krupa et al., 2017). Furthermore, researchers also vary in how they record teachers' noticing by analyzing teachers' discussions and examining teachers' written responses (i.e., reflections papers to prompts or lesson plans) (Mostofo, 2013; Sherin et al., 2011). All these variations in professional development experience might significantly affect teacher noticing (Sherin & Star, 2011).

In this regard, videos for educational purposes vary in terms of how the video is selected (from the teachers' classrooms or unknown teachers' classes), the length of the video (entire lessons or short clips), and whether the video clips are edited (Schack et al., 2013; Sherin et al., 2011; Star & Strickland, 2008; van Es & Sherin, 2002; van den Kieboom et al., 2017). Studies showed that even though classroom videos encapsulate classroom events, prospective teachers tend to focus on teachers' or students' behaviors or classroom management and struggle to pay attention to students' mathematical thinking (Star & Strickland, 2008). Star and colleagues (2011) argued that before prospective teachers can make sense of classroom features, they must first learn to pay attention to relevant elements. Classroom videos can help prospective teachers to develop their noticing skills (Santagata et al., 2007; Star& Strickland, 2008); however, they may continue to focus on the classroom environment rather than students' mathematical thinking (Superfine et al., 2015). The findings of relevant studies argued that instead of paying attention to students' thinking, the participants paid attention to a range of issues with a broad perspective at the beginning of the

study (Goldsmith & Seago, 2011; Jacobs et al., 2010; Kazemi & Franke, 2004; McDuffie et al., 2014; Sherin & Han, 2004; Sherin & van Es, 2009; Star & Strickland, 2007; van Es, 2011; van Es & Sherin, 2008). In this vein, to prevent the prospective teachers from focusing on unrelated details and to help the prospective teachers maintain students' thinking at the center (i.e., the teacher moves), micro-case video clips were used, and they included one-on-one student problem-solving interviews between the researcher and the student. Therefore, the prospective teachers' struggle to attend to students' strategies was particularly salient. Even though Jacobs et al. (2011) allowed prospective teachers to watch video clips only once, I allowed prospective teachers to watch them more than once if needed as I aimed to support their noticing of students' mathematical understanding. Furthermore, the prospective teachers had the opportunity to pause the video and reconsider the events without requiring immediate action, so they could notice the details they did not notice when they first watched the video(s) (Borko et al., 2008; van Es 2011). Hence, allowing prospective teachers to replay videos influenced group discussions and, subsequently, groups' noticing over time. For instance, in this study, some prospective teachers had contradictions about the student's understanding in the MCV4. Therefore, they wanted to rewatch the video clips to clarify the issue under discussion. examples of significant mathematical elements in students' strategies and the opportunity to interpret students' understanding as a source of prospective teachers' decision to respond to students' understanding (Sherin et al., 2009). We believe this approach holds much promise for encouraging the development of professional noticing of students' mathematical understanding. As other studies supported, participating in a video case enhanced learning environment affected teachers' noticing in the group discussions, in post-interviews after the teaching experiment, and in revising lesson plans (e.g., Carpenter et al. 1999; Jacobs et al., 2010; Schack et a., 2013; Schifter et al., 2000; Sherin & van Es, 2009; Star et al., 2011; Teuscher et al., 2017; van Es & Sherin, 2006; van Es, 2011).

While the professional noticing framework (Jacobs et al., 2010) led to changes in most prospective teachers' noticing in this study, prospective teachers do not make all the progress in the same trajectory (Goldsmith & Schifter, 1997; Scherer & Stein, 2012). The changes for two of the PSTs were neither instantaneous nor dramatic; that is, the framework did not serve as an "answer key" and was insufficient to foster noticing

(Stockero & Stenzelbarton, 2017). Instead, it appeared that the prospective teachers' noticing evolved through involvement in regular video-based group discussions that allowed them to deal with the framework components. This study demonstrated that the noticing intervention helped prospective teachers become more focused on students' mathematical thinking. These developments in professional noticing could be related to the group discussion process. In the group discussion process, the prospective teachers had the opportunity to share their thoughts with their peers, which resulted in further in-depth conversations away from the group. As a result, the prospective teachers started to develop their skills of noticing students' mathematical thinking. When the students with a lower level of noticing skill engaged with those with a higher level of noticing skill, both groups showed adaptation and revised their interpretations to better understand one another. The prospective teachers with a lower level of noticing were able to gain a deeper comprehension of the students' mathematical reasoning thanks to this group discussion.

In this study, the PSTs engaged in five weekly group discussion sessions. They all had an equal chance to participate in discussions that were likely to promote their progress in noticing. The group discussion format helped the prospective teachers learn to attend to the noteworthy mathematical details of students' thinking. The prospective teachers' increased focus on the student's mathematical understanding proved that sustained focus on individual students' mathematical thinking during group discussions was effective (Goldsmith & Seago, 2011; Ulusoy & Cakiroglu, 2021; van Es & Sherin, 2008). In this regard, the findings indicate that each prospective teacher began to construct new meanings resulting from their interaction with their peers. The change in the initial professional noticing expertise can be associated with sharing of interpretations and then validation or refusal of others in the group discussion, which included a dynamic interaction with domain-specific language. The findings pointed out that targeted group discussions that addressed a specific domain (i.e., the interpretation of the student's understanding in the context of unitizing) could lead participants to focus on students' thinking and scaffold noticing. Becoming aware of the student's mathematical thinking led prospective teachers to use a different ratio language in different contexts to make sense of the students' strategies and understanding. The video-based group discussion in the domain of proportional

reasoning supported the prospective teachers in this study in terms of using proportional language to explain the students' understanding considering the three skills in professional noticing.

In this study, I concentrated on identifying the aspects of discussion that are likely to promote prospective teachers' learning instead of merely involving exchange of thoughts or agreeing with one another. The group discussions consisted of making sense of the proportional concepts, critiquing the reasoning of others, and offering alternative interpretations of the students' understanding. The comments in the group discussions served to improve the prospective teachers' initial thinking and their ability to notice the important details of students' thinking. Thus, they led to the emergence of instructional actions to address students' mathematical understanding in the video clips. In this way, the prospective teachers were able to reconsider their initial lesson plans and develop ideas on how to revise them in light of their peers' comments and students' mathematical understanding. Several studies in the literature support what has been claimed in this study (Scherrer & Stein, 2012; Ulusoy & Cakiroglu, 2021). For example, in Session 2, PST1 proposed an instructional action based on the student's understanding in the MCV2. The comment was followed by another participant, PST6, who contradicted PST1's inference. PST1 wrote the following statement in her revised lesson plan: "One of my friends proposed an activity for the student. PST6 asked a question to reconsider her action. Because the student had a problem conceptualizing the meaning of the unit rate, she was right about her inference. I understand that constructing the unit rate concept is important to understand the relationship between quantities. In my first lesson plan, the student thought I focused only on one type. Therefore, I changed my plan and used one of the questions." This comment indicated that PST6's comments caused PST1 to revise her inferences and the original lesson plan. In other words, the group discussion provided a synchronous and asynchronous environment where the prospective teachers could validate their interpretations with others or shift their interpretations in response to the others' thinking. As Llinares and Valls (2009) stated, structured guidance (i.e., targeted tasks and specific prompts in discussions) allowed prospective teachers to reflect on and integrate multiple aspects of teaching and develop a more comprehensive perspective on teaching. In addition, the findings suggest that

facilitator that probes or challenges the prospective teachers' thinking processes is essential to encourage their participation in discussions, which increases their noticing expertise (e.g., Stockero & Stenzelbarton. 2017). This finding is of great importance because my ultimate purpose with this study is to promote a change in prospective teachers' noticing skills. In conclusion, as Ulusoy and Cakiroglu (2021) support, all prospective teachers develop the quality of their responses after video-based group discussions while suggesting instructional actions considering students' understanding.

As a result of ongoing video-based group discussions, the weekly revised lesson plans in the study revealed the progress of the PSTs' professional noticing skills. The prospective teachers emphasized the students' mathematical understanding by attending to and interpreting their thinking in their weekly revised lesson plans in all five sessions, specifically in the last two. After the first session, all prospective teachers changed their lesson plans. They made interpretations based on what they attended to in the students' strategies in the sessions. Based on their attention to the students' strategies and their interpretation of their understanding, they adopted some instructional actions discussed in the group discussion in response to the student's understanding in the micro case video clips. Then, they revised their initial lesson plans based on their peers' comments and students' mathematical understanding in the micro-case video clips. The prospective teachers used evidence of students' thinking in the video clips to propose future instructional actions. Studies revealed a correlation between various levels of professional noticing and the lesson-planning approach of teachers. In other words, increased attention to students' mathematical thinking through video-based group discussions assists teachers in defining the content of the lesson plan in a way that might reveal students' thinking and promote conceptual knowledge (Choy, 2014; Hawthorne, 2016; Stigler & Hiebert, 1999). For instance, in her weekly revised lesson plan (the first version of the revised plan), PST4 proposed future classroom actions to support students' understanding. These instructional actions were based on the student's understanding in the MCV1 (i.e., I realized that the student applied an erroneous multiplicative strategy for additive problems in the video. The reason might be the overuse of proportional methods). In the following weeks, the evidence of attending to and interpreting students' understanding in the video clips was

more apparent in revised lesson plans. During the teaching experiment, the prospective teachers deviated from their previous revised lesson plans and made significant changes to the following plans. The primary difference in most participants' responses in Sessions 5 from Sessions 1 as their ability to consider the next instructional actions based on students' thinking in the micro case video clips. Overall, they demonstrated a notable difference in responding compared to previous lesson plans. In this sense, their revised lesson plans might contribute significantly to our understanding of how their noticing skills were expanded through micro-case video analyses and group discussion.

Another noteworthy feature of this study is that the developmental trajectory by Carpenter et al. (1999) guided prospective teachers to deepen their awareness of students' thinking in the domain of proportional reasoning, which can be evidence of noticing enhancement (Ivars et al., 2018). In a similar vein, Nickerson et al. (2017) asserted that "meaningfully analyzing responses of interpreting and deciding how to respond to students' mathematical ideas requires knowledge of students' possible learning trajectories" (p. 393). In this study a domain-specific developmental trajectory was a kind of roadmap to support prospective teachers in terms of identifying learning goals, interpreting students' mathematical thinking, and responding with appropriate and more detailed instructional actions by using content-specific language in the domain of proportional reasoning (Ivars et al., 2018; Sztajn et al., 2012). The results of this study demonstrated that the developmental trajectory of a particular mathematical topic might be used by prospective teachers to begin noticing students' mathematical thinking in a particular domain and to develop professional noticing skills. Using developmental trajectory contributed to the professional noticing of students' mathematical thinking, considering the use of specific information concerning students' learning. Their professional noticing of students' understanding was developed gradually in terms of the key elements of proportional reasoning and their relationships in the developmental trajectory. The developmental trajectory led prospective teachers to synthesize students' acquisition of sophisticated proportional reasoning concepts. In this respect, this trajectory could assist prospective teachers in attending to the relevant and noteworthy elements of students' strategies and

interpreting their understanding to propose appropriate instructional suggestions. (Wilson et al., 2013; Ivars et al., 2018)

5.2. Professional Noticing of Students Thinking in the Context of Proportional Reasoning

The findings show prospective teachers' attention and interpretation skills developed through video case-based teaching experiments. Studies reported that with structured and sustained support, prospective teachers could improve their professional noticing skills (Fernández et al., 2013; Jacobs et al., 2010; Schack et al., 2013; Sherin & van Es, 2009; Stockero et al., 2017). Specifically, as a support in specific domains, developmental trajectories can assist prospective teachers' professional noticing of students' understanding in response to the critical points (i.e., misconceptions, difficulties, or critical points) of particular mathematical domains (e.g., Carpenter et al., 1999; Walkoe, 2015). In this regard, this study was situated within a well-defined mathematical content, which is proportional reasoning. In addition, the content was presented within a developmental trajectory (i.e., Carpenter et al., 1999) to illustrate the progress of learning to notice over time.

Thus far, studies have confirmed that the subject domain is seen as a promise for supporting the development of professional noticing of students' thinking (Amador et al., 2017; Callejo & Zapatera, 2016; Fernández et al., 2013; Kieboom et al., 2017; Sanchez-Matamoros et al., 2014; van Es, 2011; Walkoe, 2013). On the other hand, the results of this study did not indicate a linear progress in terms of the professional noticing skills of prospective teachers, and this study supports the findings of Amador et al. (2017). They observed a prospective teacher over three years in two subject areas. Her participant's noticing skill developed in both mathematics and science, but her responding skill was more prominent in mathematics. Sherin (2011) supported this finding by asking questions such as, "How context-specific is noticing expertise? If a mathematics teacher has expertise in noticing, will he or she have noticing expertise in all mathematical domains and instructional contexts?" (p. 11). Just as the nature of noticing expertise varies across content domains, there was a variation between topics within a content domain. A noteworthy feature of the current study is concentration

on a well-defined mathematical content domain that contains details about students' understanding, allowing prospective teachers to engage with these noteworthy mathematical details.

Consideration of a particular mathematical domain and its associated objectives in the current study may have improved the progress of the prospective teachers in terms of focusing on and making sense of students' mathematical understanding regarding additive-multiplicative relationships, unitizing-norming, and proportional relationships during the whole teaching experiment process. The result of the study revealed that although prospective teachers' professional noticing of students' understanding developed in general, this trend showed a variation in the light of the key understandings of proportional reasoning. This variation was mainly seen in video-based group discussions and revised lesson plans. Such finding may suggest that researchers need to study the professional noticing of students' mathematical understanding in detail for each key understanding within specific domains.

5.2.1. Additive and multiplicative reasoning

The PSTs in the current study had difficulty noticing Student 1's thinking in the context of additive and multiplicative situations of the Racetrack problem. In particular, they failed to attend to the student's strategy and interpret the student's understanding in the MCV1. The results revealed the difficulty of all prospective teachers in identifying the relevant elements of the student's mathematical thinking concerning the transition from additive to multiplicative thinking during the pre-clinical interviews. These prospective teachers struggled to define the constant additive relationship between the given quantities. Before the analysis of the MCV1, all prospective teachers provided *limited evidence* of attention to Student 1's strategy in the context of additive and multiplicative reasoning. Also, all teachers provided *limited evidence* of interpreting Student 1's understanding. In addition, their proposed instructional actions were *general* and *case-specific and undetailed.* One explanation for this finding might be the PSTs' limited understanding of the difference between additive and multiplicative reasoning (van Dooren et al., 2010). Similar to previous studies, this finding indicates that prospective teachers have difficulty

discriminating between additive and multiplicative situations and have an inadequate understanding of how to support students' understanding of these concepts (Sowder & Philipp, 1995; Thompson & Thompson, 1994, 1996). This might be the reason why the prospective teachers in our study could not provide evidence of attending to the student's strategy in the MCV1. As previous studies pointed out, attending to the details of students' noteworthy strategies is the first step in interpreting their understanding (Jacobs et al., 2010). Therefore, these prospective teachers could not properly interpret students' understanding in the context of additive and multiplicative reasoning.

On the other hand, although some prospective teachers could recognize the additive and multiplicative situation in the Racetrack problem, they still had difficulties attending to Student 1's strategy and interpreting his understanding. They could not explain why the Racetrack problem was not proportional and did not propose case-specific and detailed instructional actions for the student in MCV1. As mentioned above, three interrelated skills had a nested relationship, and "deciding how to respond on the basis of children's understanding can occur only if teachers interpret children's understandings, and these interpretations can be made only if teachers attend to the details of children's strategies" (Jacobs et al., 2010, p.197).

After the video-based group discussion, four prospective teachers described the noteworthy details in the student's strategy and interpreted her understanding at a robust level. However, some still had difficulties noticing the students' additive and multiplicative reasoning. In particular, they failed to interpret the issue of applying erroneous multiplicative strategies for the additive problem. This result may confirm the evidence from previous studies that teachers' content knowledge is a prerequisite for developing professional noticing of students' mathematical thinking (Ainley & Luntley, 2007; Dreher & Kuntze, 2015). This finding indicates that PSTs may need knowledge of the curriculum and students' developmental trajectories in the domain of proportional reasoning, which is not generally addressed in teacher training programs.

However, prospective teachers who provided robust evidence in interpreting Student 1's understanding made specific claims and inferences about his understanding in the video clip and explained the reasons, such as "the student used multiplicative procedures for this additive problem. The reason might be the familiarity of missing value problems and lack of knowledge in discrimination of additive and multiplicative situations in our curriculum (PST2)". Although there was no evidence of familiarity with missing value problems in the video clip, this prospective teacher could recognize the distinction between additive and multiplicative situations, which is not explored in the curriculum (MoNE, 2018). The curriculum does not include all key understandings and critical components of proportional reasoning, such as additive and multiplicative reasoning (Sowder et al., 1998). Therefore, the prospective teachers in the study stated that they made revisions to their initial lesson plans based on what they noticed after Session 1.

In their initial lesson plans, none of the prospective teachers proposed instructional actions related to additive and multiplicative reasoning in the Racetrack problem . They proposed ten instructional actions on the basis of Student 1s understanding of the MCV1. 53% of the actions were case-specific and detailed, and 42% of them were case-specific and undetailed. The proposed instructional actions were mostly based on the difficulty of transition between additive and multiplicative situations. In other words, the actions were specific and addressed the difficulty experienced by Student 1. For instance, the prospective teachers chose to "modify the numbers in the problem context to examine the student's approach" by using different numbers in the problem context as the student made a mistake due to the integer relationship between the given numbers in the problem; however, she did not explain all the details about how the numbers with non-integer relationships would help the student discriminate between additive and multiplicative situations. Our findings indicated that prospective teachers struggled to discriminate between additive and multiplicative situations as students. They described the student's strategy but did not make relevant inferences about the student's thinking.

Nonetheless, what led teachers to consider the features of proportionality was talking about the difficulty of transition from additive to multiplicative reasoning and

potential reasons, such as lack of knowledge about additive and multiplicative reasoning in the Turkish curriculum, the tendency to apply cross-product algorithms in additive situations, and the overuse of missing-value problems. Previous studies have also underlined the benefits of group discussion about students' work. The discussion process led prospective teachers with a limited level of noticing to develop a new understanding of Student 1's mathematical thinking in this context.

5.2.2. Unitizing and Norming

Studies indicated that the transition from additive to multiplicative reasoning requires the reconceptualization of unit notation (Hiebert & Behr, 1988). Multiplication reasoning is about constructing new units called composite units that can be iterated (Steffe, 2002). This process needs skills to unitize and re-unitize the given quantities with ease and flexibly (Lamon, 1994). To re-unitize is to transfer a different unit of measure while preserving the equivalent relationship with the original unit. Multiplicative thinking needs the coordination of composite units and. studies have shown that students experienced problems coordinating the units within the task, thus resulting in incorrect answers (Singh, 2000).

Considering the Grocery problem, as mentioned before, Student 2 solved the problem incorrectly. This student tried to find the price of one banana in each grocery, but she found the number of bananas for one lira, thus she found the wrong grocery (i.e., Grocery B) to be cheaper. Studies reported that students prefer the unit rate strategy before receiving instruction (Lamon, 1989; Singh, 2000). Although finding a single unit is the first preference of students, they may not construct the meaning of the unit rate strategy. This was the case in our study as well. When the prospective teachers in our study were asked about potential difficulties students may experience in the Grocery problem, none of them pointed to the difficulty of unit ratio construction. Then, when they watched the video clip, 4 out of 6 provided limited evidence of attending to the student's strategy. While 5 out of 6 prospective teachers made analyses with limited evidence in the interpretation of Student 2's understanding, none of them proposed case-specific and detailed instructional actions.

As for the Balloon problem, Student 3 solved the problem correctly by using composite units; she used the price for three balloons (composite units) to find the price for 24 balloons. Before the video analysis, 5 out of 6 prospective teachers attended to the student's strategy providing robust evidence. While half of them provided robust evidence of the interpretation of Student 3's understanding, they mostly proposed case-specific and undetailed instructional actions. Both problems concentrated on unitizing and norming processes, and the professional noticing skill levels of the participants were different. The reason might be related to how the prospective teachers conceptualized the unit ratio and composite unit.

After the video analysis of each problem, the professional noticing skills of the prospective teachers improved. 4 out of 6 prospective teachers provided robust evidence of attending to the student's strategy, and 4 out of 6 provided robust evidence of interpreting the student's understanding in the video clip of the Grocery problem . The prospective teachers proposed 14 instructional actions, and 53% of them were case-specific and detailed. For the Balloon problem, while 83% of the prospective teachers provided robust evidence of attending to Student 3's strategy, 67% of them interpreted the student's understanding by providing robust evidence. For Student 3, they proposed 11 instructional actions, and 55% of them were case-specific and detailed.

The findings of this study indicated that the prospective teachers had difficulty with the unit rate approach, as seen in the Grocery problem. Therefore, they were not able to interpret Student 2's understanding by providing robust evidence. They mostly proposed case-specific and undetailed instructional actions. After group discussion, they realized that the student's incorrect answer was related to a lack of knowledge about interpreting the (multiplicative) relationships between quantities (i.e., the unit types). Therefore, some prospective teachers proposed instructional actions (e.g., ratio tables, qualitative reasoning) for this student to make sense of the meaning of the unit rate approach within various contexts.

Lamon (1993a) highlighted the significance of constructing composite units and argued that using composite units is more effective than using single units. Similarly,

the findings showed that the prospective teachers were flexible in unitizing and re-unitizing quantities before and after the video analysis of the Balloon Problem. However, the results of this study indicate that the construction of the meaning of the unit rate concept was more difficult than the construction of a composite unit (Lamon, 1993; Steinthorsdottir & Sriraman, 2009). This finding opposes the research arguing that unit rate was easier than composite units, and the prospective teachers had difficulties reuniting quantities (LaRochelle, 2018). These different findings may be due to familiarity with the composite unit concept. Making sense of the unit rate approach (i.e., the number of bananas per lira and the total price per banana) may be difficult and unfamiliar for prospective teachers because this type of unit rate (the number of bananas per lira) is not used in our daily routines, but it is still a unit rate type.

Another shift that emerged from the analysis of this study was related to the change in language. While evaluating the students' understanding, the prospective teachers used the language "linked quantities, composite units, build up strategies and iterating." In the first session, although attention and interpretation improved, the prospective teachers' descriptions of the students' strategies lacked specific proportional language and tended to be vague. That is further evidence that these sessions entailed a language change and fostered teachers to use more proportional language to explain the reasons for their thinking. For instance, the language to compare the quantities multiplicatively includes "for every, out of every" (Lamon, 1995). Thus, it can be stated that "all forms of teacher support, appropriate tasks and tools, peer-to-peer discourse, and the language necessary to specify and build ideas" are central to students' progress (Confrey et al., 2014, p. 721). The findings of this study support that the progress of using proportional language influenced prospective teachers' thinking and allowed them to provide meaningful explanations while focusing on noticing students' thinking in the video clips (Krupa et al., 2017). Researchers have found that learning trajectories helps use a common language to describe student thinking (Fernandez et al., 2003; Ivars et al., 2017; Ivars et al., 2018). Thus, it can be said that micro-case video clips that serve as a shared artifact for the development of a common language have impacted on student learning (Seidel & Sturmer, 2014; Santagata & Guarino, 2011; van Es, 2020). This study indicates that using a developmental trajectory to notice

students' thinking provides prospective teachers with a specific language to refer to students' understandings, allowing them to concentrate on the pertinent features of their reasoning. In this vein, prospective teachers may pinpoint what they notice during their professional development programs and connect their interpretation of what they attend to with their instructional decisions.

5.2.3. Proportional Relationships

Studies have indicated that understanding proportional relationships (i.e., invariance-covariance, part-part-whole relationships) within a problem context is an indicator of higher-order reasoning, and students can make judgments based on proportional reasoning (Piaget & Inhelder, 1975). In this vein, the Photograph and Mixture problems were chosen to show the proportional relationships that require preservation of the relationship between quantities regardless of whether they form an integer.

The video clip of the Photograph problem (MCV4) included Student 4's correct solution. Before participating in the video clip analysis, 5 out of 6 prospective teachers attended to the student's strategy and provided robust evidence, and 4 of them provided robust evidence of interpreting the student's understanding. The prospective teachers provided 12 instructional actions, and 8 of these actions (67%) were case-specific anddetailed. For the video clip of the Mixture problem (MCV5), all prospective teachers provided robust evidence of attention to Student 5's strategy and interpretation of this student's understanding in the context of part-part-whole relationships. 67% of the prospective teachers proposed case-specific and detailed instructional actions based on the student's understanding in the micro-case video clip.

After the video analysis, while attending to the student's strategy during the group discussions of the MCV4 and MCV5, they simultaneously interpreted the student's understanding with evidence. All participants' responses provided robust evidence of attention to the student strategy, but four provided robust evidence of interpretation of the student's understanding in the MCV4. They proposed 13 instructional actions, and 87% of them were case-specific and detailed. After the video-based group discussion for the MCV5, all prospective teachers attended to the student's strategy with robust

evidence and interpreted her understanding by providing robust evidence. Their proposed instructional actions were mostly (78%) case-specific and detailed.

The findings of this study indicated that in both cases, the prospective teachers focused on alternative instructional actions, including seeking more information on the students' understanding or extending their understanding of invariance and covariance and part-part-whole relationships, respectively. For the photograph problem, PST5 questioned Student 4's understanding of additive and multiplicative relationships for similarity context in the video-based group discussion, so she presented an additional correspondence task. Since she stated that the Photograph problem video clip did not present evidence of how this student conceptualized additive and multiplicative reasoning, her instructional suggestions aimed to reveal the student's existing understanding. On the other hand, PST2 aimed to extend the student's understanding by highlighting the distortion and slope concepts. She stated that "understanding the slope within similarity context might allow the student to extend her understanding. Therefore, I would use unit squares to visualize the rate of change on the graph." This prospective teacher claimed that the student had a sense of invariation and covariation, and she provided evidence from the video clip as a reference. Therefore, she utilized ratio tables or graphs to extend the student's understandingin the MCV4.

For the mixture problem, the prospective teachers' responses included additional tasks such as using non-scalar numbers, multiple representations (e.g., graphs, ratio tables), and in-depth and focused instructional steps based on Student 5's understanding. While the student utilized part-part ratios to find the stronger taste, some prospective teachers questioned how this student constructed part-whole ratios. They expressed their hesitations by saying, "Equalizing the denominators to find the bigger one might be a routinized application." Therefore, they proposed instructional actions by expressing the meaning of quantities and variables in part-part and part-whole contexts. For example, Mixture A contains two glasses of lemon juice in five glasses of lemonade or 2/5 glass of lemon juice per a glass of lemonade (the referent unit is 2/5 of one glass of mixture). By writing the corresponding part-whole ratios, they could encourage the student to create equivalent ratios to recognize the options of covariance and invariance. During the interview, the prospective teachers reported that

they had not considered orange/whole mix as a referent unit of the whole. Prior research has supported that students could not compare the two mixtures; Mixture A (3 glasses of orange juice-2 glasses of water) and Mixture B (4 glasses of orange juice and three glasses of water) because this comparison demanded multiplicative and relative reasoning (Lamon, 1994; Noelting, 1980a, 1980b). As prospective teachers noticed, the numbers used in a task are also a determinant of the kinds of proportional reasoning that are required (Noelting 1980a, 1980b).

Interestingly, none of the prospective teachers used these proportional terms and key understandings in their pre-interviews and initial lesson plans. Even though some proposed a mixture problem in lesson plans, there was no explicit statement about their rationale. The prospective teachers in the study proposed instructional actions which aimed to organize proportional situations by using tables and different numbers in light of the importance of understanding proportional relationships by using correct language and on the basis of the student's understanding in each clip. In video clip-based discussions of the Photograph problem, teachers used proportional language/terms while explaining their rationale, such as equivalent ratios and iterating linked composite units. The results of this study indicated that while prospective teachers' professional noticing skills improved, their proportional language also improved. Previous studies stressed that the domain of proportional reasoning entails highly content-specific vocabulary (Lamon, 2006; MacGregor, 2002; Vanluydt et al., 2021). The findings of this study explicitly dipslayed the association between proportional language and reasoning abilities. Regarding the first research question of this study, I observed considerable progress in the use of proportional language as the professional noticing skills of the prospective teachers improved. These results align with the results of Krupa et al. (2017) on the relationship between the language and professional noticing of prospective teachers.

5.3. Implications of the Study

In the current study, the main concern was the development of prospective teachers' professional noticing of the students' thinking in the domain of proportional reasoning. In light of the results of the current study, some implications and suggestions can be

made for researchers, mathematics educators, and curriculum developers who aim to conduct a study using a video case-based professional environment for prospective and in-service teacher education. These implications and suggestions may give insights into mathematics educators' viewpoints regarding the need to fill the gaps in mathematics teacher education and professional development programs to promote prospective teachers' professional noticing expertise.

The first implication of the findings is related to the use of various types of micro-case video clips in professional development programs. The results of many studies demonstrated that prospective teachers experience considerable challenges in learning how to notice student thinking (Peterson & Leatham, 2009; Stockero, 2008). To address these shortcomings, researchers aim to provide opportunities for prospective teachers to develop their noticing skills since it has been indicated that teacher noticing can be taught (e.g., Krupa et al., 2017). In recent years, educators have focused on video-case-based professional development programs for teacher education. In the current study, it has been shown that micro-case video clips had a significant influence on PST's professional noticing abilities before teaching in real classrooms (e.g., Clarke et al., 2011; Crespo & Nicol, 2003; Jacobs & Empson, 2016). In this vein, the findings of this study have implications for how to support prospective teachers to be better able to attend to the mathematically noteworthy details of students' strategies, then interpret their understanding, and propose future instructional actions in the context of proportional reasoning. Specifically, developing expertise in deciding how to respond is challenging; however, it can be accomplished using task-based activities that target responding and long-term professional development (Jacobs et al., 2011; Star et al., 2011; Stockero, 2014; Ulusoy, 2020). Task-based activities such as analyses of focused micro-case video clips and weekly lesson plan revisions can lead to an improvement in noticing. Therefore, it can be suggested that mathematics teacher educators at universities could open courses and provide a sustained learning process using different contexts and task-based activities to improve PSTs' noticing of students' mathematical thinking through analysis and discussion of micro-case video clips.

This study highlights the critical role of socially constructed learning environments. Video-based group discussions can provide PSTs with opportunities such as working collaboratively with peers and sharing opinions about what they attend to, how they interpret what they attend to, and how they decide to respond on the basis of students' thinking. In essence, prospective teachers are more likely to select and ignore on the basis of their sense-making (i.e., attending), and that sense-making (i.e., interpreting) seems to shape their subsequent instructional actions (i.e., deciding how to respond). The evidence from the current study suggests that focused video analysis experience supports prospective teachers' ability to notice students' mathematical understanding. Discussions on various explanations help PSTs engage in a more in-depth analysis of student thinking and examine the video clips in more detail to provide more evidence to support their claims. We consider such opportunities to be highly promising for supporting mathematics teacher educators and impacting prospective teachers significantly. In this regard, this study suggests that it is important to have socially constructed learning environments in teacher education because an active, socially negotiated process helps prospective teachers notice multiple valid interpretations of students' understanding and the value of further investigation. Multiple shared interpretations in group discussions can lead prospective teachers to reconsider their future instructions and revise them based on a mutual understanding of particular issues (Goodwin, 1994; Grossman et al., 2009; Sherin, 2007). Mathematics educators could consider designing courses that provide video analysis in mathematical domains. At this point, it can be recommended to initiate video-based professional development programs, which can further support PSTs' development of professional noticing.

Studies reported that novice teachers tend to strictly follow their lesson plans, while experienced teachers are more adaptable and responsive to students' demands during instruction (Berliner, 2001). Teacher educators often propose classroom artifacts such as classroom videos for improving noticing expertise that, unfortunately, PSTs do not have (Mason, 2002; Osmanoglu, 2010). This study revealed that group discussions and weekly lesson plan revisions provided prospective teachers with an opportunity to reconsider, reconstruct, and revise their instructional decisions in initial lesson plans. Over the course of professional development, prospective teachers concentrate on students' thinking in their lesson plan revisions, and they re-approach students'

understanding in video clips with the curiosity and mindset to examine more deeply the reasoning behind students' solutions even though these solutions seem incorrect or incomplete (Goldsmith & Seago, 2017). Therefore, in order to facilitate prospective teachers' professional noticing, it might be useful to give them the opportunity to implement their revised lesson plans in real mathematics classrooms to observe the effects of the changes they make.

Another significant contribution of the current study is the descriptions for low and high levels of professional noticing expertise in the domain of proportional reasoning. A noteworthy feature of the studies is their concentration on a particular mathematical content domain (e.g., Fernández et al., 2013; Fisher et al., 2019; Sánchez-Matamoros et al., 2015; Pouta et al., 2020; van den Kieboom et al., 2017) and the extent to which teachers demonstrate varying levels of professional noticing expertise in that domain. In this study, we designed a course aimed to promote PSTs' noticing of students' thinking related to the key understandings of proportional reasoning and demonstrated that PSTs have high and low levels of professional noticing of students' thinking within key understandings of a particular topic. Hence, it can be stated that this study has made an important contribution to the literature by addressing professional noticing within a particular domain and across different mathematical domains. Noticing interventions can be designed to further explore students' thinking within a specific domain, and the key understandings of each mathematical domain are significant as they serve as a map for intervention.

5.4. Limitation of the Study and Suggestions for Further Research

This study focused on developing prospective teachers' professional noticing of students' thinking in the context of proportional reasoning. Micro-case video clips were used to prompt prospective teachers' noticing expertise in a situated and contextualized manner. These video clips included one-on-one student interviews. Using only one type of video clip rather than classroom videos as an authentic representation of classroom practice may be a limitation of the current study. However, this preference was a deliberate decision because this study highlights the importance of PSTs being prepared to notice, irrespective of the video type. Improving noticing

skill is the main aim here because in the future, PSTs may make more logical decisions thanks to the increase in their awareness about noticing through interventions (Mason, 2002). Future studies might still be recommended to explore the development of prospective teachers' professional noticing of students' thinking using different video clips such as micro-case video clips and classroom videos.

In this study, the prospective teachers did not interact with students. This might be another limitation of this study that requires further investigation. Future research may include interaction between prospective teachers and students because previous studies posited that interaction with students gives teachers the opportunity to revise their lesson plans in the moment as they implement them, which especially contributes to the development of the skill of deciding how to respond (Philip et al., 2017).

The current study concentrated on proportional reasoning, a particular mathematical domain. Proportional reasoning is a complex subject for students to learn and for teachers to teach (e.g., difference between additive and multiplicative situations). Therefore, the study aimed to provide specific support to promote prospective teachers' noticing skills so that they can attend to and interpret students' thinking about the key understandings of proportional reasoning and make appropriate instructional decisions. The choice of this domain allowed the prospective teachers in the study and the researchers to learn about and deeply notice students' thinking about the key understandings of proportional reasoning. On the other hand, this preference can be seen as a limitation as PSTs may not go beyond and notice students' thinking in other mathematical domains. Therefore, mathematics educators should seriously consider designing courses that engage prospective teachers across a variety of mathematical domains via micro-case-based development programs.

Finally, this study concentrated only on prospective teachers' professional noticing of students' thinking in the context of proportional reasoning, and it was conducted with six prospective teachers. The number of participants in this study was limited in terms of showing the development of prospective teachers' noticing skills in the domain of proportional reasoning. Therefore, future studies might investigate noticing with more prospective teachers to identify the development of prospective

teachers' professional noticing in different domains. However, it must be noted that the researcher purposefully selected a small sample size to deeply investigate the participants' professional noticing of students' mathematical thinking individually and holistically in the domain of proportional reasoning.

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APPENDICES

A. PRE/POST INTERVIEW TASKS

Parkur Problemi

Rukiye ve Neşe bir pistin etrafında eşit hızla koşmaktadır. Koşmaya önce başlayan Neşe, 4 tur koştuğunda Rukiye 2 tur koşmuştur. Buna göre Rukiye 6 tur koştuğunda, Neşe kaç tur koşmuş olacaktır? (Sorunun çözümünü ayrıntılı şekilde yazınız)



Manav Problemi

Ayşe A manavından aldığı 4 adet muza 3,6 TL ödemiştir. Berna ise B manavından aldığı 3 adet muza 3,3 TL ödemiştir. Hangi manavdan muz almak daha hesaplıdır? (Sorunun çözümünü ayrıntılı şekilde yazınız)



Balon Problemi

Derya, Ahmet ve Kaan üç tane balona 1,5 TL ödemişlerdir. 24 kişilik sınıflarına balon aldıkları takdirde kaç lira ödemeleri gerekmektedir? (Sorunun çözümünü ayrıntılı şekilde yazınız)

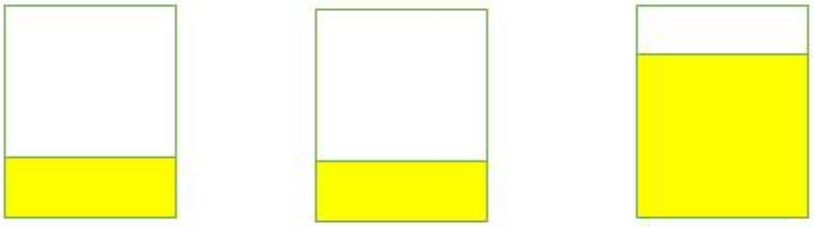


Fotoğraf problemi



Aynur, kardeşi Ahmet ile fotoğrafının boyutunu fotokopi makinesinde yukarıda olduğu gibi büyütmüştür. Buna göre büyütülmüş fotoğrafın alanı orijinal fotoğrafın alanının kaç katıdır? (Sorunun çözümünü ayrıntılı şekilde yazınız.)

Karışım Problemi



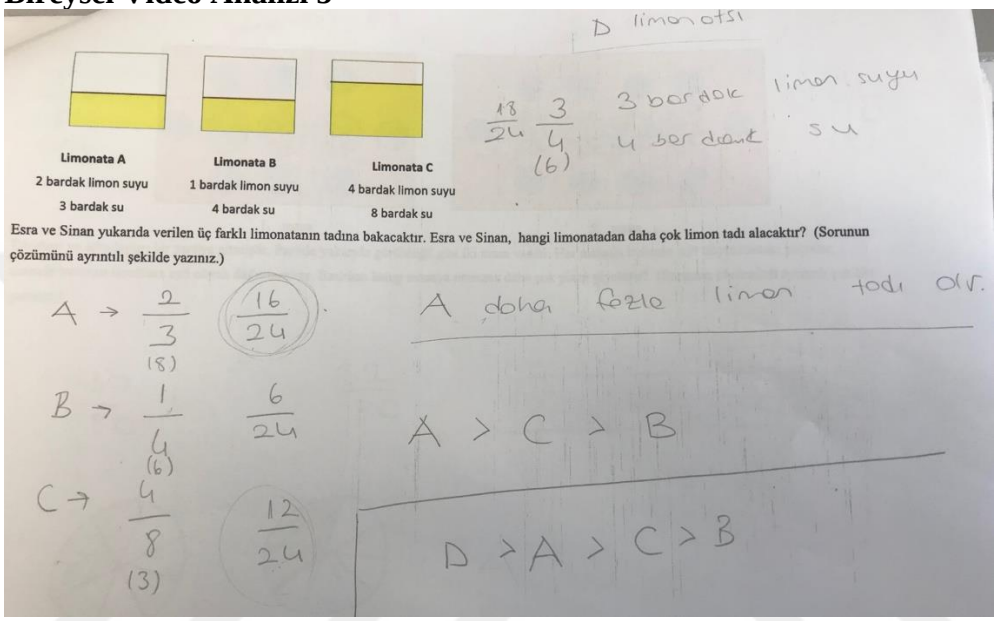
Limonata A	**Limonata B**	**Limonata C**
2 Bardak limon suyu	1 Bardak limon suyu	4 Bardak limon suyu
3 Bardak su	4 Bardak su	8 Bardak su

Esra ve Sinan yukarıda verilen üç farklı limonatanın tadına bakacaktır. Esra ve Sinan, hangi limonatadan daha çok limon tadı alacaktır? (Sorunun çözümünü ayrıntılı şekilde yazınız)

B. AN EXAMPLE OF REFLECTION PAPER

Adı soyadı
Tarih

Bireysel Video Analizi 5



1. Öğrencinin soruyu nasıl çözdüğünü detaylı bir şekilde anlatınız?
2. Bu çocuğun düşünüşü hakkında neler öğrendiğinizi detaylı şekilde açıklayınız? Bu çocuğun çözümünü nasıl yorumlarsınız?
3. Siz bu çocuğun öğretmeni olsanız, bir sonraki aşamada çocuğa ne tür bir öğretimsel yaklaşımlar sergilerdiniz?

C. SAMPLE INFORMED CONSENT FORM

1956

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY
06531 ANKARA-TURKEY

Matematik ve Fen Bilimleri Eğitimi Bölümü
Department of Mathematics and Science Education

Tel: 90 (312) 210 40 86
Faks:90 (312) 210 79 71

EK – 1 Gönüllü Katılım Formu

Değerli katılımcı,

Ben Öznur Soyak. Orta Doğu Teknik Üniversitesi Eğitim Fakültesi, İlköğretim Fen ve Matematik Eğitimi Anabilim Dalı'nda devam ettiğim doktora eğitimimde tez aşamasına gelmiş bulunuyorum. Bu çalışmada, tez danışmanlarım Prof. Dr. Erdinç Çakıroğlu ve Prof. Dr. Mine Işıksal Bostan ve ile birlikte, ortaokul öğrencilerinin yazılı cevaplarını ve matematiksel düşünüşünü içeren videolarını kullanarak ilköğretim matematik öğretmenliği son sınıf öğretmen adaylarının farkındalıklarının geliştirilmesi amaçlanmaktadır.

Araştırmaya katılım gönüllü olup, katılımcıların sonradan vazgeçmesi halinde herhangi olumsuz bir sonuç oluşmayacaktır. Araştırma sırasında toplanan veriler sadece araştırmacının bilgisi dâhilinde olup gerek diğer katılımcılar gerekse başka şahıslar tarafından bilinmeyecektir. Araştırma raporunda okul, katılımcı öğrenci ve öğretmenlerin ismi hiçbir şekilde aynen geçmeyecek, isim kullanılması gerekirse takma isim kullanılacaktır. Elde edilecek bulgular tez çalışmasında ve bilimsel yayımlarda akademik etik kurallarına dikkat edilerek kullanılacaktır. Çalışma süresince katılımcılar için potansiyel bir risk öngörülmemektedir. Ancak, katılım sırasında farklı amaçlarla toplanan veya alınan dersin gerekleri olarak toplanacak verilerin bilimsel çalışma ve tez çalışması amaçları çerçevesinde kullanılmamasını isteyebilirsiniz. Bu durum ders performansınızın değerlendirilmesinde kesinlikle negatif bir durum oluşturmayacaktır.

Çalışma hakkında daha fazla bilgi almak için ODTÜ Eğitim Fakültesi Matematik ve Fen Bilimleri Eğitimi Bölümü öğretim üyesi Prof. Dr. Mine Işıksal Bostan ve Öznur Soyak ile iletişim kurabilirsiniz. Bu çalışmaya katıldığınız için şimdiden teşekkür ederiz

Bu çalışmaya tamamen gönüllü olarak katılıyorum ve istediğim zaman yarıda kesip çıkabileceğimi biliyorum. Verdiğim bilgilerin bilimsel amaçlı yayımlarda kullanılmasını kabul ediyorum. (Formu doldurup imzaladıktan sonra uygulayıcıya geri veriniz).

İsim Soyad	Tarih	İmza
	----/----/-----	

Examples of instructional actions presented in lesson plans

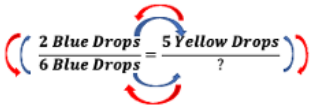
Example parts of …

Initial lesson plan (before video analysis)	Revised lesson plan (after video analysis, W3-W4)

Seher decides to mix two colors to reach to one shade of the green for cupcakes. She tries to obtain the shade of the green she wants by mixing blue and yellow in different quantity, and finally she reaches to the recipe. According to the recipe for green cupcakes, she mixes 2 drops of blue and 5 drops of yellow food coloring. However, she wants to increase the blue food coloring about 4 more drops for making enough cupcakes. How many drops of yellow food coloring should be needed to use to keep the shade of the green same?

- **Part-to-part Interpretations**

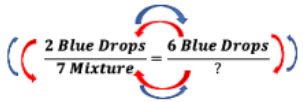
$$\frac{\textbf{2 Blue Drops}}{\textbf{6 Blue Drops}} = \frac{\textbf{5 Yellow Drops}}{?}$$



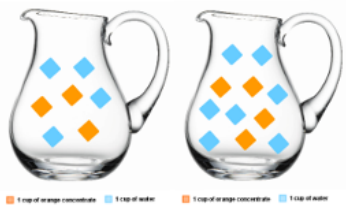
- $\frac{1}{3} = \frac{5}{x}$ x=15 yellow drops or $\frac{3}{1} = \frac{x}{5}$ x=15 yellow drops.
- $\frac{2}{5} = \frac{6}{x}$ x=15 yellow drops or $\frac{5}{2} = \frac{x}{6}$ x=15 yellow drops.

- **Part-to-whole Interpretations**

$$\frac{\textbf{2 Blue Drops}}{\textbf{7 Mixture}} = \frac{\textbf{6 Blue Drops}}{?}$$



- $\frac{2}{7} = \frac{6}{x}$ x=21 mixture, 21-6=15 or $\frac{7}{2} = \frac{x}{6}$ x=21 mixture, 21-6=15.
- $\frac{2}{6} = \frac{7}{x}$ x=21 mixture, 21-6=15 or $\frac{6}{2} = \frac{x}{7}$ x=21 mixture, 21-6=15.



According to the picture, an orange square represents a cup of orange concentrate and a blue square represents a cup of water to prepare an orange juice. There are two different orange juice in terms of amount of water and orange concentrate. Which fruit juice does have more orange flavour?

Expected Answer 1:

Creating ratio tables is one of the expected answers. Students can see the multiplicative relationship between two quantities clearly thanks to ratio table. In the following solution, students create a ratio table showing the relationship between the amount of water and orange concentrate in each mixture. In other words, they try to see how the amount of orange concentrate varies when we increase the amount of water in each mixture or vice versa. In here, we can say that they seek the part to part relationship in this question in order to find which mixture has more orange flavour because they create a ratio between orange concentrate and the water. In this way, students equalize the amount of orange concentrates in each two mixture. By using the properties of the ratio table, they could establish the multiplicative relationship. Also, it can be said that they use the composed unit or covariation method by creating these tables. Because of multiplicative relationship in ratio, these two quantities (the amount of water and orange concentrate) vary together. It is covariation method. In here, quantities are iterated several times to find a common point in these two mixture. Students use the scaling up in these ratio tables while iterating the quantities. In other words, they increase the amount of the water and orange concentrate in the same ratio. When 12 cups of orange concentrate are used in the first mixture after the iteration, 16 cups of water are used due to multiplicative relationship. Furthermore, when 12 cups of orange concentrate are used in the second mixture, 21 cups of water are used. If we compare these, we can say that 12 cups of orange concentrate are used in each mixture, but the amounts of water in each mixture are different. In here, because 21 cups of water are used in the second mixture while 16 cups of water are used in the first mixture, the first mixture has more orange flavour than second one. More water are added to the same amount orange concentrate in each mixture. (21 cups of water > 16 cups of water)

Example parts of …

Revised lesson plan (after video analysis, W3-W4)

First Mixture	Orange Concentrate	3	6	9	12
	Water	4	8	12	16

Second Mixture	Orange Concentrate	4	8	12
	Water	7	14	21

Expected Answer 5:

I. Mixture

$$\frac{3 \text{ Cups of Orange Concentrate}}{4 \text{ Cups of Water}} = \text{Orange Concentrate per 1 Cup of Water}$$

II. Mixture

$$\frac{4 \text{ Cups of Orange Concentrate}}{7 \text{ Cups of Water}} = \text{Orange Concentrate per 1 Cup of Water}$$

Students can solve this question by finding the unit rate. If they find the ratio of orange concentrate to water, they can reach the amount of orange concentrate per 1 cup of water. In here, students divide the amount of orange concentrate by the amount of water. By doing this, they use the within ratio in this question because they establish a ratio between the orange concentrate and water in the first mixture, and they do the same for the second mixture. If we compare the ratios, it is found that $\frac{3}{4}$ is greater than $\frac{4}{7}$. This means that we use more orange concentrate for 1 cup of water in the first mixture. In other words, when we add 1 cup of water to the mixtures, we add more orange concentrate to the first mixture than the second mixture. Therefore, the first mixture has more orange flavour than the second one.

Expected Answer 6:

I. Mixture

$$\frac{4 \text{ Cups of Water}}{3 \text{ Cups of Orange Concentrate}} = \text{Water Amount per 1 Cup of Orange Concentrate}$$

II. Mixture

$$\frac{7 \text{ Cups of Water}}{4 \text{ Cups of Orange Concentrate}} = \text{Water Amount per 1 Cup of Orange Concentrate}$$

Students can solve this question by finding the unit rate. If they find the ratio of water to orange concentrate, they can reach the amount of water per 1 cup of orange concentrate. In here, students divide the amount of water by the amount of orange concentrate. By doing this, they use the within ratio in this question because they establish a ratio between the orange concentrate and water in the first mixture, and they do the same for the second mixture. If we compare the ratios, it is found that $\frac{7}{4}$ is greater than $\frac{4}{3}$. This means that we use more water for 1 cup of orange concentrate in the second mixture. In other words, when we add 1 cup of orange concentrate to the mixtures, we add more water to the second mixture than the first mixture. Therefore, the first mixture has more orange flavour than the second one.

3) Sue and Julie are running around a stadium. When Sue takes 3 laps, and Julie has 9 laps, if Julie takes 15 laps, how many rounds will Sue have?

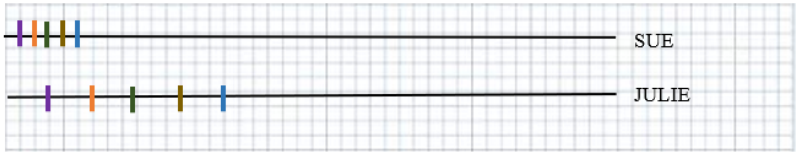
- Teacher says: You will work on groups for this question. I will give you 5 minutes to think and solve this question. I will ask you to explain some of the solutions I chose on the board. Your time is starting.

  Expected Answers:

  - Multiplicative reasoning: $\frac{3}{9} = \frac{x}{15}$, then 3*15=9*x. So, x is 4. (true)
  - Additive reasoning: Sue is up for 6 laps of Julie. So, if Julie is 15 laps, then Sue is 15+6=21 laps. (wrong)

- Teacher asks questions while discussion;
  - What are the similarities between this question and the previous question?
  - what are the differences between this question and the previous question?
  - How did you understand Sue and Julie have different rate of running?
  - What kind of relationship does Sue and Julie's rate have? Why?
  - Why there is not additive relationship?
  - How can we understand it is multiplicative reasoning rather than additive?
  - What is the multiplicative reasoning?

LAPS	**1**	**2**	**3**	**4**	**5**	**6**	**7**
SUE	1 (x3)	2 (x3)	3 (x3)	4 (x3)	5 (x3)	6 (x3)	7 (x3)
JULIE	3	6	9	12	15	18	21



[redacted] March 18, 2019
Yukardaki additive sorunun ardından, aynı concept te multiplicative soru eklemek istedim çünkü bugün stajda yukardaki soruyu sınıfta çözdüm ve çocuklar aralarında toplamsal bir ilişki olduğunu anladıktan (arlarındaki farkın hep 6 olduğunu) sonra bile hala '' yani oran-orantı var yani demi dediler. İki çokluk arasında toplamsal veya çarpımsal bir ilişki olabilceğinin ve farklerının ne olduğunu bilmiyorlar. Orantıyı da anlamlandıramamışlar tıpkı videodaki öğrenci de olduğu gibi. Grup tartışmasında öğrencilerin ezbere stratejiler kullanmasından kaynaklı verilen değişkenler arasındaki ilişkileri göremediklerini farkettim. Bu yüzden öğrencilerin düşünüşlerini de ortaya çıkarabilmek için bu soruları kullanmayı amaçladım. Bugünkü deneyimimden sonra additive reasoning sorunun hemen arkasından multiplicative reasoning gerektiren verip, aralarındaki farkı ve katsal ilişkiyi daha rahat anlamlandırmaları için bu soruyu ekledim.

[redacted]
This prospective teacher proposed two instruction suggestions while concentrating on the students' thinking in the school experience. She provided the details about proposed instructional actions for whom and for what purposes. Furthermore, this prospective teacher made revisions and explained the instructional strategy related to the student's understanding of the video by referencing the group discussion. Therefore, this instructional action was categorized as case-specific-detailed.

Reply Resolve

F. APPROVAL OF THE METU HUMAN SUBJECTS ETHICS COMMITTEE

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

DUMLUPINAR BULVARI 06800
ÇANKAYA ANKARA/TÜRKEY
T: +90 312 210 22 91
F: +90 312 210 79 59
ueam@metu.edu.tr
www.ueam.metu.edu.tr

Sayı: 28620816 /81

20 Şubat 2019

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Prof.Dr. Mine Işıksal BOSTAN

Danışmanlığını yaptığınız ***Öznur SOYAK'ın*** **"Ortaokul matematik öğretmen adaylarının öğrenci düşünüşlerini içeren video ve yazılı kaynak temelli eğitim ortamlarında orantısal düşünme farkındalıklarının gelişimi"** başlıklı araştırması İnsan Araştırmaları Etik Kurulu tarafından uygun görülmüş ve **072-ODTÜ-2019** protokol numarası ile onaylanmıştır.

Saygılarımızla bilgilerinize sunarız.

Prof. Dr. Tülin GENÇÖZ
Başkan

Prof. Dr. Ayhan SOL
Üye

Prof. Dr. Ayhan Gürbüz DEMİR
Üye

Prof. Dr. Yaşar KONDAKÇI
Üye

Doç. Dr. Emre SELÇUK
Üye

Doç. Dr. Pınar KAYGAN
Üye

Dr. Öğr. Üyesi Ali Emre TURGUT
Üye

G. CURRICULUM VITAE

Middle East Technical University Faculty of Education Department of Mathematics and Science Education Cankaya/Ankara, Turkey 06800

Professional Experience

2016 – 2018

Research and Teaching Assistant / METU Elementary Mathematics Education

ASSISTED COURSES

- ESME341 Methods of Teaching Mathematics I
- ESME342 Methods of Teaching Mathematics II
- SSME455 Mathematical Modeling for Teachers

Education

2015– 2023

Doctor of Philosophy / METU Elementary Mathematics Education

Thesis Title:

Developing Prospective Teachers' Professional Noticing of Students' Proportional Reasoning Through Video Case Enhanced Learning Environment

Courses Taken

- Quantitative Data Analysis in Education
- Test Construction in Sci. and Math. Educ.
- Cognitive Development in Science Math. Educ.
- Design Research in Education
- Analysis of Research in Elementary Math. Science Educ.
- Digital Tools in Qualitative Inquiry
- Advanced Educational Research
- Independent Study in Elementary Math. Science Educ.
- Technology in Mathematics Education
- Advanced Study of Problem Solving in Mathematics and Science Education
- Qualitative Research in Elementary Education

2011–2012

Master of Arts / University of Leicester, UK, International Education

Thesis:

The Analyzing the Students' and Teachers' Perspectives Toward Tablet PCs in a Selected Primary School in Bingol Turkey , 2012.

Courses Taken

- Contemporary Issues in International Education
- Innovation & Reform Lead Module
- Dissertation
- Introduction to Educational Research Methods
- Innovation & Reform Specialist Module

2010–2011

English Preparatory Program / Indiana University, USA

2005–2009

Bachelor of Science / Balikesir University

Faculty of Education, Elementary Mathematics Education

References

Publications

Journal Publications Covered in the SCI-Expanded List

[1] Karatas, F.I., **Soyak, Ö**. and Alp, A., 2017, February. The investigation of mathematical problem-solving processes of fifth grade students in small groups. In *CERME 10*.

[2] Bikmaz, F.H., ÇELEB, Ö., Aslıhan, A.T.A., Eren, Ö.Z.E.R., **Soyak, Ö.** and Reçber, H., 2010. Scaffolding strategies applied by student teachers to teach mathematics. *The International Journal of Research in Teacher Education*, *1*(3), pp.25-36.

[3] **Soyak, Ö.** and Isiksal, M., 2017, February. Middle school students' difficulties in proportional reasoning. In *CERME 10*.

[4] **Soyak, Ö.** and Yildiz, D.G, 2018, September. Prospective Middle School Mathematics Teachers' Productive Noticing of Misconceptions in/during Lesson Planning. In *ECER 2018 European Conference on Educational Research*.

[5] **(National Conference, In Turkish)**

Soyak, Ö. and Işıksal M., 2016, September. İlköğretim Matematik Öğretmen Adaylarının Orantısal Akıl Yürütme Çözüm Stratejileri. *XII. Ulusal Fen Bilimler ve Matematik Eğitim Kongresi (UFBMEK)*, Trabzon, TURKEY.

Areas of Interest

- Mathematics Education
- Students' Reasoning
- Special Education
- Professional Development
- Number Sense
- Teaching and Learning

Skills

- Microsoft Office
- Nvivo
- GeoGebra
- SPSS
- The Geometer's Sketchpad
- MAXQDA

H. TURKISH SUMMARY / TÜRKÇE ÖZET

VİDEO VAKALARIYLA ZENGİNLEŞTİRİLMİŞ ÖĞRENME ORTAMI İLE ÖĞRETMEN ADAYLARININ ÖĞRENCİLERİN ORANTISAL AKIL YÜRÜTMELERİNİ FARK ETME BECERİLERİNİN GELİŞTİRİLMESİ

Giriş

Fark etme becerisi son on yılda kapsamlı bir şekilde matematik eğitimcileri tarafından ilgi duyulan kavramlardan biri olarak karşımıza çıkmaktadır (Jacobs ve Spangler, 2017). Fark etme insanların karmaşık durumlara nasıl odaklandıkları ve bu durumları nasıl anlamlandırdıkları olarak tanımlanmaktadır (Jacobs vd., 2010). Bu terim ya genel durumları ya da belirli bir durumun farklı özelliklerini tanımlar. Fark etme becerisi çeşitli açılardan tanımlanmış olsa da tüm tanımlar ortak olarak öğretimle ilgili karar alma sürecince gerçekleşen olayları tanımlamanın ve bu ilgili olayları yorumlamanın önemini vurgulamaktadır (Jacobs vd., 2010; Mason, 2002, 2011; Sherin vd., 2011; van Es, 2008). Ancak, öğretim çok boyutlu bir etkinlik olduğundan, öğretmenler her sınıf olayını göremeyebilir ve yanıtlayamayabilir. Aynı olaylar gözlemlenmesine rağmen çıkarılan anlam ve yorumlar farklı olabilmektedir (Jacobs vd., 2010). Neyin fark edildiğinin net bir şekilde ortaya konması fark etme becerisi için gerekli olup yeterli değildir. Neyin fark edildiği, fark edilen detayların ya da olayların nasıl anlamlandırıldığı ve bu bağlamla nasıl öğretimsel kararların alındığı birbiriyle ilişkili olarak ele alınmalıdır (Jacobs vd., 2010; Star ve Strickland, 2008; van Es, 2011; van Es ve Sherin, 2002). Öğretmenler, öğrencilerin matematiksel düşüncelerine ve önemli sınıf olaylarına dikkat etme ve bunları yorumlamayı öğrenmede yardıma ihtiyaç duyarlar. Başka bir deyişle, fark etme becerisi, öğrenci düşünmesini desteklemek için geliştirilmesi gereken bir uygulamadır. Çalışmalar bu sürecin doğal bir şekilde gelişmediğini, fakat öğrenilebildiğini göstermiştir. Fark etme becerileri van Es ve Sherin tarafından iki yönlü olarak ele alınmaktadır. Bunlar, "(a) öğrenme ortamında neyin önemli olduğunu tanımlama, (b) bu durumlar hakkında akıl yürütmek için

öğrenme ve öğretme bilgisinde yararlanma" (van Es, 2011, p. 138) şeklinde ifade edilmiştir.

Öğretmenlerin öğrenme ortamında vermiş olduğu kararlar, gözlemledikleri durumların doğasından ve bu gözleme dayalı yapılan yorumların karmaşıklığından etkilenmektedir (Mason, 2002, 2011). Bu doğrultuda araştırmacılar, öğrencilerin stratejilerine katılmalarını, öğrenci düşüncelerini yorumlamalarını ve bunlara yanıt verme becerisini geliştirmeye yönelik öğretmenlerin fark etme becerilerine odaklanmışlardır (Jacobs vd., 2010; Mason, 2002, 2011; Sherin vd., 2011; van Es ve Sherin, 2008).

Çalışmalar, öğretmen adaylarının fark etmeyi öğrenme süreçlerinde önemli zorluklarla karşılaştıklarını göstermiştir (Peterson ve Leatham, 2009; Stockero, 2008). Öğretmen adaylarının öğrencilerle sınırlı deneyimleri, öğrenci düşünüşlerine ilişkin bilgilerinin sınırlı olmasına neden olmaktadır. Bu nedenle öğretmen adaylarının öğrenci düşüncelerini fark etmeyi öğrenmeleri ve bu fark etme becerilerini geliştirmeleri desteklenmelidir (Stockero ve Rupnow, 2017). Öğretmen adayları öğrenci düşüncesine nasıl yanıt vereceklerini ve onu genişleteceklerini öğrendikçe, öğrenci düşüncesi öğretmen öğrenimi için önemli bir kaynak haline gelmektedir. Bu doğrultuda öğretmen yetiştirme programlarında öğretmen adaylarının fark etme becerisinin geliştirilmesi büyük önem kazanmıştır.

Yapılan araştırmalar, fark etme becerilerinin spesifik matematik öğrenme alanlarında öğrenilmesinin önemini vurgulamıştır (Carpenter vd., 1988, 1989; Dreher ve Kuntze, 2015; Jacobs vd., 2010). Bu nedenle, bu çalışmada öğretmen adaylarının mesleki fark etme becerilerinin gelişimini desteklemek için spesifik matematik konusu olarak oran ve orantı konusu seçilmiştir. Oran ve orantı konusunu anlamlandırmanın temel şartının orantısal akıl yürütme becerisine sahip olunmasıdır. Orantısal akıl yürütme, ilkokul ve ortaokul matematiği de dahil olmak üzere okul matematiğinin temelinde yer aldığı için çocukların matematiksel gelişiminde önemli bir rol oynamaktadır (Lamon, 2007). Amerikan Ulusal Matematik Öğretmenleri Konseyi (NCTM) orantısal akıl yürütmenin önemine vurgu yaparak geliştirilmesinin desteklenmesi gerekliliğini ifade etmiştir (NCTM, 1989).

Lesh ve arkadaşları (1988) orantısal akıl yürütmeyi kovaryasyon ve çarpımsal karşılaştırmaların anlamlandırma yeteneği olarak tanımlamaktadır. Orantısal akıl ürütmenin temel özellikleri bilinmesine rağmen, okullarda ve ders kitaplarındaki tanımı yüzeysel ve sınırlı şekilde verilmektedir. Ulusal ve uluslararası ders kitaplarında orantısal akıl yürütme $\frac{a}{b}=\frac{c}{d}$ şeklinde içler dışlar algoritması şeklinde verilmekte olup verilmeyen değişkeni bulma şeklinde ezber metotlarla öğrencilere verilmektedir (Duatepe vd., 2005; Karplus vd ., 1983; Lamon, 1994; Lesh, vd., 1988). Lamon (1995), ders bu kavramın öğrenci anlamaları için yeterli temel atılmadan önce sembollerin tanıtıldığı tek bir bölümde öğretildiğine dikkat çekmiştir. Ancak orantısal durumlar içeren problemlerin ezberlenmiş algoritmalar kullanılarak çözülmesi, öğrencilerin orantısal akıl yürütme becerisine sahip olduklarının bir kanıtı olarak görülmemesi gerektiği ifade edilmektedir (Cramer ve Post, 1993a, 1993b; Hart, 1988). Bunun nedeni orantısal akıl yürütmenin verilen değişkenler arasındaki ilişkilerin keşfedilmesi ve anlamlandırılması orantısal akıl yürütme becerisi ile mümkün iken, ulusal ve uluslararası çalışmalar öğrencilerin bu becerilerinin yeterince gelişmediğini, bu yüzden verilen bağlam içerisindeki ilişkilerin anlamlandıramadıklarını ve birçok kavram yanılgılarına sahip olduklarını ortaya koymuştur (Çelik ve Yetkin Özdemir, 2011; Duatepe vd., 2005; Kayhan vd., 2004; Lamon, 2007; Tourniaire ve Pulos, 1985).

Sadece öğrenciler değil öğretmenler ve öğretmen adayları da orantısal akıl yürütme becerilerinde benzer sınırlılıklara ve kavram yanılgılarına sahiptir (Lamon, 2007). Bu nedenle öğretmen adayları sınıflara girmeden önce öğrencilerin orantısal akıl yürütme bağlamında nasıl düşündüklerini öğrenmeleri gerekmektedir. Araştırmalar, öğrencilerin nasıl düşündükleri üzerine inşa edilen ve düşünme şekillerini ortaya çıkarmaya yönelik öğretimsel kararların öğrencilerin ilgili matematik konusunu anlamalarını ve anlamlandırmalarını desteklediğini ortaya koymuştur (Fennema vd., 1993).

Bu çalışma, orantısal akıl yürütme bağlamında öğretmen adaylarının öğrencilerin düşüncelerini fark etme becerilerinin nasıl geliştirilebileceği konusunda araştırmacılara rehberlik etme potansiyeline sahiptir. Öğretmen adaylarının fark etme becerilerinin gelişimi orantısal akıl yürütmenin temel bileşenleri ışığında ele alınmıştır; (a) Toplamsal ve çarpımsal düşünme, (b) Birimleme ve birleşik birimleri

biçimlendirme ve (c) orantılı durumları anlamlandırma. Bu bileşenler Carpenter ve meslektaşları tarafından geliştirilen orantısal akıl yürütme rotasına dayanmaktadır (Carpenter vd., 1999). Bu öğrenme rotası 4 seviyeden oluşmaktadır. Seviye 1'de öğrenciler çoğunlukla toplamsal farka odaklanmaktadır ve rastgele işlemler yaparak problemleri çözmeye çalışmaktadır. Seviye 2'de öğrenciler oranı sadece birim faktör yaklaşımına indirgeyerek, eşdeğer oranları birimleri tekrarlı toplayarak veya tam sayılar ile çarparak oluştururlar. Ancak bu seviyedeki öğrenciler, değişkenler ile tamsayı barındırmayan işlemleri gerçekleştiremezler. Seviye 3'te öğrenciler Seviye 2' de olduğu gibi oranı sadece birim faktör yaklaşımına indirgeyerek algılasalar da, değişkenler ile tamsayı barındırmayan işlemleri de gerçekleştirebilirler. Bu seviyedeki öğrenciler, toplamsal ve çarpımsal tekrarlamalı arttırma veya azaltma (build up-build down) stratejileri kullanabilmektedir. Seviye 4'te ise öğrenciler oranı birim faktör yaklaşımına indirgemeden değişkenler arasındaki orantısal ilişkileri anlamlandırır. Bu yaklaşım öğrencilere stratejilerde esneklik sağlamaktadır. Öğrenciler verilen bağlamlar içerisindeki orantısal durumları anlamlandırarak, en uygun stratejiyi seçebilme becerilerine sahiptir.

Orantısal akıl yürütme becerilerinin gelişiminde yaşanan en büyük zorluklardan biri toplamsal düşünmeden çarpımsal düşünmeye geçiş sürecidir ve orantısal durumlar içeren problemleri çözmek için toplamsal stratejileri tercih ettikleri birçok çalışma ile ortaya konmuştur (ör., Duatepe vd., 2005; Fernández vd., 2012; Hart, 1988; Jacobs ve Willis, 2001; Kaput ve West, 1994; Karplus vd., 1983; Lamon, 1993, 1994; Noelting, 1980a, 1980b; Resnick ve Singer, 1993; Tourniaire, 1986; van Dooren vd., 2010). İlkokulun ilk yıllarından itibaren öğrenciler toplamsal düşünme gerektiren problemlerle karşılaştıkları için çarpımsal düşünme gerektiren problemlere de toplama işlemi ya da tekrarlı toplama işlemi uygulamaktadır (van Dooren vd., 2010). Bu bağlamda, toplamsal düşünmenin çarpımsal düşünmeye temel oluşturması ancak sistematik öğretimden sonra ortaya çıkmaktadır (van Dooren vd., 2009, s.191). Ayrıca, özellikle ilköğretim düzeyi ve sonrasındaki öğrencilerin (13 yaş ve sonrası) toplamsal düşünmeyi gerektiren problemler için de çarpma işlemi kullanarak çözdükleri görülmüştür (Cramer vd., 1993).

Son (2013) öğretmen adaylarının öğrencilerin benzer geometrik şekillerde verilmeyen kenarının uzunluğu bulurken yaptıkları hatalara verdikleri yanıtları incelemiştir. Öğretmen adaylarının çoğu yanlış çözümlerde işlemsel hatalara odaklanmıştır ve öğretmenlerin toplamsal ve çarpımsal düşünmeyi tanımlamadaki kavramsal eksiklerinden dolayı öğrencilerin anlamalarını yorumlayamadıkları gözlemlenmiştir. Bu yüzden öğretmenlerin öğrencilerin toplamsal ve çarpımsal ilişkileri nasıl anlamlandırdıklarını fark etmeleri becerilerinin gelişmesi için ilk öğretim deneyi oturumu Carpenter ve arkadaşlarının da önerdiği gibi toplamsal düşünme ve çarpımsal düşünmenin anlamlandırılıp ayırt edilebilmesine odaklanmaktadır.

Birimleme ve birleşik birimleri biçimlendirme orantısal akıl yürütmenin öğrenilmesi ve yapılandırılmasında kritik bileşenler olarak ortaya konmuştur (Carpenter vd., 1999; Lamon, 1995). Lamon' a göre birimleme ve biçimlendirme verilen bağlamda 1 birimi anlamlandırma veya bu bir birimi referans alarak yeni durumu yorumlama olarak tanımlanmıştır (Lamon, 1994). Yapılan çalışmalar öğrencilerin birim faktör yaklaşımını kullanma eğiliminde olduklarını göstermektedir. Fakat bu yaklaşım her problemin çözümü için ideal bir strateji olmadığını göstermektedir (Tourniaire & Pulos, 1995). Öğrencilerin orantısal ilişkileri anlamlandırabilmesi oran ve orantının öğretilmesinden önce sezgisel birimleme (unitizing) ve birimleri biçimlendirme (norming) stratejilerine sahip olmasına ve bu durumun ışığında birleşik birim oluşturma (composite unit), birleşik birimleri bağlama (linking composite units) ve birleşik birimleri bağlama (iterating composite units) becerilerinin gelişimine dayanmaktadır (Battista ve Borrow, 1995; Kaput ve West, 1994; Lamon, 1994, 1995; Steffe, 1988, 1994).

Alanyazın taramasında, öğrencilerin orantısal akıl yürütmenin bir parçası olarak birimleme ve birimleri biçimlendirmeye yönelik düşüncelerinin öğretmen adayları tarafından nasıl fark edildikleri ele alınmıştır (Buforn ve diğ., 2022). Bu çalışmada öğretmen adaylarının yarısından fazlasının (91 öğretmenden 65'i) orantısal problemleri çözmek için tekli birimleri kullandıkları (birim faktör yaklaşımı) görülmüştür. Öte yandan, birim oran yaklaşımında hata yapan öğretmen adaylarının, birimleme sürecinde de güçlükler yaşadığını ortaya koymuştur.

Sonuç olarak, çalışmalar birimleme ve birimleri biçimlendirmenin orantısal akıl yürütmenin gelişimi için kritik olduğunu ortaya koymuştur (ör., Lamon, 1994; Steffe, 1994). Buradan yola çıkarak, öğretim deneyinin ikinci ve üçüncü oturumlarında birleşik birimler oluşturarak ve bu birleşik birimleri yeni durumlarda kullanarak öğrencilerin değişkenler arasındaki kovaryasyon (covariance) ve değişmezlik (invariance) ilişkilerini ve deneyimledikleri zorlukları nasıl anlamlandırdıkları ve öğretmen adaylarının bu bağlamda öğrenci düşüncelerini fark etme becerilerinin gelişimi hedeflenmiştir.

Carpenter ve arkadaşları öğrenme rotasının son seviyesinde öğrencilerin değişkenleri arasındaki orantısal ilişkileri anlamlandırmalarına odaklanmıştır. Bu sayede öğrenciler verilen bağlamlar içerisindeki orantısal durumları anlamlandırarak, en uygun stratejiyi seçebileceklerdir (Carpenter vd., 1999). Orantısal ilişkilerin anlamlandırılması, öğrencilerin orantısal akıl yürütmeye dayalı olarak yargılarda bulunmalarına izin veren üst düzey akıl yürütmenin kritik bir parçası olarak görülmüştür (Piaget & Inhelder, 1975). Çocuklarda orantısal muhakeme deneyleri, nitel yargılardan (ör., az ya da çok, ileri ya da yakın) sayıları kullanarak nicel yorumlara ve farklı bir bağlamdaki karşılaştırma problemlerine (ör., ölçekleme ya da karışım problemleri) kadar çeşitli görevleri içerir (Ayan ve Işıksal, 2019; Bruner ve Kenney, 1966; Cramer ve Post, 1993). Bu çalışmaların hepsi ortak bir temayı paylaşmaktadır; orantısal durumlar içeren problemleri çözmek için, ilişkiler arasındaki ilişkilerin kavranması (Piaget ve Inhelder, 1975). Alanyazın taramasında, orantısal ilişkiler bazı çalışmalarda aynı ve farklı ölçüm uzayı bağlamlarında (ör., Lamon, 1994; Lesh et al., 1988; Vergnaud, 1994), bazı çalışmalarda ise parça-parça-bütün bağlamlarında içinde incelenmiştir (ör., Spinillo ve Bryant,1991). Öğretim deneyinin dördüncü ve beşinci oturumlarında orantısal ilişkilerin spesifik olarak aynı/farklı ölçüm uzaylarına ait çokluklar arasındaki ilişkiler ve parça-parça/ parça-bütün ilişkileri bağlamlarında öğrenciler tarafından anlamlandırılmalarının ve deneyimledikleri zorlukların, öğretmen adayları tarafından fark etme becerilerinin ortaya çıkarılması hedeflenmiştir.

Bu çalışmada sınıf videolarının sınırlılıkları ışığında, mikro durum video kliplerinin üretilmesi ve kullanılmasının öğretmen adaylarının fark etme becerilerini geliştirmek için verimli bir yaklaşım olabileceği önerilmiştir. Bu bağlamda, video formatı

duraklatmaya ve yeniden izlemeye izin verdiği için öğretmen adaylarının öğrencilerin orantısal muhakemelerine odaklanabilmeleri ve videoda izlenilen öğretim etkileşimlerini ihtiyaç duyulması halinde yavaşlatabilmeleri için mikro durum video kliplerinden yararlanılmıştır. Mikro durum video klipleri genellikle sadece bir öğrencinin müdahale edilmeksizin problem çözme sürecini içermektedir ve video klipteki her öğrenci çeşitli orantısal muhakeme problemlerine odaklanmaktadır. Sonuç olarak, bu çalışma, video vakalarıyla zenginleştirilmiş öğrenme ortamı ile öğretmen adaylarının öğrencilerin orantısal akıl yürütmelerini fark etme becerilerinin gelişimini incelemeyi amaçlamıştır. Çalışma aşağıdaki araştırma sorularını ele almaktadır:

1. Öğretmen adaylarının, video vakalarıyla zenginleştirilmiş bir öğrenme ortamına katıldıklarında, öğrencilerin orantısal akıl yürütmelerini fark etme becerileri nasıldır?

   a) Öğretmen adaylarının, mikro durum video kliplerinde öğrencilerin stratejilerindeki matematiksel ayrıntılara dikkat etme becerileri nasıldır?
   b) Mikro durum video kliplerinde öğretmen adayları öğrencilerin düşüncelerini nasıl yorumlamaktadır?
   c) Öğretmen adayları, öğrencilerin mikro durum video kliplerindeki düşüncelerine karşılık olarak hangi öğretimsel yaklaşımları önerirler?

Araştırmanın Önemi

Öğretim, öğretmenlerin bazı eylemlere odaklanmasını ve bazılarını görmezden gelmesini gerektiren karmaşık bir süreçtir (Teuscher vd., 2017). Sherin ve Star'ın (2011) vurguladığı gibi, "bir öğretmen, duyusal verilerin, karmaşasıyla bombardımana tutulur" (s. 69). Bu bağlamda, öğretmenlerin öğrencilerin öğrenmesini geliştiren kararlar almak için bu verileri filtrelemesi gerekir. Bu ifade, "öğretmenin en göze çarpan özelliklerini" yönetmenin merkezinde yer aldığı için öğretmenin fark etme becerilerinin önemini vurgular (Doyle, 1977, s. 52). Son yıllarda, matematik eğitimcileri, öğretmenlerin her olaya katılmanın ve yanıt vermenin mümkün olmadığı sınıf olaylarını nasıl anlamlandırdığını anlamak için yapıları fark etmeye ilgi göstermiştir (Jacobs vd., 2011). Bazı araştırmalar sadece öğretmenlerin dikkatine

odaklanırken (Star vd., 2011), bazıları ise öğretmenlerin sınıfta gözlemledikleri hakkında nasıl akıl yürüttüklerini ele almıştır (Sherin, 2007; van Es, 2011). Jacobs ve arkadaşları (2010) çalışmalarının odak noktasını çocukların matematiksel düşünmelerini fark etme becerilerine taşımış ve öğretmenlerin fark etme becerilerini üç adımda ele almıştır: (a) öğrencilerin stratejilerine dikkat etme, (b) öğrenci anlamalarını yorumlama ve (c) öğrenci anlamalarını dikkate alarak nasıl yanıt vereceğine karar verme (Jacobs vd., 2010, s. 172). Fark etme becerileri sadece öğretmenlerin değil aynı zamanda öğretmen adaylarının da sahip olması gereken mesleki becerilerden birisidir (Mason, 2002; Stigler ve Hiebert, 1999). Çalışmalar yeterli tecrübeye sahip olmayan öğretmen adaylarının öğrenciye ve öğretimsel durumlara odaklanmakta zorluk yaşadıklarını göstermektedir (Jacobs vd., 2010; Santaga vd., 2007; Star ve Strickland 2008; van Es 2011). Diğer taraftan, sınıf ortamındaki öğrenme ortamlarında ortaya çıkan olaylardan önce, öğrencilerin matematiksel düşünmelerini fark etmeyi öğrenmeleri gerektiği vurgulanmaktadır (Star vd., 2011). Her ne kadar üniversitelerdeki öğretmen yetiştirme programlarının önemli bir parçası olan okul deneyimi, öğretmen adaylarının fark etme becerilerinin gelişimine katkı sağlasa da yeterli olmamaktadır (Ishler ve Kay, 1981). Dördüncü sınıf öğretmen adaylarının gözlemlediklerini anlamlandırarak yorumlama ve bu doğrultuda öğretimsel kararlar almada zorluk yaşadıkları gözlemlenmiştir (Jacobs vd., 2011, s. 99). Araştırmalar, fark etme becerilerinin öğrenilebileceğini (Jacobs vd., 2010; Star vd., 2011; Stockero, 2014) ancak bu becerilerin doğal yollarla gelişemediğini göstermiştir (Stockero, 2014; Star ve Strickland, 2008). Bu nedenle bu çalışmada, öğretmen adaylarının fark etme becerilerinin gelişimine yardımcı olmak için öğrenci video vakalarıyla zenginleştirilmiş bir öğretim deneyi tasarlanmıştır (van Es ve Sherin, 2005). Öğretim deneyinde öğrencilerin orantısal durumlar içeren problemleri çözme süreçlerinin yer aldığı mikro durum video klipler kullanılmıştır ve bu video klipler aracılığı öğretmen adaylarının farkındalık becerilerinin geliştirilmesi amaçlanmıştır.

Mesleki fark etme becerisi, alana özgü bir uzmanlıktır (Jacobs ve Empson, 2016); bu nedenle, öğretmenlerin öğrencilerin düşüncelerini anlamlandırma ve öğrencinin düşüncesine yönelik nasıl cevap vereceğine karar verme yeterliliği matematiksel alana göre değişiklik göstermektedir (Fernandez vd., 2011; LaRochelle, 2018; Walkoe, 2013). Çalışmaların çoğunluğu fark etme becerilerini belirli bir alana

odaklanmaksızın genel bir şekilde incelemektedir (van Es ve Sherin, 2002; Goldsmith ve Seago, 2011). Bu bağlamda oran ve orantı konusu seçilmiştir. Orantısal düşünme öğrencilerin matematiksel gelişiminde bir dönüm noktasıdır (Lesh vd., 1988). Bununla ilgili olarak, NCTM'nin Müfredat ve Değerlendirme Standartları (1989) orantısal düşünmenin öneminin altını çizer: "Orantısal akıl yürütme çok büyük bir öneme sahiptir bu yüzden geliştirilmesini sağlamak için zaman ve çabayı hak eder. Öğrencilerin orantısal akıl yürütme yoluyla modellenebilecek ve çözülebilecek birçok problem durumunu görmeleri gerekir" (NCTM, 1989, s. 82). Türkiye'de matematik ders kitapları oran ve orantının tanımıyla başlar ve içler-dışlar algoritmasının kullanımının öğretilmesi ile devam eder. Fakat öğrencilere, verilen nicelikler arasındaki çarpımsal ilişkiler anlamlandırılmadan içler-dışlar algoritması öğretilir (Lamon, 1995). Ayrıca, ülkemizde Milli Eğitim Bakanlığı Ortaokul Matematik Dersi Öğretim Programı'nda (MEB, 2013, 2018) orantısal akıl yürütmenin tüm anahtar öğrenmeleri (ör., toplamsal ve çarpımsal düşünme) içermemektedir. Bu bağlamda, çalışmanın ulusal literatüre katkı sağlayacağı düşünülmektedir.

Yukarıda belirtilen durumları dikkate alarak, öğretmen adaylarının odağını öğrencilerin orantısal akıl yürütmelerine yönlendirmek için orantısal akıl yürütmenin temel bileşenlerine dayalı öğrencilerin bireysel problem çözme süreçlerini içeren çeşitli video kliplerden yararlandık. Bu temel bileşenler öğrencilere, Carpenter ve meslektaşları (1999) tarafından geliştirilen orantısal akıl yürütme rotasına dayanan beş problem ile tanıtıldı. Bu anlamda, mevcut çalışmanın, orantısal akıl yürütme bağlamında klinik öğrenci görüşmelerinin videolarını izleyerek ve bunlar üzerinde derinlemesine düşünerek, öğretmen adaylarının öğrencilerin matematiksel düşünmelerini fark etme becerilerinin gelişmesine fayda sağlayacağı düşünülmektedir.

Yöntem

Bu çalışma, orantısal muhakeme bağlamında video vaka tabanlı öğretim deneyleri aracılığıyla öğretmen adaylarının öğrencilerin matematiksel anlamalarını mesleki fark etme becerilerinin gelişimini incelemektedir. Öğretim deneylerinin temel amacı, öğrencilerin matematiksel öğrenmeleri ve muhakemeleri hakkında bilgi vermek ve bu doğrultuda öğretimsel stratejilerini yönlendirerek öğrencilere daha iyi bir öğrenme

ortamı sunmaktır (Steffe vd., 2000). Bu noktada klinik görüşmelerden ve deneysel çalışmalardan farklılık göstermektedir. Yapılandırmacı öğretim deneyi, Piaget'nin klinik görüşmelerinden türetilmiş ve öğrencilerin matematik bilgisine odaklanmaktadır (Steffe vd., 2000). Cobb ve meslektaşları bu yaklaşımı, öğrencilerin ve öğretmenlerin işbirliği içinde geliştirilen alternatif sınıf ortamlarında öğrenci öğrenmelerini araştırmak için genişletmiştir ve süreci bireysel yapılandırma ve sosyal öğrenme çerçevesinde incelemiştir (Cobb, 2000, s. 312). Önceki araştırmalar, öğrenmenin hem bireysel yapılandırma hem de sosyal etkileşim süreçlerini içerdiğini ve sosyal etkileşimlerin öğrencilerin matematiksel gelişimi için katalizör olarak görüldüğünü göstermektedir (Cobb ve Bauersfeld, 1995; Cobb ve Yackel, 1996). Bu nedenle, sosyal bağlam olarak 9 haftalık öğretim deneyi oturumları, öğretmen adaylarının birbirleriyle ve araştırmacıyla etkileşim yoluyla öğrencilerin matematiksel anlamalarını fark etmelerini geliştirmek için tasarlanmıştır ve öğretmen adaylarının fark etme becerilerinin gelişimi orantısal muhakeme bağlamında detaylı bir şekilde incelenmiştir.

Katılımcılar

Bu çalışmanın bağlamını öğretmen yetiştirme programları altında bulundan ilköğretim matematik öğretmenliği programı oluşturmaktadır. Öğretmen adaylarının öğrenci düşüncelerini fark etme becerilerini geliştirmek amacıyla dördüncü sınıf öğretmen adayları için "İlköğretim Fen Bilimleri ve Matematik Eğitiminde Projeler" adlı seçmeli ders açılarak tasarlanmıştır. Bu ders 2018-2019 bahar döneminde açılmış olup dokuz haftalık süresinde veri toplanmıştır. Bu çalışma için Ankara'da bir devlet üniversitesinde İlköğretim Matematik Öğretmenliği Programı'nda (İME) dördüncü sınıf öğretmen adaylarına seçmeli ders açılmıştır ve bu derse altı öğretmen adayı katılmıştır. Amaçlı örneklem tekniğinden faydalanılarak katılımcılar seçilmiştir. Bu teknik, çalışma öncesinde belirlenen kriterleri sağlayan bireylerin seçimini kapsamaktadır (Gall vd., 2007). Bu dersin sadece son sınıf öğrencileri için açılmış olmasının nedeni, öğretmen adaylarının Matematik Öğretim Yöntemleri 1 ve 2, Okul Deneyimi 1 derslerini almış ve Okul Deneyimi 2 dersini de hali hazırda alıyor olmalarıdır. Sonuç olarak, sunduğumuz seçmeli dersi beş kadın ve bir erkek dördüncü sınıf öğretmen adayı almıştır ve bu altı katılımcı ile bu çalışma gerçekleştirilmiştir.

Verilerin Toplanması

Bu çalışmanın verileri 9 haftalık süreçte aşamalar şeklinde toplanmıştır. Tablo 1 bu aşamaları ve bu aşamalarda kullanılan veri toplama araçlarını göstermektedir.

Tablo 1. Öğretim Deneyi Süreci

Öğretim Deneyi Süreci	**Veri Toplama Araçları**
Öğretim deneyi öncesi	Klinik ön görüşme İlk ders planları
Video vaka odaklı öğretim deneyi oturumları	Video vaka odaklı yansıtıcı düşünce raporları Video vaka odaklı grup tartışmaları
Öğretim deneyi sonrası	Revize edilmiş ders planları Son klinik görüşmeler

Öğretim Deneyi Hazırlık Süreci

İlgili alan yazın araştırıldığında öğrencilerin orantısal akıl yürütme stratejilerini ortaya koyan yeterli videolar bulunmadığından dolayı, micro durum video veri tabanı oluşturmak için 7. sınıf öğrencileri ile ailelerinin onayları alınarak bireysel görüşmeler yapılmış ve bu görüşmeler öğrencilerin yüzü gözükmeyecek şekilde kamera ile kayıt altına alınmıştır. Bu görüşmelerde öğrencilere beş problem sorulmuş ve farklı stratejiler kullanarak çözmeleri istenmiştir. Problemi çözmelerinin akabinde verdikleri cevaptan farklı (örneğin doğru çözüme yanlış çözüm) çözüm gösterilerek ve çözümlerin doğruluğu ve yanlışlığı hakkında bilgi vermeksizin neden çözümlerin farklı olduğu sorulmuştur.

Video veri tabanı oluşturulmasının akabinde öğretmen adayları ile klinik görüşmelere başlanmıştır. Klinik görüşmeler, katılımcılar ve araştırmacı arasındaki yüz yüze etkileşim yoluyla benzersiz ve doğrudan bilgi sağlayan önemli bir veri toplama tekniğidir (Clement, 2000). Bu görüşmeler, araştırmacıların katılımcıların düşüncelerini doğal bir şekilde ortaya çıkarıp belgelemesine ve katılımcıları görevlerin tartışılmasına dahil ederek kavramsal anlayışlar hakkında derinlemesine bilgi edinmesine olanak tanır (Ambrose vd., 2004). Öğretmen adaylarına öğrencilere sorulan problemler yöneltilmiş olup öncelikle problemleri farklı stratejiler kullanarak

çözmeleri istenmiştir. Öğretmen adaylarına "öğrenciler sence bu soruyu nasıl çözerdi ya da ne gibi hatalı çözümler yapardı?" ve "Bu soruyu şu şekilde çözen öğrencin için nasıl bir öğretimsel yaklaşım sergilerdin?" şeklinde yönergeler yardımı ile öğretmen adaylarının orantısal muhakemeyle ilgili öğrenci düşünüşlerini ve öğretimsel yaklaşımlarını ortaya çıkarma amaçlanmıştır.

Ön görüşmelerin tamamlanmasıyla öğretmen adaylarından 7. Sınıf öğrencilerine oran ve orantı konusunu nasıl öğreteceklerinin detaylarını içeren bir ders planı hazırlamaları istenmiştir ve bu ders planını katılımcılar bir hafta içerisinde araştırmacıya elektronik ortam yardımı ile göndermiştir.

Ön görüşmeler ve ders planlarından gelen veriler ışığında öğretim deneyi oturumlarında kullanılmak üzere muhtemel videolar seçilmiştir. Videoların seçiminde ve mikro durum videolarının tasarlanmasında video süreleri ve Sherin ve meslektaşları (2009) tarafından önerilen üç bileşen dikkate alınmıştır. Micro durum videoları 10 dakikanın altında olacak şekilde belirlenmiştir. Bu üç bileşen; görünüm, derinlik ve açıklık olarak isimlendirilmiştir ve bu çalışmada bu bileşenlerin seviyeleri sırasıyla *"yüksek-yüksek-yüksek"* ve *"yüksek-düşük-düşük"* düzeylerinde olan mikro durum videoları öğretmen adaylarına izletilmiştir.

Öğretim deneyinin uygulanması

Öğretim deneyinin uygulama süreci, mikro durum videolarının öğretim deneyi oturumu verilerine göre belirlenmesi, öğretim oturumlarının gerçekleştirilmesi, ders planlarının revize edilmesi ve öğretim deneyi oturumları sonrası son klinik görüşmelerin yapılması adımlarından oluşmuştur.

Öğretim deneyi oturumlarında izletilecek mikro durum videolarının seçimi ve düzenlenmesi klinik ön görüşmelerin ve ders planlarının incelenmesinin ardından gerçekleşmiştir. Seçilen videolar muhtemel videolar olmak üzere haftalık oturumlarda ortaya çıkan verilere göre izletilecek videolar netlik kazanmıştır. Bu videoların seçimindeki ölçütler; (a) video süresi, (b) öğrenci düşünüş çeşitliliği, (c) öğretmen

adaylarının oran ve orantı ile ilgili sahip oldukları bilgi şeklinde sıralanmıştır (Linsenmeier ve Sherin, 2009).

Öğretim deneyi oturumları

Öğretim deneyi oturumları beş hafta sürmüştür, Öğretmen adayları her hafta bir mikro durum videosunu grup olarak izlemiştir. Sonrasında öğretmen adaylarından videodaki öğrenci düşünüşünde neyi fark ettikleri, bu durumu nasıl fark ettikleri ve videodaki öğrencinin öğretmeni olsalardı bu öğrenciye yönelik nasıl bir öğretimsel yaklaşım sergileyeceklerini içeren bireysel olarak yansıtıcı rapor yazmaları istenmiştir. Raporların yazımı tamamlandıktan sonra grup tartışması araştırmacı tarafından başlatılmıştır. Grup tartışmalarının verimliliğini arttırmak için teorik çerçeve kullanılmıştır (van Es vd., 2014). Her bir öğretim deneyi sonrasında öğretmen adaylarından ihtiyaç duymaları halinde ders planlarında revizeler yapmaları ve bu revizeleri yaparken neden bu değişikliğe ihtiyaç duyduklarını da açıklamaları istenmiştir. Her iki oturum sonrası da benzer şekilde devam etmiştir.

Son klinik görüşmelerin yapılması ve revize edilen ders planlarının son halinin elde edilmesi

Oturumların bitmesiyle öğretmen adaylarıyla son klinik görüşmeler yapılmıştır. Bu görüşmeler yaklaşık 60 dakika sürmüş olup ön görüşmelerde sorulan oran ve orantı problemlerinin tümü ön güreşmelerde verdiği cevaplara referans verilerek sorulmuştur. Görüşmelerin sonunda öğretmen adayları revize ettikleri ders planlarının en son versiyonlarını araştırmacı ile paylaşmaları istenmiştir. Katılımcıların öğrencilerin oran ve orantı konusundaki düşüncelerini fark etme becerilerindeki değişim ve gelişim bu görüşmelerde verdiği cevaplar ve ders planlarında haftalık olarak yaptıkları revizeler incelenerek ortaya çıkarılmıştır.

Verilerin Analizi

Öğretmen adaylarının öğretim deneyi kapsamında öğrencilerin oran ve orantı konusunda düşünmelerini fark etme becerilerini ve bu becerilerde meydana gelen

değişimleri incelemek amacıyla içerik analizi tekniğinden yararlanılmıştır. İçerik analizi, tema ve kodların tanımlanmasında ya da kodlama yaparak sistematik sınıflandırma ile sürecin yorumlanmasını sağlayan bir tekniktir.

Klinik ön ve son görüşmelerin analizi

Klinik görüşmeler, araştırmacıların katılımcıların düşüncelerini doğal olarak ortaya çıkarmasına, belgelemesine ve katılımcıları görevlerin tartışılmasına dahil ederek kavramsal anlayışları hakkında derinlemesine bilgi edinmesine olanak tanır (Clement, 2000; Ambrose vd., 2004). Bu çalışmada, klinik öncesi ve sonrası görüşmelerin yapılmasında iki temel amaç benimsenmiştir; (a) oturumlar için vaka temelli video klipler seçmek ve (b) öğretmen adaylarının fark etme becerilerinin gelişim sürecini gözlemlemek. Klinik ön görüşmeler analiz edilirken tematik kodlama kullanılmış olup, bu sayede temalar ve kodlar oluşturulmuştur. Bu bağlamda öncelikle (a) verilerin tanınması, (b) ilk kodların oluşturulması, (c) temalar belirlenmesi ve (d) tema ve kodları içeren raporun hazırlanması şeklinde aşamalı olarak gerçekleşmiştir. Klinik son görüşmelerde öğretmen adaylarının mikro durum video kliplerindeki öğrenci düşünüşlerine yönelik grup tartışmalarında ve revize edilmiş ders planlarında açıklık kazandırılması gereken ifadeler katılımcılara sorularak, fark etme becerilerindeki değişim sürecini gözlemleme amacı benimsenmiştir.

Öğretim deneyi oturumları sonunda elde edilen verilen analizi

Öğretmen adaylarının öğrencilerin düşünmelerini orantısal muhakeme bağlamında fark etme becerilerini yorumlamak için Jacobs ve arkadaşları (2010) tarafından geliştirilen teorik çerçeveden faydalanılmıştır. Bu teorik çerçevede öğrencilerin matematiksel düşüncelerine odaklanılmış olup fark etme becerisi birbiri ile ilişkili üç bileşende (a) öğrencilerin stratejilerine dikkat etme, (b) öğrenci anlamalarını yorumlama ve (c) öğrenci anlamalarını dikkate alarak nasıl yanıt vereceğine karar verme becerileri incelemiştir. İlk beceri (attending), öğrencilerin stratejilerindeki konuyla ilgili matematiksel detaylara ne derece dikkat ettiklerini ele almaktadır. Bu çalışmada bu beceri sınırlı (limited) ve güçlü (robust) olarak kodlanmıştır. Öğrenci anlamalarını yorumlama becerisi (interpreting), öğretmenlerin öğrenci

stratejilerindeki anlamalarını nasıl yorumladıklarını ve bu yorumların mevcut çalışmalarla ele almaktadır. Bu çalışmada bu beceri sınırlı (limited) ve güçlü (robust) olarak kodlanmıştır. Üçüncü beceri ise (deciding how to respond), öğretmenlerin öğrencilere cevap verirken, öğrencilerinin matematiksel anlamalarını ne derecede dikkate aldıklarını ve bir sonraki adıma karar verirken planladıkları öğretimsel yaklaşımların öğrencilerin düşünceleriyle tutarlılıklarını ele almaktadır. Bu çalışmada Jacobs ve arkadaşlarının kodlamalarından farklı olarak üçüncü bileşen (a) genel (general), (b) vaka odaklı-detay içermeyen (case-specific-undetailed) ve (c) vaka odaklı-detay içeren (case-specific-detailed) şeklinde kodlanmıştır. Bu üç beceri birbiri ile iç içe ilişkili olup fark etmenin anlamlı olması için üç bileşen de birlikte bulunmalıdır (Jacobs vd., 2010). Bu bağlamda öğretmen adaylarının video vaka odaklı yansıtıcı düşünce raporları, video vaka odaklı grup tartışmaları ve revize edilmiş ders planları bu teorik çerçeve kullanılarak (professional noticing of students' mathematical thinking) analiz edilmiştir.

Bulgular

Bu çalışma, video vakalarıyla zenginleştirilmiş öğrenme ortamı ile öğretmen adaylarının öğrencilerin orantısal akıl yürütmelerini fark etme becerilerinin doğasını ve gelişimsel sürecini incelemeyi amaçlamıştır. Bu bölümde, bulgular orantısal akıl yürütmenin temel bileşenlerine dayalı olarak üç bölümde incelenmiştir. Bu bileşenler Carpenter ve arkadaşları (1999) tarafından (a) toplamsal ve çarpımsal düşünme, (b) birimleme ve birleşik birimleri biçimlendirme ve (c) orantılı ilişkileri anlamlandırma şeklinde önerilmiştir. Beş problem bu üç bileşen doğrultusunda seçilmiş ve gruplandırılmıştır. Öğretmen adaylarının fark etme becerilerinin doğası ve gelişimsel süreci Jacobs ve arkadaşlarının teorik çerçevesi altında yorumlanmıştır. Her bölümde, öğretmen adaylarının orantısal akıl yürütme bağlamında öğrenci düşünmesine ilişkin mevcut fark etme becerileri, klinik ön görüşmeler, ilk ders planları ve haftalık video vaka odaklı yansıtıcı düşünce raporları kullanılarak analiz edilmiştir. Ardından, video vaka odaklı grup tartışmaları, haftalık gözden geçirilmiş ders planlarından ve öğretim deneyi oturumlarının sonunda yapılan son klinik görüşmelerden elde edilen veriler kullanılarak, öğretim deneyi boyunca öğretmen adaylarının öğrencilerin matematiksel düşüncelerini fark becerilerinin gelişimsel süreci incelenmiştir.

Micro durum videoları (MCVs)	Problemler	Öğrenciler	Tanım
Toplamsal ve çarpımsal düşünme MCV1	Parkur problemi	Öğrenci 1 (S1)	MCV1'de, öğrenci çarpımsal strateji kullanarak problemi yanlış bir şekilde çözdü.
Birimleme ve birleşik birimleri biçimlendirme MCV2, MCV3	Manav problemi	Öğrenci 2 (S2)	MCV2'de öğrenci, birim oran stratejisi kullanarak problemi yanlış bir şekilde çözdü
	Balon problemi	Öğrenci 3 (S3)	MCV3'te Öğrenci, birleşik birim strateji kullanarak problemi doğru bir şekilde çözdü.
Orantısal ilişkileri anlamlandırma MCV4, MCV5	Fotoğraf problemi	Öğrenci 4 (S4)	MCV4'te öğrenci, aynı ve farklı ölçüm uzaylarında yapılan kıyaslama stratejileri kullanarak problemi doğru bir şekilde çözdü.
	Karışım problemi	Öğrenci 5 (S5)	MCV5'te öğrenci, parça-parça-bütün stratejileri kullanarak problemi doğru bir şekilde çözdü.

Bu çalışma, öğretmen adaylarının video vakalarıyla zenginleştirilmiş öğrenme ortamlarına katılmadan önce öğrencilerin stratejilerine dikkat etme ve öğrenci anlamalarını yorumlama becerilerinin yeterli olmadığını göstermiştir. Öğretmen adayları mikro durum videolarındaki öğrencilerin stratejilerindeki matematiksel ayrıntılara odaklanmaları sınırlı şekilde gerçekleşti. Bu bağlamda, öğrencilerin çözümleri yeniden oluşturulmak istendiği taktirde öğretmen adaylarının verdikleri yanıtlar yetersiz bilgi içerdiği görülmüştür. Öğrencilerin stratejilerine dikkat gösterdiğine dair sınırlı kanıt gösteren yorumlar, tipik olarak problem çözme sürecine atıfta bulunurken, öğrencilerin problemi nasıl çözdüklerine dair detayları dışarıda bırakmıştır. Öğretmen adayları yanıtlarında, öğrencilerin anlamalarına ilişkin iddialarını desteklemek için yeterli olmayan kanıtlar kullanarak öğrencilerin matematiksel düşüncelerini yorumlamaya odaklanmışlardır. Katılımcıların öğrenci düşünüşünden anladıkları hakkında doğrulanmamış, ispat edilmemiş veya basitleştirilmiş iddialarda bulunmuşlardır (kanıt eksikliği-lack of evidence). Öğretmen adayları öğrencinin matematiksel düşüncesine yönelik yorumlarda bulunsa da bu yorumlar genel olarak tanımlayıcı bir bakış açısıyla yapılma eğilimindeydi. Bununla birlikte, öğretmen adaylarının büyük bir çoğunluğu vakaya özgü ve

detaylandırılmamış öğretimsel yaklaşımları önermiştir. Öğretmen adayları öğrencilerin anlamalarına odaklanmış olsalar da yanıtları önerilen öğretimsel yaklaşımları için sınırlı ayrıntılar ve genel gerekçeler içeriyordu. Öğretmen adayları bu yaklaşımların neden videodaki öğrenci(ler) için uygun adımlar olduğunu açıklamadan önerilerde bulundu. Ayrıca bazı öğretmen adaylarının önerileri çok genel öğretimsel yaklaşımları içermektedir. Daha önceki araştırmalarda da belirtildiği gibi, bu çalışma öğrenci düşüncesini fark etmeyi öğrenmenin öğretmen adayları için zor olduğunu göstermiştir (Amador, 2021; Erickson, 2011; Jacobs vd., 2010; Star vd., 2011; Star ve Strickland, 2008; van Es ve Şirin, 2002).

Bu çalışmanın sonuçları, öğretmen adaylarının fark etme becerilerinin video vakalarıyla zenginleştirilmiş öğrenme ortamı aracılığıyla geliştiğini göstermiştir. Mikro durum video klip analizlerine katılarak, öğretmen adaylarının orantılı muhakeme bağlamında öğrencilerin düşünmelerini fark etme becerileri önemli ölçüde geliştiğini ortaya koymuştur. Özellikle öğrencilerin stratejilerindeki matematiksel detaylara dikkat etme ve öğrencilerin anlamalarını onların videodaki eylem ve sözlerine dair kanıtlar sunarak yorumlayabilme becerilerindeki gelişim daha hızlı gerçekleşmiştir. Bu gelişimsel durum mevcut çalışmalar tarafından da desteklenmektedir. Araştırmanın sonuçları, öğretmen adaylarının öğrencilerin anlamalarına nasıl cevap vereceklerine karar vermede zorlandıklarını ortaya koysa da mikro durum video destekli öğretim deneyi oturumları sayesinde tüm öğretmen adaylarının cevap verme becerilerinde gelişme olduğunu ortaya koymuştur. Bu bağlamda, öğretmen adaylarının çoğu, öğretim deneyi oturumları boyunca mikro vaka video kliplerinde öğrencilerin stratejilerindeki matematiksel detaylara daha fazla dikkat ettikleri, bu detayları ifade ederken öğrencilerin stratejinden kanıtlar gösterdikleri gözlemlenmiştir. Bu durum, öğretmen adaylarının öğrencilerin stratejilerinin ayrıntılı, spesifik ve doğru açıklamaya çalıştıklarını göstermiştir. Diğer bir deyişle, öğretmen adayları mikro vaka videolarını inceledikten ve öğrencilerin düşünceleri hakkında video temelli grup tartışmaları yaptıktan sonra, büyük ölçüde vakaya özel ve ayrıntılı matematiksel öğretimsel yaklaşımlar sağlamıştır.

Tartışma ve Sonuç

Bu çalışmadan elde edilen bulgular öğretim deneyi oturumlarının başında öğretmen adaylarının mikro durum video kliplerdeki öğrencilerin stratejilerindeki matematiksel detaylara dikkat etmedikleri göstermiştir. Video vakalarıyla zenginleştirilmiş oturumlar sayesinde, öğretmen adaylarının fark etme becerilerinin, olaylar dizisini genel olarak fark etmekten ziyade, önemli ayrıntılara ve orantısal muhakeme bağlamındaki ilgili olaylara odaklanma eğiliminde oldukları keşfedilmiştir (Sherin ve Han, 2004; van Es ve Sherin, 2011). Öğretmen adayları öğrencilerin stratejilerinin ayrıntılarına odaklanarak öğrencilerin ne yaptıklarına dair açık ve doğru açıklamalar yapmaya başlamış ve ifadelerini de öğrencilerin stratejilerinden örnekler ile gerekçelendirmiştir. Öğretim deneyi oturumlarının ilk haftasında öğretmen adaylarının yorumlarının genel olarak tanımlama ve değerlendirme eğiliminde olduğu gözlemlenmiştir. Yorumlarını destekleyici öğrenci düşünüşlerine yönelik herhangi bir gerekçe göstermemişler ve sadece öğrencilerin çözümlerini yeniden ifade etmişlerdir. İkinci oturumdan sonra, öğretmen adayları öğrencinin eylem ve sözlerine dair kanıtlar sunarak öğrencilerin anlamalarını yorumlamaya başlamıştır. Araştırmanın bu bulgusu, öğretmen adaylarının öğrencilerin düşüncelerine daha fazla dikkat ettiklerinin ve öğrencilerin düşüncesini anlamlandırma ve yorumlama pratiğine uyum sağladıklarının bir kanıtı olarak değerlendirilmiştir (Teuscher vd., 2017).

Öğretmen adaylarının öğrencilerin nasıl düşünmüş olabileceklerini ve nedenlerini gerekçeleyerek yorumlamalarının, verecekleri yanıtlarda da farklılaşmaya neden olduğu gözlemlenmiştir. Bu doğrultuda, öğretmen adaylarının video kliplerdeki öğrencilerin sadece anlamlandırdıklarını değil, anlamlandıramadıklarına da odaklanmaya başladıkları ve bu ifadelerini destekleyici gerekçeler ile sunmaya başladıkları görülmüştür. Benzer şekilde Males (2017) öğretmen adaylarının sonraki oturumlarda öğrencilerin neleri anlamlandırdıkları ve neleri anlamlandıramadıklarına daha fazla odaklandıklarını ortaya koymuştur. Bu gelişim, odaklı ve sürdürülebilir bir destek sunması açısından video-durum temelli çalışmaların katkısını vurgulayan çalışmaları destekler nitelikte olmuştur. Öğrencilerin orantısal akıl yürütme gelişimini destekleyen bir öğrenme yörüngesi takip edilmesine ve farklı bireysel öğrenci

çözümleri (ör., doğru/yanlış çözümler) içeren problem çözme videolarının odaklı analizine atfedilebilir.

Bu çalışmadan elde edilen veriler, öğretmen adaylarının diğer çalışmalarda olduğu gibi öğrencilerin anlamalarına nasıl karşılık vereceklerine karar vermede zorluk yaşadıklarını ortaya koymuştur (Jacobs vd., 2010), ancak video klip destekli öğretim deneyi oturumları sayesinde tüm öğretmen adaylarının yanıt verme becerilerinde gelişme gözlemlenmiştir (Krupa vd., 2017; Sánchez-Matamoros vd., 2019). Öğretmen adaylarının öğretim deneyi oturumlarındaki mikro durum videolarını incelemesi ve video temelli grup tartışmaları yapması ile birlikte, videodaki öğrenci düşünüşüne özel ve daha fazla ayrıntı içeren matematiksel öğretimsel yaklaşım planları içeren tavsiyelerde bulunmuşlardır. Örneğin, çalışmanın başında önerilen öğretimsel yaklaşım planları ayrıntı içermemekle birlikte, basamaklarına dair gerekçeler de barındırmıyordu. Bu bağlamda fark ettikleri noktaları anlamlandırmaya yönelik yorumlarının eksikliği ve gerekçesizliği de dikkat çekmektedir. Grup tartışmaları ile birlikte öğretmen adayları öğrencilerin düşünüşlerini anlamlandırmaya ve yorumlamaya daha fazla odaklanmaları ve bu odaklanmanın farklı yorumlamaları da ortaya çıkarmasıyla beraber, farklı öğretimsel yaklaşım planlarının da ortaya çıkmasını sağlamıştır. Önerilen planlar incelendiğinde, bu planların daha fazla ayrıntı içerdiği ve her bir adımın gerekçelendirmeye çalışıldığı görülmüştür. Grup tartışması aşamasında öğretmen adaylarının karar verdiği öğretimsel yaklaşım planlarını paylaşmaları ve diğer öğretmen adaylarının hem paylaşılan hem de kendi planları üzerinde yorum yapmaları, öğretmen adaylarının sonraki öğretim yöntemlerini daha ayrıntılı yönergelerle ve bu ayrıntıları da destekleyici gerekçelerle sunmaya başlamalarını sağlamıştır. Bununla birlikte, grup tartışmalarının, öğretmen adaylarının yönergelerini yeniden gözden geçirmelerine ve tartışılan kritik olayların özelinde planların revize etmelerine neden olduğunu da ders planlarında ve son klinik görüşmelerinde ifade etmişlerdir. Bu bulgular önceki çalışmalarla da örtüşmektedir (ör., Amador vd., 2011; Jacobs vd., 2010; Philip vd., 2011). Bu doğrultuda, matematik eğitimcileri, matematiksel alanlarda öğrenci düşüncesine odaklanan seçmeli dersler tasarlamayı düşünebilir, böylece video vaka temelli mesleki gelişim programlarının öğretmen adaylarının fark etme becerilerinin gelişimini matematik konuları özelinde daha fazla desteklenlemesi sağlanabilir.

Mevcut araştırmaların, öğretmen adaylarının öğrenci düşünmelerini fark etme becerilerinin gelişimini spesifik matematik öğrenme alanlarında destekledikleri görülmüştür (Amador vd., 2017; Callejo ve Zapatera, 2016; Fernández vd., 2013; Kieboom vd., 2017; Sanchez-Matamoros vd., 2014; van Es, 2011; Walkoe, 2013). Bu amaçla, fark etme becerisinin gelişimsel sürecini inceleyebilmek için Carpenter ve meslektaşlarının orantısal akıl yürütme rotası takip edilmiştir. Fakat çalışmanın sonuçları öğretmen adaylarının fark etme becerileri gelişimi açısından doğrusal bir ilerleme göstermemiştir ve bu çalışma Amador ve arkadaşlarının bulgularını desteklemektedir. (Amador vd., 2017). Çalışma sonuçlarının ortaya çıkardığı önemli bir sonuç da fark etme becerisinin gelişimi alanlar ve aynı alanın konuları arasında farklılık gösterdiği gibi aynı konunun bileşenleri için de farklılık göstermesidir. Bu çalışma öğretmen adayları orantısal akıl yürütme konusunda kapsamlı araştırmalar yapmışlar, öğrencilerin olası düşünme şekillerini gerekçelendirerek tahmin etmeye ve bu doğrultuda ders planlarını revize etmeye odaklanmışlardır (Fernandez ve Chokshi, 2002). Bu doğrultuda, birbirinden farklı matematik konuları özelinde ve her bir konunun bileşenlerini de dikkate alarak çalışmaların yapılması tavsiye edilmiştir (Walkoe, 2013).

Bu çalışmadan elde edilen veriler, öğretmen adaylarının öğrenci düşünüşlerini fark etme becerilerinin video vakalarıyla zenginleştirilmiş öğrenme ortamları ile geliştirilebileceğini ortaya koymuştur. Yapılan çalışmalarda çoğunlukla sınıf durum videoları kullanılmış olup (ör., Sherin vd., 2011) son yıllarda öğrenci düşünüşüne odaklanan video durumlarının kullanıldığı çalışmalar yapılmaya başlanmıştır (ör., Jacobs vd., 2010). Bazı çalışmaların aksine öğretmen adaylarının "Bu çalışmanın bulguları ışığında video vakalarıyla zenginleştirilmiş öğrenme ortamlarında bireysel öğrenci video analizleri ile başlayıp ve sınıf durum videolarının da sürece dahil edilmesi önerilmektedir. Alanyazındaki bazı çalışmaların aksine, öğretim deneyi sonunda fark etme becerisinin üç bileşeninde de gelişme gözlemlenmiş olup öğretmen adayları öğrenci stratejilerine daha fazla odaklanmışlar, öğrenci anlamalarını yorumlamışlar ve bu yorumlarını somut verilerde gerekçelendirmişlerdir. Mevcut çalışmalar, öğretmen adaylarının öğrenci anlamalarını dikkate alarak nasıl yanıt vereceğine karar verme becerilerinin gelişiminde zorlukları ortaya çıkarmıştır. Bu çalışmanın bulguları öğretmen adaylarının bu becerilerinde de gelişmenin olduğunu

ortaya koymuştur. Bu duruma öğretmen adaylarının öğretim deneyi ortamlarında bireysel öğrenci videolarında öğrenci düşünüşüne odaklanmaları, ve arkadaşlarıyla tartışma ortamında fikir alışverişinde bulunarak farklı yorumları dinleme deneyimlemeleri rol oynamış olabilir. Öğretmen adaylarının sınıf durum videolarını analiz ederken, öğrenci düşüncesinden ziyade sınıf ortamındaki öğrenci konuşmalarına, sınıfın fiziki yapısına veya sınıf yönetimine dikkatini verebilirler. Fakat bireysel öğrenci video analizleri ve öğretmen adaylarının birbirleri ile fikir paylaşımları birleştiğinde, sosyal öğrenme ortamının fark etme becerilerinin gelişimini desteklediği düşünülmektedir. Karar verme becerisindeki gelişim doğrusal bir gelişimin olmadığı ise videolarda spesifik bir matematik konusunun işlenmesiyle ortaya çıkmıştır. Bu durum da fark etme becerisinin konu odaklı olduğunu desteklemektedir (ör., Fernández vd., 2013; Fisher vd., 2019; Sánchez-Matamoros vd., 2015; van den Kieboom vd., 2017). Bu bağlamda üniversitelerde öğretmen adaylarına öğretim yöntemi derslerinde öğrenci düşünüşüne spesifik farklı matematik konularında odaklanan mikro durum videoları ve grup tartışmaları aracılığı ile öğretmen adaylarının fikir alışverişinde bulunacağı ortamlar yaratılarak (ör.,seçmeli dersler) bu çalışmada olduğu gibi öğretmen adaylarının fark etme becerilerinin haftalık gelişimsel süreçlerinin takibi yazılı (ör., haftalık revise edilmiş ders planları) veya sözlü (ör., haftalık ders sonu kısa görüşmeler) verilerin toplanmasıyla yapılabilir. Bu çalışmanın sınırlılıklarından biri olan öğrencilerle etkileşim eksikliği ise açılan seçmeli dersin sonunda verilen görevlerin öğrencilerle görüşme yapılarak uygulanmasıyla desteklenebilir.

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YAZARIN / AUTHOR

Soyadı / Surname : Soyak
Adı / Name : Öznur
Bölümü / Department: İlköğretim / Elementary Education

TEZİN ADI / TITLE OF THE THESIS (**İngilizce** / English): DEVELOPING PROSPECTIVE TEACHERS' PROFESSIONAL NOTICING OF STUDENTS' PROPORTIONAL REASONING THROUGH VIDEO CASE ENHANCED LEARNING ENVIRONMENT

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