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Explicit representations of dihedral Gauss hypergeometric functions

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Abstract
Solutions of the Gauss hypergeometric equation, two of whose characteristic exponent differences are half-odd-integers and the third is an arbitrary complex number, are called dihedral Gauss hypergeometric functions. They can be expressed in terms of elementary functions since they belong to Schwarz's dihedral class. In this article, explicit trigonometric and algebraic identities for dihedral Gauss hypergeometric functions in terms of the Jacobi polynomials with some pre-factors are presented. To this end, four different sets of linearly independent solutions of a specific ordinary differential equation are obtained where the first three sets contain dihedral Gauss hypergeometric functions and the fourth one consists of elementary functions. Dihedral identities are then obtained by expressing the hypergeometric solutions as linear combinations of the elementary ones. Moreover, dihedral identities are extended to a more general class of Gauss hypergeometric functions.

Keywords Trigonometric Pöschl–Teller potential · Dihedral Gauss hypergeometric functions · Jacobi polynomials

Mathematics Subject Classification Primary 33C05 · 33C45 · 34L40

1 Introduction

It is well-known that the Gauss hypergeometric function ${}_2F_1\,(a, b; c; z)$ can be represented in terms of elementary functions in z for some special values of the parameters. Schwarz listed fifteen cases for which hypergeometric functions can be expressed algebraically [9]. In other words, Schwarz tabulated the parameters for which the hypergeometric equation has a finite monodromy group, or equivalently has two independent solutions that are algebraic functions of z. These parameters are known as the characteristic exponent differences $\{1 - c, c - a - b, a - b\}$ of the Gauss hyperge-

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ometric equation. Schwarz's class I is known as dihedral class since the monodromy group is a finite dihedral group. However, Gauss hypergeometric functions with a dihedral monodromy group (either finite or infinite order) are referred to as dihedral hypergeometric functions [13]. Therefore, the characteristic exponent differences of the hypergeometric equation for the dihedral Gauss hypergeometric functions can be expressed as $\{1-c, c-a-b, a-b\} = \{m+\frac{1}{2}, n+\frac{1}{2}, \alpha\}$ in an unordered way and up to a sign where m and n are non negative integers and α is a complex constant with possibly some specific restrictions on it. If the characteristic exponent α is an irrational number, the monodromy group is an infinite dihedral group, if it is a non-integer rational number, the monodromy group is a finite dihedral group. If α is an integer, the monodromy group is either a cyclic group of order two (hence isomorphic to $\mathbb{Z}/2\mathbb{Z}$) or an infinite dihedral group [13].

Vidūnas [13] expressed dihedral functions in terms of terminating Appell's F_2 or F_3 double sums which is the first and most comprehensive and important study on expressing dihedral Gauss hypergeometric functions in terms of algebraic functions. Furthermore, general trigonometric identities for dihedral functions were also derived in [13].

In this study, our aim is to express dihedral Gauss hypergeometric functions in terms of the Jacobi polynomials of both trigonometric and algebraic arguments with some pre-factors. In fact, instead of terminating double sums, here we express dihedral Gauss hypergeometric functions as a finite sum containing the combinations of the complex exponentials $e^{\pm i\alpha x}$ with the Jacobi polynomials $P_k^{(\pm\alpha,\mp\alpha)}(t)$ where the degree k is either m or n and the trigonometric argument t is either $t = i\tan x$ or $t = i\cot x$. To this end, we follow the idea of [1] where the general solution of the certain ordinary differential equation, namely the Schrödinger equation with squared secant potential (or a part of the trigonometric Pöschl–Teller potential) [8]

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{n(n+1)}{\cos^2 x}\right] y_n(x;\lambda) = \lambda y_n(x;\lambda), \quad x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), \tag{1.1}$$

is obtained in two different ways (by keeping the eigenvalue constant $\lambda = \alpha^2 > 0$ as a general real parameter); first by transforming the equation into a Gauss hypergeometric equation whose characteristic exponent differences are in the unordered unsigned set $\{\frac{1}{2}, n+\frac{1}{2}, \alpha\}$ and second by transforming it to the Jacobi differential equation by using another set of transformations on both independent and dependent variables. First approach leads to the fundamental set of solutions in terms of specific dihedral functions ($m = 0$) and the second one gives the linearly independent solutions $\{e^{i\alpha x}P_n^{(-\alpha,\alpha)}(i\tan x), e^{-i\alpha x}P_n^{(\alpha,-\alpha)}(i\tan x)\}$. Then the identities are obtained by expressing the hypergeometric solutions as the linear combinations of the solutions obtained by the second approach. Therefore, for the general dihedral hypergeometric functions, by mimicking this idea, we make use of the Schrödinger equation with the

trigonometric Pöschl–Teller potential [8]

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{m(m+1)}{\sin^2 x}+\frac{n(n+1)}{\cos^2 x}\right] y_{m,n}(x;\lambda)=\lambda y_{m,n}(x;\lambda),\quad x\in\left(0,\frac{\pi}{2}\right), \tag{1.2}$$

since it can be transformed into a hypergeometric differential equation with characteristic exponent differences $\{m+\frac{1}{2}, n+\frac{1}{2}, \alpha\}$ up to permutations and sign changes where $\lambda=\alpha^2$ is a positive real parameter. That is, the solutions of (1.2) are dihedral Gauss hypergeometric functions. Then it remains to obtain the solutions in terms of elementary functions. This can be accomplished by another set of transformations on both dependent and independent variables leading to the ordinary differential equation for the Appell's $F_2(\xi, 2-\xi)$ function which is derived by Vidūnas in [12]. Returning back to the original variables via back transformations we obtain the solutions in terms of terminating F_2 sums with trigonometric arguments multiplied by some trigonometric pre-factors. We show that these solutions can be written as a finite sum involving the Jacobi polynomials of trigonometric argument with some trigonometric pre-factors. The general trigonometric dihedral identities are then obtained by expressing the hypergeometric solutions in terms of these solutions. On the other hand, the algebraic dihedral identities are obtained by a change of variable in the trigonometric ones.

The last equation was also considered by Darboux, in the second book of the series *Leçons sur la théorie générale des surfaces* [3, p. 212] where the parameter $\lambda=\alpha^2$ is replaced by $-h$. He obtained the two-parameter recurrence formula

$$y_{m+1,n+1}=y'_{m,n}+((n+1)\tan x-(m+1)\cot x)\,y_{m,n}, \tag{1.3}$$

for the general solution with the initializations

$$y_{m,-1}=y_{m,0},\qquad y_{-1,n}=y_{0,n},\qquad y_{0,0}=Ae^{i\alpha x}+Be^{-i\alpha x}, \tag{1.4}$$

in which $i\alpha$ stands for $\sqrt{h}$ in his notation. For non-zero α values, he deduced that the general solution has the form

$$y_{m,n}=e^{i\alpha x}\mathsf{P}_{m,n}+e^{-i\alpha x}\mathsf{Q}_{m,n}, \tag{1.5}$$

where $\mathsf{P}_{m,n}$ and $\mathsf{Q}_{m,n}$ are some polynomials of $\tan x$, $\cot x$ and $\sqrt{h}=i\alpha$. In this article, we will obtain explicit representations for the polynomials $\mathsf{P}_{m,n}$ and $\mathsf{Q}_{m,n}$, too.

Therefore, after reviewing the special cases where the characteristic exponent differences are in the unordered and unsigned sets $\{\frac{1}{2}, \frac{1}{2}, \alpha\}$, $\{\frac{1}{2}, n+\frac{1}{2}, \alpha\}$ and $\{n+\frac{1}{2}, n+\frac{1}{2}, \alpha\}$ in Sect. 2, we derive hypergeometric and elementary solutions of (1.2) in Sect. 3. Section 4 devoted to the derivation of the general trigonometric and algebraic dihedral identities for which the exponent differences are in the set $\{m+\frac{1}{2}, n+\frac{1}{2}, \alpha\}$ up to permutations and sign changes. In addition, we extend the identities to a more general class of Gauss hypergeometric functions where one of the non negative integers m or n is replaced by a complex parameter γ. Some corollaries

inferred from dihedral identities are given in Sect. 5. Finally, Sect. 6 concludes the paper with some comments.

2 Special cases of dihedral functions

The simplest dihedral functions are obtained when $m = n = 0$ which are the solutions of the Gauss hypergeometric equation with the characteristic exponent differences $\{\frac{1}{2}, \frac{1}{2}, \alpha\}$ up to permutations and sign changes. The most common identities we encounter in literature [4, 5] belong to this class. Another class of dihedral functions are obtained when $m = 0$ and n is a non negative integer which are the solutions of the hypergeometric equation with the characteristic exponent differences $\{\frac{1}{2}, n + \frac{1}{2}, \alpha\}$ up to permutations and sign changes. They are expressed in terms of terminating ${}_2F_1$ sums by Vidūnas under the title "The simple cases" in Sect. 4 of [13]. On the other hand, Alıcı [1] expressed these functions as combinations of the Jacobi polynomials with complex exponentials in trigonometric form which can be rewritten in algebraic form as well. The identities for the third class of simple dihedral functions in which $m = n$ can be obtained from the previous one via quadratic transformations. Clearly, for this class the characteristic exponent differences are in the set $\{n + \frac{1}{2}, n + \frac{1}{2}, \alpha\}$ up to permutations and sign changes. This case was also considered by Vidūnas under the title "The simple cases" in Sect. 4 of [13].

Therefore, in this section, for the sake of completeness we review the special cases where the characteristic exponent differences of the Gauss hypergeometric equation are in the sets $\{\frac{1}{2}, \frac{1}{2}, \alpha\}$, $\{\frac{1}{2}, n + \frac{1}{2}, \alpha\}$ and $\{n + \frac{1}{2}, n + \frac{1}{2}, \alpha\}$ up to permutations and sign changes.

2.1 The case $m = n = 0$

In this case, the hypergeometric identities can be derived by making use of the simple ordinary differential equation

$$-y''(x;\lambda) = \lambda y(x;\lambda), \quad \lambda = \alpha^2 > 0, \quad x \in \left(0, \frac{\pi}{2}\right), \tag{2.1}$$

which can be transformed into the form

$$\xi(1-\xi)y'' + \left(\frac{1}{2} - \xi\right)y' + \frac{1}{4}\alpha^2 y = 0, \quad \xi \in (0,1), \tag{2.2}$$

by using the transformation $\xi = \sin^2 x$. The last equation is a Gauss hypergeometric equation

$$\xi(1-\xi)u'' + (c - (a+b+1)\xi)\,u' - abu = 0, \tag{2.3}$$

with $a = -\alpha/2$, $b = \alpha/2$ and $c = 1/2$. It is known that the last equation has a set of linearly independent solutions around each regular singular points $\xi = 0, 1, \infty$. It admits a pair of linearly independent solutions [4, (15.10.2)–(15.10.7)], [5, pp. 42-43]

$$u_1(\xi;\alpha^2) = {}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\ \middle|\ \xi\right),$$
$$u_2(\xi;\alpha^2) = \xi^{1-c}\,{}_2F_1\left(\begin{matrix}a-c+1,\ b-c+1\\ 2-c\end{matrix}\ \middle|\ \xi\right), \tag{2.4}$$

around the regular singularity $\xi = 0$ with exponents 0 and $1-c$, or another set of linearly independent solutions

$$u_3(\xi;\alpha^2) = {}_2F_1\left(\begin{matrix}a,\ b\\ a+b-c+1\end{matrix}\ \middle|\ 1-\xi\right),$$
$$u_4(\xi;\alpha^2) = (1-\xi)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a,\ c-b\\ c-a-b+1\end{matrix}\ \middle|\ 1-\xi\right), \tag{2.5}$$

around the regular singularity $\xi = 1$ with exponents 0 and $c-a-b$. It admits another pair of linearly independent solutions

$$u_5(\xi;\alpha^2) = \xi^{-a}\,{}_2F_1\left(\begin{matrix}a,\ a-c+1\\ a-b+1\end{matrix}\ \middle|\ \frac{1}{\xi}\right),$$
$$u_6(\xi;\alpha^2) = \xi^{-b}\,{}_2F_1\left(\begin{matrix}b,\ b-c+1\\ b-a+1\end{matrix}\ \middle|\ \frac{1}{\xi}\right), \tag{2.6}$$

for $a \neq b$, around the third regular singularity at infinity with exponents a and b, which are valid as $\xi \to \infty$. Thus, returning back to the original variable x via $\xi = \sin^2 x$, we see that (2.1) has a set of linearly independent hypergeometric solutions

$$y_1(x;\alpha) = {}_2F_1\left(\begin{matrix}-\frac{\alpha}{2},\ \frac{\alpha}{2}\\ \frac{1}{2}\end{matrix}\ \middle|\ \sin^2 x\right),$$
$$y_2(x;\alpha) = \sin x\ {}_2F_1\left(\begin{matrix}\frac{1-\alpha}{2},\ \frac{1+\alpha}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ \sin^2 x\right), \tag{2.7}$$

around $x = 0$,

$$y_3(x;\alpha) = {}_2F_1\left(\begin{matrix}-\frac{\alpha}{2},\ \frac{\alpha}{2}\\ \frac{1}{2}\end{matrix}\ \middle|\ \cos^2 x\right),$$
$$y_4(x;\alpha) = \cos x\ {}_2F_1\left(\begin{matrix}\frac{1-\alpha}{2},\ \frac{1+\alpha}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ \cos^2 x\right), \tag{2.8}$$

around $x = \pi/2$ and

$$y_{5,6}(x;\alpha) = \sin^{\pm\alpha} x\ {}_2F_1\left(\begin{matrix}\mp\frac{\alpha}{2},\ \frac{1\mp\alpha}{2}\\ 1\mp\alpha\end{matrix}\ \middle|\ \frac{1}{\sin^2 x}\right), \tag{2.9}$$

which are valid as $\xi = \sin^2 x \to \infty$, i.e., as $x \to \pm i\infty$. Notice for the above six hypergeometric functions that the characteristic exponent differences of their hyper-

geometric equation are in the set $\{\frac{1}{2}, \frac{1}{2}, \alpha\}$ up to permutations and sign changes. On the other hand, (2.1) has another set of linearly independent elementary solutions $\left\{e^{-i\alpha x}, e^{i\alpha x}\right\}$ from which we may extract the real ones as

$$\{\sin(\alpha x), \cos(\alpha x)\}. \tag{2.10}$$

Therefore, the hypergeometric solutions in (2.7)–(2.9) can be represented as a linear combination of the elementary solutions in (2.10). For example, the first solution in (2.7) can be expressed in terms of the elementary ones in (2.10) as

$$\begin{aligned} {}_2F_1\left(\begin{matrix}-\frac{\alpha}{2}, & \frac{\alpha}{2} \\ \frac{1}{2} & \end{matrix}\,\middle|\, \sin^2 x\right) &= \cos x\ {}_2F_1\left(\begin{matrix}\frac{1-\alpha}{2}, & \frac{1+\alpha}{2} \\ \frac{1}{2} & \end{matrix}\,\middle|\, \sin^2 x\right) \\ &= \cos^{\pm\alpha} x\ {}_2F_1\left(\begin{matrix}\mp\frac{\alpha}{2}, & \frac{1\mp\alpha}{2} \\ \frac{1}{2} & \end{matrix}\,\middle|\, -\tan^2 x\right) = \cos(\alpha x), \end{aligned} \tag{2.11}$$

where we have used the Euler's and Pfaff's transformations [4, (15.8.1)], [5, p. 47]

$$\begin{aligned} {}_2F_1\left(\begin{matrix}a, & b \\ c & \end{matrix}\,\middle|\, z\right) &= (1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a, & c-b \\ c & \end{matrix}\,\middle|\, z\right) \\ &= (1-z)^{-a}\,{}_2F_1\left(\begin{matrix}a, & c-b \\ c & \end{matrix}\,\middle|\, \frac{z}{z-1}\right) \\ &= (1-z)^{-b}\,{}_2F_1\left(\begin{matrix}c-a, & b \\ c & \end{matrix}\,\middle|\, \frac{z}{z-1}\right). \end{aligned} \tag{2.12}$$

Similarly, the second solution in (2.7) can be written in terms of the elementary ones in (2.10) as

$$\begin{aligned} {}_2F_1\left(\begin{matrix}\frac{1+\alpha}{2}, & \frac{1-\alpha}{2} \\ \frac{3}{2} & \end{matrix}\,\middle|\, \sin^2 x\right) &= \cos x\ {}_2F_1\left(\begin{matrix}\frac{2-\alpha}{2}, & \frac{2+\alpha}{2} \\ \frac{3}{2} & \end{matrix}\,\middle|\, \sin^2 x\right) \\ &= \cos^{\pm\alpha-1} x\ {}_2F_1\left(\begin{matrix}\frac{1\mp\alpha}{2}, & \frac{2\mp\alpha}{2} \\ \frac{3}{2} & \end{matrix}\,\middle|\, -\tan^2 x\right) = \frac{\sin(\alpha x)}{\alpha \sin x}. \end{aligned} \tag{2.13}$$

The solutions in (2.8) do not yield any new identities but the shifted ($x \to \frac{\pi}{2} - x$) versions of the last two formulas. Finally, the hypergeometric solutions in (2.9) together with the elementary ones in (2.10) lead to the identities

$$\begin{aligned} {}_2F_1\left(\begin{matrix}\frac{\alpha}{2}, & \frac{\alpha+1}{2} \\ \alpha+1 & \end{matrix}\,\middle|\, \sec^2 x\right) &= i\tan x\ {}_2F_1\left(\begin{matrix}\frac{\alpha+1}{2}, & \frac{\alpha+2}{2} \\ \alpha+1 & \end{matrix}\,\middle|\, \sec^2 x\right) \\ &= (2\cos x)^{\alpha} e^{-i\alpha x} = \left(\frac{1+i\tan x}{2}\right)^{-\alpha}, \end{aligned} \tag{2.14}$$

and

$$(-i\tan x)^{-\alpha}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+1}{2},\frac{\alpha}{2}\\ \alpha+1\end{matrix}\,\middle|\,\csc^2 x\right)=(-i\tan x)^{-\alpha-1}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+2}{2},\frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\,\middle|\,\csc^2 x\right)$$
$$=(2\cos x)^{\alpha}e^{i\alpha x}=\left(\frac{1-i\tan x}{2}\right)^{-\alpha},\tag{2.15}$$

where $x\in\left(0,\frac{\pi}{2}\right)$ and $\alpha\notin\mathbb{Z}^-$. The last identity is obtained from the previous one by letting $x\to\frac{\pi}{2}-x$ and rearranging. Direct application of Pfaff's transformations to (2.14) leads to an equivalent identities of (2.15) which are valid on $\left(-\frac{\pi}{2},0\right)$. Here, α can be replaced by $-\alpha$ to obtain another four identities but they are structurally identical.

The identities in (2.11) and (2.13)–(2.15) can be expressed in an equivalent algebraic forms, too. For example, replacement of $z=-\tan^2 x$ in (2.11) leads to

$${}_2F_1\left(\begin{matrix}\frac{\alpha}{2},&-\frac{\alpha}{2}\\ &\frac{1}{2}\end{matrix}\,\middle|\,\frac{z}{z-1}\right)=(1-z)^{-\frac{1}{2}}\,{}_2F_1\left(\begin{matrix}\frac{1-\alpha}{2},&\frac{1+\alpha}{2}\\ &\frac{1}{2}\end{matrix}\,\middle|\,\frac{z}{z-1}\right)$$
$$=(1-z)^{\mp\frac{1}{2}\alpha}{}_2F_1\left(\begin{matrix}\mp\frac{\alpha}{2},&\frac{1\mp\alpha}{2}\\ &\frac{1}{2}\end{matrix}\,\middle|\,z\right)=\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\alpha/2}\right\}^{+}_{\sqrt{z}},\tag{2.16}$$

where the last one may be restated as it usually appears in literature [5, 13]

$${}_2F_1\left(\begin{matrix}\frac{\alpha}{2},&\frac{1+\alpha}{2}\\ &\frac{1}{2}\end{matrix}\,\middle|\,z\right)=\left\{\left(1-\sqrt{z}\right)^{-\alpha}\right\}^{+}_{\sqrt{z}}.\tag{2.17}$$

Here, for the sake of brevity, we use the special notation

$$\{A(t)\}^{\pm}_{t}=\frac{1}{2}\big(A(t)\pm A(-t)\big),\tag{2.18}$$

of Maier [6] which indicates the extraction of the even $(+)$ or the odd $(-)$ parts of $A(t)$ under the action $t\to-t$. The notation will be used throughout the paper whenever possible. Equivalent representation

$${}_2F_1\left(\begin{matrix}\frac{\alpha}{2},&-\frac{\alpha}{2}\\ &\frac{1}{2}\end{matrix}\,\middle|\,z\right)=(1-z)^{\frac{1}{2}}\,{}_2F_1\left(\begin{matrix}\frac{1-\alpha}{2},&\frac{1+\alpha}{2}\\ &\frac{1}{2}\end{matrix}\,\middle|\,z\right)$$
$$=(1-z)^{\pm\frac{1}{2}\alpha}{}_2F_1\left(\begin{matrix}\mp\frac{\alpha}{2},&\frac{1\mp\alpha}{2}\\ &\frac{1}{2}\end{matrix}\,\middle|\,\frac{z}{z-1}\right)\tag{2.19}$$
$$=\left\{\left(\sqrt{1-z}+\sqrt{-z}\right)^{\alpha}\right\}^{+}_{\alpha},$$

of (2.16) may be obtained by letting $z=\sin^2 x$ in (2.11), or replacing z with $z/(z-1)$ in (2.16). Similarly, an algebraic form of the identities in (2.13) read as

$$\begin{aligned}(1-z)^{-\frac{1}{2}}\,{}_2F_1\left(\begin{matrix}\frac{1+\alpha}{2},\ \frac{1-\alpha}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right)&=(1-z)^{-1}\,{}_2F_1\left(\begin{matrix}\frac{2-\alpha}{2},\ \frac{2+\alpha}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\&=(1-z)^{\mp\frac{1}{2}\alpha}\,{}_2F_1\left(\begin{matrix}\frac{1\mp\alpha}{2},\ \frac{2\mp\alpha}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ z\right)\\&=-\frac{1}{\alpha\sqrt{z}}\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\alpha/2}\right\}^{-}_{\sqrt{z}},\end{aligned}\tag{2.20}$$

upon letting $z=-\tan^2 x$, in which the last one may be rewritten as

$${}_2F_1\left(\begin{matrix}\frac{1+\alpha}{2}\frac{2+\alpha}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ z\right)=\frac{1}{\alpha\sqrt{z}}\ \left\{\left(1-\sqrt{z}\right)^{-\alpha}\right\}^{-}_{\sqrt{z}},\tag{2.21}$$

which is the form more frequently encountered in literature [5, 13]. Alternatively, letting $z=\sin^2 x$ in (2.13), or replacing z with $z/(z-1)$ in (2.20), we get an equivalent representation

$$\begin{aligned}{}_2F_1\left(\begin{matrix}\frac{1+\alpha}{2},\ \frac{1-\alpha}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ z\right)&=(1-z)^{\frac{1}{2}}\,{}_2F_1\left(\begin{matrix}\frac{2-\alpha}{2},\ \frac{2+\alpha}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ z\right)\\&=(1-z)^{-\frac{1}{2}(1\mp\alpha)}\,{}_2F_1\left(\begin{matrix}\frac{1\mp\alpha}{2},\ \frac{2\mp\alpha}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\&=\frac{1}{\alpha\sqrt{-z}}\left\{\left(\sqrt{1-z}+\sqrt{-z}\right)^{\alpha}\right\}^{-}_{\alpha}.\end{aligned}\tag{2.22}$$

Finally, letting $z=\sec^2 x$ in (2.14) and then applying Pfaff's transformations to the resulting algebraic identities we obtain the unified formula

$$\begin{aligned}{}_2F_1\left(\begin{matrix}\frac{\alpha}{2},\ \frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\ \middle|\ z\right)&=(1-z)^{\frac{1}{2}}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+1}{2},\ \frac{\alpha+2}{2}\\ \alpha+1\end{matrix}\ \middle|\ z\right)\\&=(1-z)^{-\frac{1}{2}\alpha}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+1}{2},\ \frac{\alpha}{2}\\ \alpha+1\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\&=(1-z)^{-\frac{1}{2}(\alpha+1)}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+2}{2},\ \frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\&=\left(\frac{1+\sqrt{1-z}}{2}\right)^{-\alpha},\quad \alpha\in\mathbb{C}\setminus\mathbb{Z}^{-}.\end{aligned}\tag{2.23}$$

which is invariant under the replacement of z by $z/(z-1)$.

On the other hand, in (2.13) or in (2.20)–(2.22) the right-hand sides are undetermined when $\alpha = 0$. In this case, taking limit in (2.13) as $\alpha \to 0$ we obtain

$$ {}_2F_1\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ \sin^2 x\right) = \cos x\ {}_2F_1\left(\begin{matrix}1,\ 1\\ \frac{3}{2}\end{matrix}\ \middle|\ \sin^2 x\right) = \sec x\ {}_2F_1\left(\begin{matrix}\frac{1}{2},\ 1\\ \frac{3}{2}\end{matrix}\ \middle|\ -\tan^2 x\right) = \frac{x}{\sin x}, \tag{2.24} $$

or in an algebraic form

$$ \begin{aligned} {}_2F_1\left(\begin{matrix}\frac{1}{2},\ 1\\ \frac{3}{2}\end{matrix}\ \middle|\ z\right) &= (1-z)^{-\frac{1}{2}}\ {}_2F_1\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right) = (1-z)^{-1}\ {}_2F_1\left(\begin{matrix}1,\ 1\\ \frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\ &= \frac{1}{\sqrt{z}}\left\{\log\left(1+\sqrt{z}\right)\right\}_{\sqrt{z}}^{-} = \frac{\arctan\left(\sqrt{-z}\right)}{\sqrt{-z}} = \frac{\operatorname{arctanh}\left(\sqrt{z}\right)}{\sqrt{z}}, \end{aligned} \tag{2.25} $$

where the equivalent representations of the right-hand sides are obtained from (2.21) as $\alpha \to 0$ and (2.24) by letting $z = -\tan^2 x$, respectively. Alternatively, from (2.22) as $\alpha \to 0$ and (2.24) we may write

$$ \begin{aligned} (1-z)^{-\frac{1}{2}}\ {}_2F_1\left(\begin{matrix}\frac{1}{2},\ 1\\ \frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right) &= {}_2F_1\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ z\right) = (1-z)^{\frac{1}{2}}\ {}_2F_1\left(\begin{matrix}1,\ 1\\ \frac{3}{2}\end{matrix}\ \middle|\ z\right)\\ &= \frac{\log\left(\sqrt{1-z}+\sqrt{-z}\right)}{\sqrt{-z}} = \frac{\arcsin\left(\sqrt{z}\right)}{\sqrt{z}}, \end{aligned} \tag{2.26} $$

which together with (2.25) implies the known equality

$$ \arctan z = \arcsin\left(\frac{z}{\sqrt{z^2+1}}\right), \tag{2.27} $$

for $iz \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$.

Remark 1 The identities in (2.11), (2.13)–(2.15) and (2.24) can be written with hyperbolic pre-factors and arguments. However, in order not to fill the content with similar formulas just an example will be enough. For instance, in (2.14) letting $x \to -iz$ we get

$$ \begin{aligned} {}_2F_1\left(\begin{matrix}\frac{\alpha}{2},\ \frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\ \middle|\ \operatorname{sech}^2 z\right) &= \tanh z\ {}_2F_1\left(\begin{matrix}\frac{\alpha+1}{2},\ \frac{\alpha+2}{2}\\ \alpha+1\end{matrix}\ \middle|\ \operatorname{sech}^2 z\right)\\ &= (\tanh z)^{-\alpha}\ {}_2F_1\left(\begin{matrix}\frac{\alpha+1}{2},\ \frac{\alpha}{2}\\ \alpha+1\end{matrix}\ \middle|\ -\operatorname{csch} z\right)\\ &= (\tanh z)^{-\alpha-1}\ {}_2F_1\left(\begin{matrix}\frac{\alpha+2}{2},\ \frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\ \middle|\ -\operatorname{csch} z\right)\\ &= (2\cosh z)^{\alpha} e^{-\alpha z} = \left(\frac{1+\tanh z}{2}\right)^{-\alpha}, \quad \alpha \notin \mathbb{Z}^-, \end{aligned} \tag{2.28} $$

which are valid for $\mathrm{Re}(z) > 0$. A part of the above formula, more specifically the one containing the second hypergeometric function in (2.28), can be found in [5, p. 39].

2.2 The case $m = 0, n \geq 0$

This subclass includes dihedral hypergeometric functions whose hypergeometric equation has characteristic exponent differences $\{\frac{1}{2}, n+\frac{1}{2}, \alpha\}$ up to a sign and permutations. In Sect. 4 of [13] Vidūnas presented algebraic identities in terms of terminating ${}_2F_1$ functions. In addition, in Sect. 6 of [13] Vidūnas provided trigonometric versions of these identities including several attractive examples at the end of Sect. 6.1. Similar trigonometric identites were obtained in [1] that can also be written in algebraic terms. We will review them in this subsection. They were obtained by making use of equation (1.1) which could also be obtained by means of (1.2) with $m \geq 0$, $n = 0$. The latter is known as the squared cosecant potential which is nothing but the shifted $(x \to \pi/2 - x)$ version of the squared secant potential in (1.1). So, the cases $m = 0$, $n \geq 0$ and $m \geq 0$, $n = 0$ are equivalent. The squared secant potential has a set of linearly independent solutions [1]

$$\left\{ e^{i\alpha x} P_n^{(\alpha,-\alpha)}(-i\tan x),\ e^{-i\alpha x} P_n^{(\alpha,-\alpha)}(i\tan x) \right\}, \tag{2.29}$$

besides their dihedral hypergeometric solutions [1]

$$\begin{aligned} y_n^{(1)}(x;\alpha^2) &= \cos^{n+1} x\ {}_2F_1\left(\begin{matrix} \frac{n-\alpha+1}{2}, & \frac{n+\alpha+1}{2} \\ \frac{1}{2} \end{matrix} \middle| \sin^2 x \right), \\ y_n^{(2)}(x;\alpha^2) &= \sin x \cos^{n+1} x\ {}_2F_1\left(\begin{matrix} \frac{n-\alpha+2}{2}, & \frac{n+\alpha+2}{2} \\ \frac{3}{2} \end{matrix} \middle| \sin^2 x \right), \end{aligned} \tag{2.30}$$

around the point $x = 0$

$$\begin{aligned} y_n^{(3)}(x;\alpha^2) &= \cos^{n+1} x\ {}_2F_1\left(\begin{matrix} \frac{n-\alpha+1}{2}, & \frac{n+\alpha+1}{2} \\ n+\frac{3}{2} \end{matrix} \middle| \cos^2 x \right), \\ y_n^{(4)}(x;\alpha^2) &= \cos^{-n} x\ {}_2F_1\left(\begin{matrix} \frac{-n+\alpha}{2}, & \frac{-n-\alpha}{2} \\ \frac{1}{2}-n \end{matrix} \middle| \cos^2 x \right), \end{aligned} \tag{2.31}$$

around the point $x = \pi/2$ and

$$\begin{aligned} y_n^{(5)}(x;\alpha^2) &= \cot^{n+1} x \sin^{\alpha} x\ {}_2F_1\left(\begin{matrix} \frac{n-\alpha+1}{2}, & \frac{n-\alpha+2}{2} \\ 1-\alpha \end{matrix} \middle| \csc^2 x \right), \\ y_n^{(6)}(x;\alpha^2) &= \cot^{n+1} x \sin^{-\alpha} x\ {}_2F_1\left(\begin{matrix} \frac{n+\alpha+1}{2}, & \frac{n+\alpha+2}{2} \\ 1+\alpha \end{matrix} \middle| \csc^2 x \right). \end{aligned} \tag{2.32}$$

which are valid as $\xi = \sin^2 x \to \infty$, i.e., as $x \to \pm i\infty$. Similarly, the squared cosecant potential assumes the set

$$\left\{ e^{-i\alpha x} P_m^{(\alpha,-\alpha)}(-i\cot x),\ e^{i\alpha x} P_m^{(\alpha,-\alpha)}(i\cot x) \right\}, \tag{2.33}$$

as linearly independent solutions and the above hypergeometric solutions with n replaced by m and x by $\pi/2 - x$. Clearly, for the hypergeometric functions in (2.30)–(2.32), the characteristic exponent differences of their hypergeometric equation are in the set $\{\frac{1}{2}, n + \frac{1}{2}, \alpha\}$ up to permutation and sign changes. Expressing the hypergeometric solutions in (2.30) in terms of the elementary ones in (2.29), the identities

$$\begin{aligned}
&\cos^{n+1} x\ {}_2F_1\left(\begin{matrix} \frac{n-\alpha+1}{2}, & \frac{n+\alpha+1}{2} \\ & \frac{1}{2} \end{matrix} \,\middle|\, \sin^2 x \right) \\
&\quad = \cos^{-n} x\ {}_2F_1\left(\begin{matrix} \frac{-n+\alpha}{2}, & \frac{-n-\alpha}{2} \\ & \frac{1}{2} \end{matrix} \,\middle|\, \sin^2 x \right) \\
&\quad = \cos^{\pm\alpha} x\ {}_2F_1\left(\begin{matrix} \frac{-n\mp\alpha}{2}, & \frac{n\mp\alpha+1}{2} \\ & \frac{1}{2} \end{matrix} \,\middle|\, -\tan^2 x \right) \\
&\quad = \frac{n!}{2^n} \frac{1}{\left(\frac{\alpha-n+1}{2}\right)_n} \left\{ e^{i\alpha x} P_n^{(\alpha,-\alpha)}(-i\tan x) \right\}_x^+,
\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}
&\cos^{n+1} x\ {}_2F_1\left(\begin{matrix} \frac{n-\alpha+2}{2}, & \frac{n+\alpha+2}{2} \\ & \frac{3}{2} \end{matrix} \,\middle|\, \sin^2 x \right) \\
&\quad = \cos^{-n} x\ {}_2F_1\left(\begin{matrix} \frac{-n+\alpha+1}{2}, & \frac{-n-\alpha+1}{2} \\ & \frac{3}{2} \end{matrix} \,\middle|\, \sin^2 x \right) \\
&\quad = \cos^{\pm\alpha-1} x\ {}_2F_1\left(\begin{matrix} \frac{-n\mp\alpha+1}{2}, & \frac{n\mp\alpha+2}{2} \\ & \frac{3}{2} \end{matrix} \,\middle|\, -\tan^2 x \right) \\
&\quad = \frac{n!}{2^{n+1}} \frac{(i\sin x)^{-1}}{\left(\frac{\alpha-n}{2}\right)_{n+1}} \left\{ e^{i\alpha x} P_n^{(\alpha,-\alpha)}(-i\tan x) \right\}_x^-,
\end{aligned} \tag{2.35}$$

were obtained in [1] where $P_n^{(\alpha,-\alpha)}(.)$ are the Jacobi polynomials $P_n^{(\alpha,\beta)}(.)$ with $\beta = -\alpha$ and $(a)_k$ is the so-called Pochhammer's symbol defined by

$$(a)_k = a(a+1)(a+2)\ldots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \qquad (a)_0 = 1. \tag{2.36}$$

The above formulas in (2.34) and (2.35) are valid for $n \in \mathbb{N}_0, \alpha \in \mathbb{C}$ and $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Similarly, the identities

$${}_2F_1\left(\begin{matrix} \frac{-n+\alpha}{2}, & \frac{-n-\alpha}{2} \\ & \frac{1}{2} - n \end{matrix} \,\middle|\, \sin^2 x \right)$$

$$
\begin{aligned}
&= \cos x \; {}_2F_1\left(\begin{matrix}\frac{-n-\alpha+1}{2}, & \frac{-n+\alpha+1}{2} \\ \frac{1}{2}-n & \end{matrix}\,\middle|\, \sin^2 x\right) \\
&= \cos^{n\pm\alpha} x \; {}_2F_1\left(\begin{matrix}\frac{-n\mp\alpha+1}{2}, & \frac{-n\mp\alpha}{2} \\ \frac{1}{2}-n & \end{matrix}\,\middle|\, -\tan^2 x\right) \\
&= \frac{n!}{2^n\left(\frac{1}{2}\right)_n}(-i\sin x)^n\left\{e^{i\alpha x}P_n^{(\alpha,-\alpha)}(i\cot x)\right\}_{\alpha}^{+},
\end{aligned} \tag{2.37}
$$

and

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}\frac{n-\alpha+1}{2}, & \frac{n+\alpha+1}{2} \\ n+\frac{3}{2} & \end{matrix}\,\middle|\, \sin^2 x\right) \\
&= \cos x \; {}_2F_1\left(\begin{matrix}\frac{n+\alpha+2}{2}, & \frac{n-\alpha+2}{2} \\ n+\frac{3}{2} & \end{matrix}\,\middle|\, \sin^2 x\right) \\
&= \cos^{\pm\alpha-n-1} x \; {}_2F_1\left(\begin{matrix}\frac{n\mp\alpha+1}{2}, & \frac{n\mp\alpha+2}{2} \\ n+\frac{3}{2} & \end{matrix}\,\middle|\, -\tan^2 x\right) \\
&= \frac{n!2^{n+1}\left(\frac{1}{2}\right)_{n+1}}{(\alpha-n)_{2n+1}}(i\sin x)^{-n-1}\left\{e^{i\alpha x}P_n^{(\alpha,-\alpha)}(i\cot x)\right\}_{\alpha}^{-},
\end{aligned} \tag{2.38}
$$

were obtained in [1] by representing the hypergeometric solutions in (2.31) as the linear combinations of the elementary ones in (2.29) where $n \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$ and $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Now, if we define the sets

$$
\mathcal{S}_n = \begin{cases} \{\pm 1, \pm 3, \ldots, \pm(n-1)\}, & \text{if } n \text{ is even} \\ \{\pm 1, \pm 3, \ldots, \pm n\}, & \text{if } n \text{ is odd} \end{cases}, \tag{2.39}
$$

for $n \in \mathbb{N}$ with $S_0 = \emptyset$ and

$$
\mathcal{T}_n = \begin{cases} \{0, \pm 2, \pm 4, \ldots, \pm n\}, & \text{if } n \text{ is even} \\ \{0, \pm 2, \pm 4, \ldots, \pm(n-1)\}, & \text{if } n \text{ is odd} \end{cases}, \tag{2.40}
$$

for $n \in \mathbb{N}_0$, then it is clear from (2.34) that the Pochhammer symbol in the denominator of the right-hand side becomes zero if α is in the set $\mathcal{S}_n$ when n is even and α belongs to the set $\mathcal{T}_n$ when n is odd. But in this case the hypergeometric functions on the left side are defined. Thus, the numerator of the right-hand side must accordingly be zero, too. This means that one needs to look at the limiting cases when $\alpha \to k \in \mathcal{S}_n$ if n is even and $\alpha \to k \in \mathcal{T}_n$ if n is odd in order to remove the indeterminacy. Similarly, in (2.35) one needs to take limit as $\alpha \to k \in \mathcal{S}_n$ if n is odd and $\alpha \to k \in \mathcal{T}_n$ if n is even to remove the indeterminacy. Likewise, in (2.38), one needs to look at the limiting cases when α is an element of the set

$$
\mathcal{U}_n = \{0, \pm 1, \pm 2, \ldots, \pm n\}, \tag{2.41}
$$

in order to remove the indeterminacy. Finally, by comparing the hypergeometric solutions in (2.32) with the elementary ones in (2.29), we get

$$\begin{aligned}
&(-i\tan x)^{n-\alpha}\ {}_2F_1\left(\begin{matrix}\frac{-n+\alpha+1}{2},\ \frac{-n+\alpha}{2}\\ \alpha+1\end{matrix}\ \middle|\csc^2 x\right)\\
&\quad=(-i\tan x)^{-(n+\alpha+1)}\ {}_2F_1\left(\begin{matrix}\frac{n+\alpha+1}{2},\ \frac{n+\alpha+2}{2}\\ \alpha+1\end{matrix}\ \middle|\csc^2 x\right)\\
&\quad=\frac{n!}{(\alpha+1)_n}\left(\frac{1-i\tan x}{2}\right)^{-\alpha}P_n^{(\alpha,-\alpha)}(-i\tan x).
\end{aligned}\tag{2.42}$$

and

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}\frac{n+\alpha+1}{2},\ \frac{-n+\alpha}{2}\\ \alpha+1\end{matrix}\ \middle|\sec^2 x\right)\\
&\quad=i\tan x\ {}_2F_1\left(\begin{matrix}\frac{-n+\alpha+1}{2},\ \frac{n+\alpha+2}{2}\\ \alpha+1\end{matrix}\ \middle|\sec^2 x\right)\\
&\quad=\frac{n!}{(\alpha+1)_n}\left(\frac{1+i\tan x}{2}\right)^{-\alpha}P_n^{(\alpha,-\alpha)}(i\tan x),
\end{aligned}\tag{2.43}$$

in which $n\in\mathbb{N}_0$, $\alpha\in\mathbb{C}\setminus\mathbb{Z}^-$ and $x\in\left(0,\frac{\pi}{2}\right)$. They were miss derived in [1] as a single equation containing the four hypergeometric functions in the last two formulas with the false restriction on α, namely $\alpha\neq -1,-2,\ldots,-n$. Another four identities can be obtained by the replacement $\alpha\to-\alpha$ in the last equation which are valid for $\alpha\in\mathbb{C}\setminus\mathbb{Z}^+$.

Note that the right-hand sides of (2.34) and (2.35) are the real and imaginary parts of the solution $e^{i\alpha x}P_n^{(\alpha,-\alpha)}(-i\tan x)$. Thus, they can be written as

$$\begin{aligned}
\mathrm{Re}\left(e^{i\alpha x}P_n^{(\alpha,-\alpha)}(-i\tan x)\right)&=\left\{e^{i\alpha x}P_n^{(\alpha,-\alpha)}(-i\tan x)\right\}_i^+,\\
\mathrm{Im}\left(e^{i\alpha x}P_n^{(\alpha,-\alpha)}(-i\tan x)\right)&=\left\{-ie^{i\alpha x}P_n^{(\alpha,-\alpha)}(-i\tan x)\right\}_i^-,
\end{aligned}\tag{2.44}$$

by using the notation in (2.18). Similar discussion applies to the identities in (2.37) and (2.38) with the solution $(-i)^n e^{i\alpha x}P_n^{(\alpha,-\alpha)}(i\cot x)$. However, the above representations are more preferable in writing the equivalent algebraic forms of these identities. In fact, letting $z=-\tan^2 x$ in (2.34) obtain

$$\begin{aligned}
&(1-z)^{-\frac{1}{2}(n+1)}\ {}_2F_1\left(\begin{matrix}\frac{n-\alpha+1}{2},\ \frac{n+\alpha+1}{2}\\ \frac{1}{2}\end{matrix}\ \middle|\frac{z}{z-1}\right)\\
&\quad=(1-z)^{\frac{1}{2}n}\ {}_2F_1\left(\begin{matrix}\frac{-n+\alpha}{2},\ \frac{-n-\alpha}{2}\\ \frac{1}{2}\end{matrix}\ \middle|\frac{z}{z-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (1-z)^{\mp\frac{1}{2}\alpha}\ {}_2F_1\left(\begin{matrix}\frac{-n\mp\alpha}{2},\ \frac{n\mp\alpha+1}{2}\\ \frac{1}{2}\end{matrix}\ \middle|\ z\right)\\
&= \frac{n!}{2^n}\frac{1}{\left(\frac{\alpha-n+1}{2}\right)_n}\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\alpha/2} P_n^{(\alpha,-\alpha)}(\sqrt{z})\right\}^{+}_{\sqrt{z}},
\end{aligned}
\tag{2.45}
$$

where $n \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus [1, \infty)$. Again one needs to look at the limiting cases when $\alpha \to k \in \mathcal{S}_n$ if n is even and $\alpha \to k \in \mathcal{T}_n$ if n is odd where the sets $\mathcal{S}_n$ and $\mathcal{T}_n$ are defined in (2.39) and (2.40), respectively.

Likewise, letting $z = -\tan^2 x$ in (2.35) leads to

$$
\begin{aligned}
&(1-z)^{-\frac{1}{2}(n+2)}\ {}_2F_1\left(\begin{matrix}\frac{n-\alpha+2}{2},\ \frac{n+\alpha+2}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&\quad= (1-z)^{\frac{1}{2}(n-1)}\ {}_2F_1\left(\begin{matrix}\frac{-n+\alpha+1}{2},\ \frac{-n-\alpha+1}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&\quad= (1-z)^{\mp\frac{1}{2}\alpha}\ {}_2F_1\left(\begin{matrix}\frac{-n\mp\alpha+1}{2},\ \frac{n\mp\alpha+2}{2}\\ \frac{3}{2}\end{matrix}\ \middle|\ z\right)\\
&\quad= \frac{n!}{2^{n+1}}\frac{1}{\left(\frac{\alpha-n}{2}\right)_{n+1}}\left(-\sqrt{z}\right)^{-1}\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\alpha/2} P_n^{(\alpha,-\alpha)}(\sqrt{z})\right\}^{-}_{\sqrt{z}},
\end{aligned}
\tag{2.46}
$$

in which $n \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus [1, \infty)$. Here, one needs to consider the limiting cases as $\alpha \to k \in \mathcal{S}_n$ if n is odd and $\alpha \to k \in \mathcal{T}_n$ if n is even to remove the indeterminacy. Note that, no matter which branch of the square root function ($z = -\tan^2 x \Rightarrow i \tan x = \pm\sqrt{z}$) is chosen in (2.45) and (2.46) we obtain the same formulas since their right-hand sides are invariant under $\sqrt{z} \to -\sqrt{z}$.

Similarly, letting $z = -\tan^2 x$ in (2.37) and (2.38) yields

$$
\begin{aligned}
&(1-z)^{\frac{1}{2}n}\ {}_2F_1\left(\begin{matrix}\frac{-n+\alpha}{2},\ \frac{-n-\alpha}{2}\\ \frac{1}{2}-n\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&\quad= (1-z)^{\frac{1}{2}(n-1)}\ {}_2F_1\left(\begin{matrix}\frac{-n-\alpha+1}{2},\ \frac{-n+\alpha+1}{2}\\ \frac{1}{2}-n\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&\quad= (1-z)^{\mp\frac{1}{2}\alpha}\ {}_2F_1\left(\begin{matrix}\frac{-n\mp\alpha+1}{2},\ \frac{-n\mp\alpha}{2}\\ \frac{1}{2}-n\end{matrix}\ \middle|\ z\right)\\
&\quad= \frac{n!}{2^n\left(\frac{1}{2}\right)_n}\left(\sqrt{z}\right)^n\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\alpha/2} P_n^{(\alpha,-\alpha)}\left(1/\sqrt{z}\right)\right\}^{+}_{\alpha},
\end{aligned}
\tag{2.47}
$$

and

$$\begin{aligned}
&(1-z)^{-\frac{1}{2}(n+1)}\ {}_2F_1\left(\begin{matrix}\frac{n-\alpha+1}{2},\ \frac{n+\alpha+1}{2}\\ n+\frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&\quad=(1-z)^{-\frac{1}{2}(n+2)}\ {}_2F_1\left(\begin{matrix}\frac{n+\alpha+2}{2},\ \frac{n-\alpha+2}{2}\\ n+\frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&\quad=(1-z)^{\mp\frac{1}{2}\alpha}\ {}_2F_1\left(\begin{matrix}\frac{n\mp\alpha+1}{2},\ \frac{n\mp\alpha+2}{2}\\ n+\frac{3}{2}\end{matrix}\ \middle|\ z\right)\\
&\quad=\frac{n!2^{n+1}\left(\frac{1}{2}\right)_{n+1}}{(\alpha-n)_{2n+1}}\left(-\sqrt{z}\right)^{-n-1}\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\alpha/2}P_n^{(\alpha,-\alpha)}\left(1/\sqrt{z}\right)\right\}_{\alpha}^{-},
\end{aligned}\tag{2.48}$$

respectively, in which $n\in\mathbb{N}_0$, $\alpha\in\mathbb{C}$, $z\in\mathbb{C}\setminus[1,\infty)$. Note that (2.48) is valid for $\alpha\in\mathcal{U}_n$ in the limit sense because of indeterminacy where $\mathcal{U}_n$ is defined in (2.41). Again, either branch $i\tan x=\pm\sqrt{z}$ of the square root function is welcome since the last lines of (2.47) and (2.48) are invariant under the replacement $\sqrt{z}\to-\sqrt{z}$.

Finally, letting $-i\tan x=\sqrt{1-z}$ and $i\tan x=\sqrt{1-z}$ in (2.42) and (2.43), respectively, (so that $z=\sec^2 x$) we obtain the unified formula

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}\frac{n+\alpha+1}{2},\ \frac{-n+\alpha}{2}\\ \alpha+1\end{matrix}\ \middle|\ z\right)\\
&\quad=(1-z)^{\frac{1}{2}}\ {}_2F_1\left(\begin{matrix}\frac{-n+\alpha+1}{2},\ \frac{n+\alpha+2}{2}\\ \alpha+1\end{matrix}\ \middle|\ z\right)\\
&\quad=(1-z)^{\frac{1}{2}(n-\alpha)}\ {}_2F_1\left(\begin{matrix}\frac{-n+\alpha+1}{2},\ \frac{-n+\alpha}{2}\\ \alpha+1\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&\quad=(1-z)^{-\frac{1}{2}(n+\alpha+1)}\ {}_2F_1\left(\begin{matrix}\frac{n+\alpha+1}{2},\ \frac{n+\alpha+2}{2}\\ \alpha+1\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&\quad=\frac{n!}{(\alpha+1)_n}\left(\frac{1+\sqrt{1-z}}{2}\right)^{-\alpha}P_n^{(\alpha,-\alpha)}\left(\sqrt{1-z}\right).
\end{aligned}\tag{2.49}$$

where $n\in\mathbb{N}_0$, $\alpha\in\mathbb{C}\setminus\mathbb{Z}^-$ and $z\in\mathbb{C}\setminus[1,\infty)$. The last identity may also be obtained by just letting $-i\tan x=\sqrt{1-z}$ in (2.42) (or $i\tan x=\sqrt{1-z}$ in (2.43)) and then applying Pfaff's transformations to the resulting algebraic identities.

Remark 2 The hypergeometric identites of this subsection can be rewritten in hyperbolic terms, too. For example, by letting $x \to iz$, (2.37) can be rephrased as

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}\frac{-n+\alpha}{2}, \frac{-n-\alpha}{2}\\ \frac{1}{2}-n\end{matrix}\,\middle|\, -\sinh^2 z\right)\\
&\quad = \cosh z\ {}_2F_1\left(\begin{matrix}\frac{-n-\alpha+1}{2}, \frac{-n+\alpha+1}{2}\\ \frac{1}{2}-n\end{matrix}\,\middle|\, -\sinh^2 z\right)\\
&\quad = \cosh^{n\pm\alpha} z\ {}_2F_1\left(\begin{matrix}\frac{-n\mp\alpha+1}{2}, \frac{-n\mp\alpha}{2}\\ \frac{1}{2}-n\end{matrix}\,\middle|\, \tanh^2 z\right)\\
&\quad = \frac{n!}{2^n\left(\frac{1}{2}\right)_n}\sinh^n z\left\{e^{-\alpha z}P_n^{(\alpha,-\alpha)}(\coth z)\right\}_\alpha^+,
\end{aligned} \tag{2.50}$$

which are valid for $n \in \mathbb{N}_0$ and $\alpha, z \in \mathbb{C}$.

Remark 3 Notice that the above identities in this subsection reduce to the ones in the previous subsection when $n = 0$ as expected.

2.3 The case $m = n$

For this special class of dihedral functions, the characteristic exponent differences of the Gauss hypergeometric equation are in the set $\{n+\frac{1}{2}, n+\frac{1}{2}, \alpha\}$ up to permutations and sign changes. For this class, the identities can be obtained from the previous one where $m = 0$ (or $n = 0$) via quadratic transformations. This case was also considered in Sects. 4 and 6 of [13] to obtain algebraic and trigonometric identities, respectively, with the help of quadratic transformations. We also make use of quadratic transformations and two identities can be obtained from (2.37)–(2.38) by applying the identity [4, (15.8.18)]

$${}_2F_1\left(\begin{matrix}a, b\\ \frac{a+b+1}{2}\end{matrix}\,\middle|\, z\right) = {}_2F_1\left(\begin{matrix}\frac{a}{2}, \frac{b}{2}\\ \frac{a+b+1}{2}\end{matrix}\,\middle|\, 4z(1-z)\right), \quad \mathrm{Re}(z) < \frac{1}{2}, \tag{2.51}$$

or its trigonometric equivalent

$${}_2F_1\left(\begin{matrix}a, b\\ \frac{a+b+1}{2}\end{matrix}\,\middle|\, \sin^2\theta\right) = {}_2F_1\left(\begin{matrix}\frac{a}{2}, \frac{b}{2}\\ \frac{a+b+1}{2}\end{matrix}\,\middle|\, \sin^2 2\theta\right), \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}. \tag{2.52}$$

To this end, first we let $x \to 2x$ and $\alpha \to \alpha/2$ in the first hypergeometric identities of (2.37)–(2.38), then we apply (2.52) to the resulting identities and finally we make

use of the Euler's and Pfaff's transformations in (2.12) to obtain

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}-n+\frac{\alpha}{2},\ -n-\frac{\alpha}{2}\\ \frac{1}{2}-n\end{matrix}\ \middle|\sin^2 x\right)\\
&\quad=\cos^{2n+1}x\ {}_2F_1\left(\begin{matrix}\frac{1-\alpha}{2},\ \frac{1+\alpha}{2}\\ \frac{1}{2}-n\end{matrix}\ \middle|\sin^2 x\right)\\
&\quad=\cos^{2n\pm\alpha}x\ {}_2F_1\left(\begin{matrix}-n\mp\frac{\alpha}{2},\ \frac{1\mp\alpha}{2}\\ \frac{1}{2}-n\end{matrix}\ \middle|-\tan^2 x\right)\\
&\quad=\frac{n!}{2^n\left(\frac{1}{2}\right)_n}(-i\sin 2x)^n\left\{e^{i\alpha x}P_n^{(\frac{\alpha}{2},-\frac{\alpha}{2})}(i\cot 2x)\right\}_\alpha^+,
\end{aligned}\tag{2.53}$$

and

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}n+1-\frac{\alpha}{2},\ n+1+\frac{\alpha}{2}\\ n+\frac{3}{2}\end{matrix}\ \middle|\sin^2 x\right)\\
&\quad=\cos^{-2n-1}x\ {}_2F_1\left(\begin{matrix}\frac{1-\alpha}{2},\ \frac{1+\alpha}{2}\\ n+\frac{3}{2}\end{matrix}\ \middle|\sin^2 x\right)\\
&\quad=\cos^{-2n-2\pm\alpha}x\ {}_2F_1\left(\begin{matrix}n+1\mp\frac{\alpha}{2},\ \frac{1\pm\alpha}{2}\\ n+\frac{3}{2}\end{matrix}\ \middle|-\tan^2 x\right)\\
&\quad=\frac{n!2^{n+1}\left(\frac{1}{2}\right)_{n+1}}{\left(\frac{\alpha}{2}-n\right)_{2n+1}}(i\sin 2x)^{-n-1}\left\{e^{i\alpha x}P_n^{(\frac{\alpha}{2},-\frac{\alpha}{2})}(i\cot 2x)\right\}_\alpha^-,
\end{aligned}\tag{2.54}$$

which are valid when $n\in\mathbb{N}_0$, $\alpha\in\mathbb{C}$ and $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. In the last formula one needs to look at the limiting cases when $\alpha\to k\in 2\mathcal{U}_n$ in order to remove the indeterminacy, where the set $\mathcal{U}_n$ is defined in (2.41).

The third identity can be obtained from (2.42) by applying the quadratic transformation [4, (15.8.13)]

$${}_2F_1\left(\begin{matrix}a,\ b\\ 2b\end{matrix}\ \middle|z\right)=\left(1-\frac{z}{2}\right)^{-a}{}_2F_1\left(\begin{matrix}\frac{a}{2},\ \frac{a+1}{2}\\ b+\frac{1}{2}\end{matrix}\ \middle|\frac{z^2}{(2-z)^2}\right),\quad |\arg(1-z)|<\pi,\tag{2.55}$$

or its equivalent

$${}_2F_1\left(\begin{matrix}a,\ b\\ 2b\end{matrix}\ \middle|\sec^2\theta\right)=\left(1-\frac{\sec^2\theta}{2}\right)^{-a}{}_2F_1\left(\begin{matrix}\frac{a}{2},\ \frac{a+1}{2}\\ b+\frac{1}{2}\end{matrix}\ \middle|\sec^2 2\theta\right),\quad -\frac{\pi}{4}<\theta<\frac{\pi}{4},\tag{2.56}$$

to the first hypergeometric identity of (2.42). For example, shifting $x \to \frac{\pi}{2} - 2x$ and halving $\alpha \to \alpha/2$ in the first hypergeometric identity of (2.42) we obtain

$$\begin{aligned}(-i\cot 2x)^{n-\frac{\alpha}{2}}\,{}_2F_1\left(\begin{matrix}\frac{-2n+\alpha+2}{4}, & \frac{-2n+\alpha}{4}\\ & \frac{\alpha}{2}+1\end{matrix}\,\middle|\,\sec^2 2x\right)\\ =\frac{n!}{\left(\frac{\alpha}{2}+1\right)_n}\left(\frac{1-i\cot 2x}{2}\right)^{-\frac{\alpha}{2}} P_n^{(\frac{\alpha}{2},-\frac{\alpha}{2})}(-i\cot 2x),\end{aligned} \tag{2.57}$$

where now $x \in \left(0, \frac{\pi}{4}\right)$. Then by applying (2.56) to the last equation and rearranging we get

$$\begin{aligned}(-i\cot x)^{n-\alpha}\,{}_2F_1\left(\begin{matrix}-n+\frac{\alpha}{2}, & \frac{\alpha+1}{2}\\ & \alpha+1\end{matrix}\,\middle|\,\sec^2 x\right)\\ =\frac{n!}{\left(\frac{\alpha}{2}+1\right)_n}\left(\frac{1-i\cot x}{2}\right)^{-\alpha} P_n^{(\frac{\alpha}{2},-\frac{\alpha}{2})}(-i\cot 2x),\end{aligned} \tag{2.58}$$

which is valid on the larger interval $x \in \left(0, \frac{\pi}{2}\right)$. Finally, by applying Eulers' transformation to the last equation, we get

$$\begin{aligned}&(-i\cot x)^{n-\alpha}\,{}_2F_1\left(\begin{matrix}-n+\frac{\alpha}{2}, & \frac{\alpha+1}{2}\\ & \alpha+1\end{matrix}\,\middle|\,\sec^2 x\right)\\ &=(-i\cot x)^{-n-\alpha-1}\,{}_2F_1\left(\begin{matrix}n+1+\frac{\alpha}{2}, & \frac{\alpha+1}{2}\\ & \alpha+1\end{matrix}\,\middle|\,\sec^2 x\right)\\ &=\frac{n!}{\left(\frac{\alpha}{2}+1\right)_n}\left(\frac{1-i\cot x}{2}\right)^{-\alpha} P_n^{(\frac{\alpha}{2},-\frac{\alpha}{2})}(-i\cot 2x).\end{aligned} \tag{2.59}$$

Likewise, the fourth identity can be obtained by applying the quadratic transformation [4, (15.8.14)]

$${}_2F_1\left(\begin{matrix}a, & b\\ & 2b\end{matrix}\,\middle|\,z\right)=(1-z)^{-a/2}\,{}_2F_1\left(\begin{matrix}\frac{a}{2}, & b-\frac{a}{2}\\ & b+\frac{1}{2}\end{matrix}\,\middle|\,\frac{z^2}{4(z-1)}\right), \quad |\arg(1-z)|<\pi, \tag{2.60}$$

or its equivalent

$${}_2F_1\left(\begin{matrix}a, & b\\ & 2b\end{matrix}\,\middle|\,\csc^2\theta\right)=\left(1-\csc^2\theta\right)^{-a/2}\,{}_2F_1\left(\begin{matrix}\frac{a}{2}, & b-\frac{a}{2}\\ & b+\frac{1}{2}\end{matrix}\,\middle|\,\csc^2 2x\right), \quad -\frac{\pi}{4}<\theta<\frac{\pi}{4}, \tag{2.61}$$

to (2.43) exactly in the same way as in the previous derivation. This leads to

$$\begin{aligned}
&(i\cot x)^{-n}\ {}_2F_1\left(\begin{matrix}-n+\frac{\alpha}{2},\ \frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\ \middle|\csc^2 x\right)\\
&\quad=(i\cot x)^{n+1}\ {}_2F_1\left(\begin{matrix}n+1+\frac{\alpha}{2},\ \frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\ \middle|\csc^2 x\right)\\
&\quad=\frac{n!}{\left(\frac{\alpha}{2}+1\right)_n}\left(\frac{1+i\cot x}{2}\right)^{-\alpha}P_n^{\left(\frac{\alpha}{2},-\frac{\alpha}{2}\right)}(i\cot 2x),
\end{aligned}\tag{2.62}$$

where $n\in\mathbb{N}_0$, $\alpha\in\mathbb{C}\setminus\mathbb{Z}^-$ and $x\in\left(0,\frac{\pi}{2}\right)$.

Letting $z=-\tan^2 x$, leads to the following algebraic versions

$$\begin{aligned}
&(1-z)^n\ {}_2F_1\left(\begin{matrix}-n+\frac{\alpha}{2},\ -n-\frac{\alpha}{2}\\ \frac{1}{2}-n\end{matrix}\ \middle|\frac{z}{z-1}\right)\\
&\quad=(1-z)^{-\frac{1}{2}}\ {}_2F_1\left(\begin{matrix}\frac{1-\alpha}{2},\ \frac{1+\alpha}{2}\\ \frac{1}{2}-n\end{matrix}\ \middle|\frac{z}{z-1}\right)\\
&\quad=(1-z)^{\mp\frac{1}{2}\alpha}\ {}_2F_1\left(\begin{matrix}-n\mp\frac{\alpha}{2},\ \frac{1\mp\alpha}{2}\\ \frac{1}{2}-n\end{matrix}\ \middle|z\right)\\
&\quad=\frac{n!}{\left(\frac{1}{2}\right)_n}(\sqrt{z})^n\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\frac{\alpha}{2}}P_n^{\left(\frac{\alpha}{2},-\frac{\alpha}{2}\right)}\left(\frac{1+z}{2\sqrt{z}}\right)\right\}_\alpha^+,
\end{aligned}\tag{2.63}$$

and

$$\begin{aligned}
&(1-z)^{-n-1}\ {}_2F_1\left(\begin{matrix}n+1-\frac{\alpha}{2},\ n+1+\frac{\alpha}{2}\\ n+\frac{3}{2}\end{matrix}\ \middle|\frac{z}{z-1}\right)\\
&\quad=(1-z)^{-\frac{1}{2}}\ {}_2F_1\left(\begin{matrix}\frac{1-\alpha}{2},\ \frac{1+\alpha}{2}\\ n+\frac{3}{2}\end{matrix}\ \middle|\frac{z}{z-1}\right)\\
&\quad=(1-z)^{\mp\frac{1}{2}\alpha}\ {}_2F_1\left(\begin{matrix}n+1\mp\frac{\alpha}{2},\ \frac{1\pm\alpha}{2}\\ n+\frac{3}{2}\end{matrix}\ \middle|z\right)\\
&\quad=\frac{n!\left(\frac{1}{2}\right)_{n+1}}{\left(\frac{\alpha}{2}-n\right)_{2n+1}}(-\sqrt{z})^{-n-1}\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\frac{\alpha}{2}}P_n^{\left(\frac{\alpha}{2},-\frac{\alpha}{2}\right)}\left(\frac{1+z}{2\sqrt{z}}\right)\right\}_\alpha^-,
\end{aligned}\tag{2.64}$$

of (2.53) and (2.54), respectively where $n\in\mathbb{N}_0$, $\alpha\in\mathbb{C}$ and $z\in\mathbb{C}\setminus[1,\infty)$ and in the latter, one needs to look at the limiting cases when $\alpha\to k\in 2\mathcal{U}_n$ in order to remove the indeterminacy, where the set $\mathcal{U}_n$ is defined in (2.41).

Similarly, letting $z = \sec^2 x$ in (2.62) and then applying Pfaff's transformations to the resulting algebraic identities we obtain the unified formula

$$\begin{aligned}
&(1-z)^{\frac{1}{2}(n-\alpha)}\ {}_2F_1\left(\begin{matrix}-n+\frac{\alpha}{2},\ \frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&\quad=(1-z)^{-\frac{1}{2}(n+\alpha+1)}\ {}_2F_1\left(\begin{matrix}n+1+\frac{\alpha}{2},\ \frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&\quad=(1-z)^{-\frac{1}{2}n}\ {}_2F_1\left(\begin{matrix}-n+\frac{\alpha}{2},\ \frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\ \middle|\ z\right)\\
&\quad=(1-z)^{\frac{1}{2}(n+1)}\ {}_2F_1\left(\begin{matrix}n+1+\frac{\alpha}{2},\ \frac{\alpha+1}{2}\\ \alpha+1\end{matrix}\ \middle|\ z\right)\\
&\quad=\frac{n!}{\left(\frac{\alpha}{2}+1\right)_n}\left(\frac{1+\sqrt{1-z}}{2}\right)^{-\alpha}P_n^{\left(\frac{\alpha}{2},-\frac{\alpha}{2}\right)}\left(\frac{2-z}{2\sqrt{1-z}}\right),
\end{aligned}\tag{2.65}$$

which is valid for $n \in \mathbb{N}_0$, $\alpha \in \mathbb{C}\setminus\mathbb{Z}^-$ and $z \in \mathbb{C}\setminus[1,\infty)$.

Note that all the hypergeometric functions in this subsection are the solutions of the Gauss hypergeometric equation with characteristic exponent differences $\{n+\frac{1}{2}, n+\frac{1}{2}, \alpha\}$ up to permutations and sign changes. Once we have found the formulas for the general case $\{m+\frac{1}{2}, n+\frac{1}{2}, \alpha\}$, we will compare them with the results of this subsection when $m = n$ which will lead to an interesting Jacobi polynomial identity.

3 General solutions of the trigonometric Pöschl–Teller potential

In this section we will obtain the general solution of (1.2) in terms of hypergeometric and elementary functions. We will see that the hypergeometric solutions are the general dihedral functions and elementary solutions can be written as a terminating sum involving combinations of the complex exponentials with the Jacobi polynomials of trigonometric argument.

3.1 General solution in terms of Gauss hypergeometric functions

Let us keep the eigenvalue parameter $\lambda = \alpha^2 > 0$ in (1.2) as a positive real constant. The equation in (1.2) is referred to as the time independent Schrödinger equation with the trigonometric Pöschl–Teller potential after [8]. Clearly, (1.2) is invariant under $\alpha \to -\alpha, m \to -(m+1), n \to -(n+1)$ and $(m \leftrightarrow n,\ \ x \to \frac{\pi}{2} \pm x)$. By letting,

$$\xi = \sin^2 x, \qquad \xi \in (0,1), \tag{3.1}$$

we transform (1.2) to the form

$$\left[\xi(1-\xi)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+\left(\frac{1}{2}-\xi\right)\frac{\mathrm{d}}{\mathrm{d}\xi}-\frac{1}{4}\left(\frac{m(m+1)}{\xi}+\frac{n(n+1)}{1-\xi}-\alpha^2\right)\right]y(\xi;\alpha^2)=0. \tag{3.2}$$

The singularities at $\xi = 0, 1$ of the last equation recommend searching for a solution of the type

$$y(\xi;\alpha)=\xi^r(1-\xi)^s u(\xi;\alpha),\quad r,s\in\mathbb{R}, \tag{3.3}$$

which gives the possibility of determining the value of r and s in order to remove the undesired terms in (3.2) proportional to $1/\xi$ and $1/(1-\xi)$. Then, (3.2) takes the form

$$\left[\xi(1-\xi)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+\left(2r+\frac{1}{2}-(2r+2s+1)\xi\right)\frac{\mathrm{d}}{\mathrm{d}\xi}+\frac{1}{4\xi}\left(4r^2-2r-m(m+1)\right)\right.$$
$$\left.+\frac{1}{4(1-\xi)}\left(4s^2-2s-n(n+1)\right)+\frac{1}{4}\alpha^2-(r+s)^2\right]u(\xi;\alpha)=0, \tag{3.4}$$

where we have used $\xi/(1-\xi) = -1 + 1/(1-\xi)$. If r and s are chosen as the roots of the quadratic equations

$$4r^2-2r-m(m+1)=0,\qquad 4s^2-2s-n(n+1)=0, \tag{3.5}$$

that is $r = -\frac{m}{2}, \frac{m+1}{2}$ and $s = -\frac{n}{2}, \frac{n+1}{2}$, then the terms proportional to $1/\xi$ and $1/(1-\xi)$ becomes zero. In fact, it doesn't matter which root is chosen for r (resp. s) since (1.2) is invariant under $m \to -(m+1)$ (resp. $n \to -(n+1)$). Thus, choosing for example, $r = -\frac{m}{2}$ and $s = -\frac{n}{2}$ the equation in (3.4) becomes

$$\xi(1-\xi)u''+\left(\frac{1}{2}-m-(-m-n+1)\xi\right)u'-\frac{1}{4}(m+n-\alpha)(m+n+\alpha)u=0, \tag{3.6}$$

which is a Gauss hypergeometric equation introduced in (2.3) with parameters

$$a=\frac{1}{2}(\alpha-m-n),\qquad b=\frac{1}{2}(-\alpha-m-n),\qquad c=\frac{1}{2}-m. \tag{3.7}$$

Therefore, a pair of linearly independent solutions around the point $x = 0$ can be written from (2.4) as

$$\begin{aligned} y_1(x;\alpha)&=\sin^{-m}x\cos^{-n}x\ {}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2},\ \frac{-\alpha-m-n}{2}\\ \frac{1}{2}-m\end{matrix}\ \middle|\ \sin^2 x\right),\\ y_2(x;\alpha)&=\sin^{m+1}x\cos^{-n}x\ {}_2F_1\left(\begin{matrix}\frac{\alpha+m-n+1}{2},\ \frac{-\alpha+m-n+1}{2}\\ m+\frac{3}{2}\end{matrix}\ \middle|\ \sin^2 x\right), \end{aligned} \tag{3.8}$$

on returning back to the original variable x via (3.1) and (3.3). Similarly, a set of linearly independent solutions around the point $x = \pi/2$ maybe written from (2.5) as

$$\begin{aligned} y_3(x;\alpha) &= \sin^{-m} x \cos^{-n} x \; {}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2}, & \frac{-\alpha-m-n}{2} \\ \frac{1}{2}-n & \end{matrix}\middle| \cos^2 x\right), \\ y_4(x;\alpha) &= \sin^{-m} x \cos^{n+1} x \; {}_2F_1\left(\begin{matrix}\frac{-\alpha-m+n+1}{2}, & \frac{\alpha-m+n+1}{2} \\ n+\frac{3}{2} & \end{matrix}\middle| \cos^2 x\right). \end{aligned} \tag{3.9}$$

Note that, y_3 and y_4 can be obtained from y_1 and y_2 respectively, by using the invariance of the equation (1.2) under the replacements $m \leftrightarrow n$ and $x \to \frac{\pi}{2} - x$. Finally, (2.6) leads to the solutions

$$\begin{aligned} y_5(x;\alpha) &= \sin^{n-\alpha} x \cos^{-n} x \; {}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2}, & \frac{\alpha+m-n+1}{2} \\ 1+\alpha & \end{matrix}\middle| \frac{1}{\sin^2 x}\right), \\ y_6(x;\alpha) &= \sin^{n+\alpha} x \cos^{-n} x \; {}_2F_1\left(\begin{matrix}\frac{-\alpha-m-n}{2}, & \frac{-\alpha+m-n+1}{2} \\ 1-\alpha & \end{matrix}\middle| \frac{1}{\sin^2 x}\right), \end{aligned} \tag{3.10}$$

which are valid as $\xi = \sin^2 x \to \infty$, i.e., as $x \to \pm i\infty$. Clearly, the hypergeometric solutions (3.8)–(3.10) of (1.2) are general dihedral functions since they are at the same time solutions of the hypergeometric equation in (3.6) with the characteristic exponent differences $\{m + \frac{1}{2}, n + \frac{1}{2}, \alpha\}$ up to permutations and sign changes.

3.2 General Solution in terms of elementary functions

First, by letting

$$y(x;\alpha) = e^{i\alpha x} u(x;\alpha^2), \tag{3.11}$$

we transform equation (1.2) to the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2i\alpha\frac{\mathrm{d}}{\mathrm{d}x} - \frac{m(m+1)}{\sin^2 x} - \frac{n(n+1)}{\cos^2 x}\right] u(\xi;\alpha) = 0, \tag{3.12}$$

in which α appears as the coefficient of the first derivative term rather than the function itself. Then, the transformation

$$\xi = \frac{2i\cot x}{1 + i\cot x} = \frac{2}{1 - i\tan x}, \tag{3.13}$$

on the independent variable takes the last equation to the form

$$\left[(1-\xi)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - (\alpha+1)\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{m(m+1)}{(2-\xi)^2} - \frac{n(n+1)}{\xi^2}\right] u(\xi;\alpha) = 0. \tag{3.14}$$

Finally, the singularities of the last equation at the points $\xi = 0, 2$ suggest a solution of type

$$u(\xi;\alpha) = \xi^r (2-\xi)^s v(\xi;\alpha), \tag{3.15}$$

which gives the chance of specifying the values of the parameters r and s to discard the unwanted terms proportional to ξ^{-2} and $(2-\xi)^{-2}$. With this transformation the last equation takes the form

$$\left[(1-\xi)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \left(\frac{2r}{\xi} + \frac{2s}{2-\xi} - 2r - 2s - \alpha - 1\right)\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{1}{\xi^2}\big(r(r-1) - n(n+1)\big) - \frac{1}{(2-\xi)^2}\big(s(s-1) - m(m+1)\big) - (\alpha + s + r)\left(\frac{r}{\xi} - \frac{s}{2-\xi}\right)\right] v(\xi;\alpha) = 0, \tag{3.16}$$

where we have used $(1-\xi)/(2-\xi)^2 = 1/(2-\xi) - 1/(2-\xi)^2$ and $2(1-\xi)/(\xi(2-\xi)) = 1/\xi - 1/(2-\xi)$ when necessary. The terms proportional to ξ^{-2} and $(2-\xi)^{-2}$ can be removed if r and s are chosen as the roots $r = -n, n+1$ and $s = -m, m+1$ of the quadratic equations

$$r^2 - r - n(n+1) = 0, \qquad s^2 - s - m(m+1) = 0. \tag{3.17}$$

So, if we let $r = -n$ and $s = -m$ equation (3.16) takes the form

$$(1-\xi)v'' + \left(-\frac{2n}{\xi} - \frac{2m}{2-\xi} + 2m + 2n - \alpha - 1\right)v' - (\alpha - m - n)\left(-\frac{n}{\xi} + \frac{m}{2-\xi}\right)v = 0. \tag{3.18}$$

The last ordinary differential equation derived by Vidūnas in [12, (22)] and is a specialization of the system of partial differential equations for Appell's $F_2(x, y)$ function with $y = 2 - x$. Thanks to the [12, (22)] without which it wouldn't be possible to identify one solution of the last equation as $F_2\left(\begin{smallmatrix} a;\ b,\ b' \\ c,\ c' \end{smallmatrix} \middle| \xi, 2-\xi\right)$ function with parameters $a = \alpha - m - n, b = -n, b' = -m, c = -2n$ and $c' = -2m$. It is defined by the double hypergeometric series

$$\begin{aligned} F_2\left(\begin{matrix} a;\ b,\ b' \\ c,\ c' \end{matrix} \middle| x, y\right) &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(a)_{p+q}(b)_p(b')_q}{(c)_p(c')_q}\frac{x^p}{p!}\frac{y^q}{q!} \\ &= \sum_{p=0}^{\infty} \frac{(a)_p(b)_p}{(c)_p} {}_2F_1\left(\begin{matrix} a+p,\ b' \\ c' \end{matrix} \middle| y\right)\frac{x^p}{p!} \\ &= \sum_{q=0}^{\infty} \frac{(a)_q(b')_q}{(c')_q} {}_2F_1\left(\begin{matrix} a+q,\ b \\ c \end{matrix} \middle| x\right)\frac{y^q}{q!}, \end{aligned} \tag{3.19}$$

which converges absolutely for $|x| + |y| < 1$ [10, p. 23]. Thus, a solution of (3.18) can be written as

$$v(\xi;\alpha) = F_2\left(\begin{matrix} \alpha - m - n;\ -n,\ -m \\ -2n, -2m \end{matrix} \middle| \xi,\ 2-\xi\right). \tag{3.20}$$

For general values of the parameters a, b, b', c, c' the double hypergeometric series for the Appell's $F_2(\xi, 2-\xi)$ function in (3.19) does not converge for any ξ. However, in the last equation, since $b = -n$ and $b' = -m$ with n and m are non negative integers, the series for F_2 function in (3.21) terminates in both directions and hence, reduces to a finite sum of $(n+1)(m+1)$ terms. From (3.20) we obtain one solution of the original equation (1.2) as

$$\begin{aligned} Y^{(1)}_{mn}(x;\alpha) &= \frac{(1+i\cot x)^{m+n}}{(i\cot x)^n} e^{i\alpha x} F_2\left(\begin{matrix}\alpha-m-n;\ -n,\ -m\\ -2n,\ -2m\end{matrix}\ \middle|\ \frac{2i\cot x}{1+i\cot x}, \frac{2}{1+i\cot x}\right)\\ &= \frac{(1-i\tan x)^{m+n}}{(-i\tan x)^m} e^{i\alpha x} F_2\left(\begin{matrix}\alpha-m-n;\ -n,\ -m\\ -2n,\ -2m\end{matrix}\ \middle|\ \frac{2}{1-i\tan x}, \frac{2i\tan x}{i\tan x-1}\right), \end{aligned} \tag{3.21}$$

on returning back to the original variables via back transformations (3.15), (3.13) and (3.11). Now, we are going to express this finite double sum as a finite sum of the Jacobi polynomials with $(n+1)$ and/or $(m+1)$ terms. But, before we need the following lemmas.

Lemma 1 *The following identities*

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= (-1)^n P_n^{(\beta,\alpha)}(-x)\\ &= \left(\frac{1+x}{2}\right)^n P_n^{(\alpha,-2n-\alpha-\beta-1)}\left(\frac{3-x}{1+x}\right)\\ &= \left(\frac{1-x}{2}\right)^n P_n^{(-2n-\alpha-\beta-1,\beta)}\left(\frac{x+3}{x-1}\right)\\ &= (-1)^n \left(\frac{1+x}{2}\right)^n P_n^{(-2n-\alpha-\beta-1,\alpha)}\left(\frac{x-3}{x+1}\right)\\ &= (-1)^n \left(\frac{1-x}{2}\right)^n P_n^{(\beta,-2n-\alpha-\beta-1)}\left(\frac{3+x}{1-x}\right), \end{aligned} \tag{3.22}$$

hold for the Jacobi polynomials.

Proof By taking $a = n+\alpha+\beta+1$ and $c = \alpha+1$ in the formulas

$$\begin{aligned} {}_2F_1\left(\begin{matrix}-n,\ a\\ c\end{matrix}\ \middle|\ z\right) &= (1-z)^n {}_2F_1\left(\begin{matrix}-n,\ c-a\\ c\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\ &= \frac{(a)_n}{(c)_n}(-z)^n {}_2F_1\left(\begin{matrix}-n,\ 1-n-c\\ 1-n-a\end{matrix}\ \middle|\ \frac{1}{z}\right)\\ &= \frac{(a)_n}{(c)_n}(1-z)^n {}_2F_1\left(\begin{matrix}-n,\ c-a\\ 1-n-a\end{matrix}\ \middle|\ \frac{1}{1-z}\right)\\ &= \frac{(c-a)_n}{(c)_n} z^n {}_2F_1\left(\begin{matrix}-n,\ 1-n-c\\ 1-n+a-c\end{matrix}\ \middle|\ 1-\frac{1}{z}\right)\\ &= \frac{(c-a)_n}{(c)_n} {}_2F_1\left(\begin{matrix}-n,\ a\\ 1-n+a-c\end{matrix}\ \middle|\ 1-z\right), \end{aligned} \tag{3.23}$$

(19)–(23) of [11] and then using the hypergeometric representation [4, (18.5.7)], [5, p. 212]

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) \tag{3.24}$$

of the Jacobi polynomials we obtain the formulas in Lemma 1. □

The first one is well-known and the last two are application of it to the second and third formulas. We will particularly make use of the fourth one in the following lemma.

Lemma 2 *The Appell's F_2 function in (3.21) can be represented as finite sums*

$$\begin{aligned}
&\frac{(2m)!}{(m!)^2}\frac{(2n)!}{(n!)^2} F_2\left(\begin{matrix}\alpha-m-n; & -n, & -m \\ & -2n, & -2m\end{matrix}\,\middle|\, \frac{2\xi}{1+\xi}, \frac{2}{1+\xi}\right) \\
&\quad = \left(\frac{2}{1+\xi}\right)^m \sum_{k=0}^{n} \binom{n+k}{k} \frac{(\alpha-m-n)_{n-k}}{(n-k)!} \left(\frac{2\xi}{1+\xi}\right)^{n-k} P_m^{(\alpha-k,k-\alpha)}(\xi) \\
&\quad = \left(\frac{2\xi}{1+\xi}\right)^n \sum_{k=0}^{m} \binom{m+k}{k} \frac{(\alpha-m-n)_{m-k}}{(m-k)!} \left(\frac{2}{1+\xi}\right)^{m-k} P_n^{(\alpha-k,k-\alpha)}(1/\xi)\,,
\end{aligned} \tag{3.25}$$

involving the Jacobi polynomials.

Proof By using the second equation in (3.19) we write

$$\begin{aligned}
&F_2\left(\begin{matrix}\alpha-m-n; & -n, & -m \\ & -2n, & -2m\end{matrix}\,\middle|\, \frac{2\xi}{1+\xi}, \frac{2}{1+\xi}\right) \\
&\quad = \sum_{p=0}^{n} \frac{(\alpha-m-n)_p(-n)_p}{p!(-2n)_p} \left(\frac{2\xi}{1+\xi}\right)^p {}_2F_1\left(\begin{matrix}\alpha-m-n+p, & -m \\ & -2m\end{matrix}\,\middle|\, \frac{2}{1+\xi}\right),
\end{aligned} \tag{3.26}$$

in which the Gauss hypergeometric function sums to

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}\alpha-m-n+p, & -m \\ & -2m\end{matrix}\,\middle|\, \frac{2}{1+\xi}\right) &= \frac{m!}{(-2m)_m} P_m^{(-2m-1,\alpha-n+p)}\left(\frac{\xi-3}{\xi+1}\right) \\
&= \frac{(-1)^m m!}{(-2m)_m}\left(\frac{2}{1+\xi}\right)^m P_m^{(\alpha-n+p,-\alpha+n-p)}(\xi),
\end{aligned} \tag{3.27}$$

upon the use of (3.24) and the fourth formula in (3.22), respectively. Moreover, the constants appearing in (3.26) and (3.27) can be written as

$$\frac{(-n)_p}{(-2n)_p} = \binom{2n-p}{n-p}\frac{(n!)^2}{(2n)!}, \qquad \frac{(-1)^m m!}{(-2m)_m} = \frac{(m!)^2}{(2m)!}, \tag{3.28}$$

by using the Pochhammer symbol identity

$$(-a)_k = (-1)^k (a-k+1)_k. \tag{3.29}$$

Finally, inserting (3.27) and (3.28) in (3.26) and changing the summation index $p \to n-k$ we obtain the first equation in (3.25). The second one can be obtained by reversing the order of summation, or equivalently, using the last representation in (3.19). □

Theorem 1 *The linearly independent solutions of* (1.2) *can be represented in terms of elementary functions as*

$$\begin{aligned} Y_{mn}^{(1)}(x;\alpha) &= \sum_{k=0}^{n} \binom{n+k}{k} \frac{(\alpha-m-n)_{n-k}}{(n-k)!} \left(\frac{1-i\tan x}{2}\right)^k e^{i\alpha x} P_m^{(\alpha-k,k-\alpha)}(i\cot x) \\ &= \sum_{k=0}^{n} \binom{n+k}{k} \frac{(\alpha-m-n)_{n-k}}{(n-k)!(2\cos x)^k} e^{i(\alpha-k)x} P_m^{(\alpha-k,k-\alpha)}(i\cot x) \\ &= \sum_{k=0}^{m} \binom{m+k}{k} \frac{(\alpha-m-n)_{m-k}}{(m-k)!} \left(\frac{1+i\cot x}{2}\right)^k e^{i\alpha x} P_n^{(\alpha-k,k-\alpha)}(-i\tan x) \\ &= \sum_{k=0}^{m} \binom{m+k}{k} \frac{(\alpha-m-n)_{m-k}}{(m-k)!(-2i\sin x)^k} e^{i(\alpha-k)x} P_n^{(\alpha-k,k-\alpha)}(-i\tan x), \end{aligned} \tag{3.30}$$

and

$$Y_{mn}^{(2)}(x;\alpha) = \overline{Y_{mn}^{(1)}(x;\alpha)}, \tag{3.31}$$

where $\overline{w}$ *stands for the complex conjugate of* w.

Proof Since any constant multiple of a solution is also a solution for the linear homogeneous equation (1.2), we may discard the factorial coefficients in (3.25) and insert it into (3.21) with $\xi = i\cot x$ and $\xi = i\tan x$ respectively, to obtain (3.30) by keeping in mind that

$$1 - i\tan x = \frac{e^{-ix}}{\cos x}, \quad 1 + i\cot x = \frac{e^{-ix}}{-i\sin x}. \tag{3.32}$$

On the other hand, (1.2) is a linear differential equation with real coefficients having a complex valued solution $Y_{mn}^{(1)}(x;\alpha) := u(x) + iv(x)$ of a real variable x. Thus, its conjugate $Y_{mn}^{(2)}(x;\alpha) = \overline{Y_{mn}^{(1)}(x;\alpha)} := u(x) - iv(x)$ can be taken as the second linearly independent solution. Moreover, the real and imaginary parts of $Y_{mn}^{(1)}(x;\alpha)$ are the real valued solutions of (1.2). □

Clearly, when $m = n = 0$, the above solutions simplify to

$$\left\{Y_{00}^{(1)}, Y_{00}^{(2)}\right\} = \left\{e^{i\alpha x}, e^{-i\alpha x}\right\}, \tag{3.33}$$

which agree with the ones in (2.10). Similarly, for $m = 0$, $n \geq 0$ and $m \geq 0$, $n = 0$ they reduce to

$$\left\{Y_{0n}^{(1)}, Y_{0n}^{(2)}\right\} = \left\{e^{i\alpha x} P_n^{(\alpha,-\alpha)}(-i\tan x), e^{-i\alpha x} P_n^{(\alpha,-\alpha)}(i\tan x)\right\}, \tag{3.34}$$

and

$$\left\{Y_{m0}^{(1)}, Y_{m0}^{(2)}\right\} = \left\{e^{i\alpha x} P_m^{(\alpha,-\alpha)}(i\cot x), e^{-i\alpha x} P_m^{(\alpha,-\alpha)}(-i\cot x)\right\}, \tag{3.35}$$

respectively, which are in accordance with the ones in (2.29) and (2.33) in respective order.

It is well-known that any solution of the linear homogeneous ordinary differential equation (1.2) can be written as a linear combination of the linearly independent solutions in (3.30) and (3.31). Thus, we can relate the dihedral hypergeometric functions in (3.8)–(3.10) with the elementary ones in (3.30) and (3.31). The following lemma will be useful in determining the combination constants of such representations.

Lemma 3 *The solutions in* (3.30) *and* (3.31) *have the limiting values*

$$\lim_{x\to 0^+} \sin^m x\, Y_{mn}^{(1)}(x;\alpha) = i^m A_{mn}(\alpha), \tag{3.36}$$

$$\lim_{x\to 0^+} \sin^m x\, Y_{mn}^{(2)}(x;\alpha) = (-i)^m A_{mn}(\alpha), \tag{3.37}$$

as $x \to 0^+$ *and*

$$\lim_{x\to \frac{\pi}{2}^-} \cos^n x\, Y_{mn}^{(1)}(x;\alpha) = (-i)^n e^{i\alpha\pi/2} A_{nm}(\alpha), \tag{3.38}$$

$$\lim_{x\to \frac{\pi}{2}^-} \cos^n x\, Y_{mn}^{(2)}(x;\alpha) = i^n e^{-i\alpha\pi/2} A_{nm}(\alpha), \tag{3.39}$$

when $x \to \frac{\pi}{2}^-$, *respectively, where*

$$A_{mn}(\alpha) = \frac{2^m \left(\frac{1}{2}\right)_m}{m!} \frac{2^n \left(\frac{\alpha-m-n+1}{2}\right)_n}{n!}. \tag{3.40}$$

Proof By using the third or fourth representation of $Y_{mn}^{(1)}$ in (3.30), we see that only the last term ($k = m$) survives as $x \to 0^+$ which gives

$$\lim_{x\to 0^+} \sin^m x\, Y_{mn}^{(1)}(x;\alpha) = \binom{2m}{m} \left(\frac{i}{2}\right)^m P_n^{(\alpha-m,m-\alpha)}(0). \tag{3.41}$$

Then, by using the formula [1, (2.29)]

$$(-1)^n P_n^{(-a,a)}(0) = P_n^{(a,-a)}(0) = \frac{2^n \left(\frac{a-n+1}{2}\right)_n}{n!}, \tag{3.42}$$

and rearranging we obtain (3.36). Similarly, by making use of the first representation of $Y_{mn}^{(1)}$ in (3.30) we get

$$\lim_{x\to \frac{\pi}{2}^-} \cos^n x\, Y_{mn}^{(1)}(x;\alpha) = \binom{2n}{n} \left(-\frac{i}{2}\right)^n e^{i\alpha\pi/2} P_m^{(\alpha-n,n-\alpha)}(0), \tag{3.43}$$

since only the last term ($k = n$) survives as $x \to \frac{\pi}{2}^-$. Thus, by using (3.42), after a little algebra ve obtain (3.38). Clearly, (3.37) and (3.39) follow from the fact that $Y_{mn}^{(2)}$ is the complex conjugate of $Y_{mn}^{(1)}$. □

4 General dihedral Gauss hypergeometric identities

4.1 Trigonometric forms of dihedral identities

In this subsection we will derive trigonometric versions of dihedral hypergeometric identities based on the solutions of the Schrödinger equation with the trigonometric Pöschl–Teller potential in (1.2).

Theorem 2 *The following trigonometric dihedral hypergeometric identities hold for* $\alpha \in \mathbb{C}$, $m, n \in \mathbb{N}_0$ *and* $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$
\begin{aligned}
&(-i\sin x)^{-m}\cos^{-n}x\ {}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2}, & \frac{-\alpha-m-n}{2}\\ \frac{1}{2}-m & \end{matrix}\ \middle|\ \sin^2 x\right)\\
&\quad=(-i\sin x)^{-m}\cos^{n+1}x\ {}_2F_1\left(\begin{matrix}\frac{-\alpha-m+n+1}{2}, & \frac{\alpha-m+n+1}{2}\\ \frac{1}{2}-m & \end{matrix}\ \middle|\ \sin^2 x\right)\\
&\quad=(i\cot x)^{m}\cos^{\pm\alpha}x\ {}_2F_1\left(\begin{matrix}\frac{\mp\alpha-m-n}{2}, & \frac{\mp\alpha-m+n+1}{2}\\ \frac{1}{2}-m & \end{matrix}\ \middle|\ -\tan^2 x\right) \qquad (4.1)\\
&\quad=B_{mn}(\alpha)\sum_{k=0}^{m}\binom{m+k}{k}\frac{(\alpha-m-n)_{m-k}}{(m-k)!(-2i\sin x)^k}\left\{e^{i(\alpha-k)x}P_n^{(\alpha-k,k-\alpha)}(-i\tan x)\right\}_x^{\substack{+,\ m-k\ \text{even}\\ -,\ m-k\ \text{odd}}}\\
&\quad=B_{mn}(\alpha)\sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-m-n)_{n-k}}{(n-k)!(2\cos x)^k}\left\{e^{i(\alpha-k)x}P_m^{(\alpha-k,k-\alpha)}(i\cot x)\right\}_{\alpha-k}^{+},
\end{aligned}
$$

in which

$$
B_{mn}(\alpha)=\frac{m!}{2^m\left(\frac{1}{2}\right)_m}\frac{n!}{2^n\left(\frac{\alpha-m-n+1}{2}\right)_n}, \qquad (4.2)
$$

and the limiting cases when $\alpha \to r \in \mathcal{S}_{m,n-1} = \{m\pm1, m\pm3, \ldots, m\pm(n-1)\}$ *with* $\mathcal{S}_{m,-1}=\emptyset$ *if* n *is even and* $\alpha \to r \in \mathcal{T}_{m,n-1} = \{m, m\pm2, m\pm4, \ldots, m\pm(n-1)\}$ *if* n *is odd should be considered in order to remove the indeterminacy on the right-hand side.*

Proof Any solution of (1.2) can be written as a linear combination of the linearly independent solutions $Y_{mn}^{(1)}(x;\alpha)$ and $Y_{mn}^{(2)}(x;\alpha)$ in (3.30) and (3.31), respectively. Thus, the first hypergeometric solution $y_1(x;\alpha)$ in (3.8) can be written as

$$
\begin{aligned}
&\sin^{-m}x\cos^{-n}x\ {}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2}, & \frac{-\alpha-m-n}{2}\\ \frac{1}{2}-m & \end{matrix}\ \middle|\ \sin^2 x\right) \qquad (4.3)\\
&\quad=C_{mn}^{(1)}(\alpha)Y_{mn}^{(1)}(x;\alpha)+C_{mn}^{(2)}(\alpha)Y_{mn}^{(2)}(x;\alpha),
\end{aligned}
$$

where $C^{(1)}_{mn}(\alpha)$ and $C^{(2)}_{mn}(\alpha)$ are combination constants to be determined. Multiplying both sides by $\sin^m x$ and taking limit as $x \to 0^+$ lead to

$$C^{(1)}_{mn}(\alpha) + (-1)^m C^{(2)}_{mn}(\alpha) = \frac{(-i)^m}{A_{mn}(\alpha)}, \tag{4.4}$$

where $A_{mn}(\alpha)$ is defined in (3.36). Likewise, multiplying both sides of (4.3) by $\cos^n x$ and taking limit as $x \to \frac{\pi}{2}^-$, we obtain

$$e^{i\alpha\pi/2} C^{(1)}_{mn}(\alpha) + (-1)^n e^{-i\alpha\pi/2} C^{(2)}_{mn}(\alpha) = \frac{i^n}{A_{nm}(\alpha)}\, {}_2F_1\left(\begin{matrix} \frac{\alpha-m-n}{2}, & \frac{-\alpha-m-n}{2} \\ \frac{1}{2} - m & \end{matrix} \,\middle|\, 1 \right). \tag{4.5}$$

Simplification of the right-hand side leads to

$$e^{i\alpha\pi/2} C^{(1)}_{mn}(\alpha) + (-1)^n e^{-i\alpha\pi/2} C^{(2)}_{mn}(\alpha) = \frac{(-i)^m}{A_{mn}(\alpha)} \begin{cases} \cos\frac{\alpha\pi}{2}, & m+n \ \text{ even} \\ i\sin\frac{\alpha\pi}{2}, & m+n \ \text{ odd} \end{cases}, \tag{4.6}$$

whose derivation can be found in Appendix 1. Then, in either case where $m + n$ is even or odd, (4.4) and (4.6) gives

$$C^{(1)}_{mn}(\alpha) = \frac{(-i)^m}{2A_{mn}(\alpha)} = \frac{1}{2}(-i)^m B_{mn}(\alpha), \qquad C^{(2)}_{mn}(\alpha) = \frac{i^m}{2A_{mn}(\alpha)} = \frac{1}{2} i^m B_{mn}(\alpha). \tag{4.7}$$

Inserting these values in (4.3) and rearranging we obtain

$$\begin{aligned} \sin^{-m} x \cos^{-n} x\, {}_2F_1\left(\begin{matrix} \frac{\alpha-m-n}{2}, & \frac{-\alpha-m-n}{2} \\ \frac{1}{2} - m & \end{matrix} \,\middle|\, \sin^2 x \right) &= B_{mn}(\alpha) \mathrm{Re}\left((-i)^m Y^{(1)}_{mn}(x;\alpha) \right) \\ &= B_{mn}(\alpha) \left\{ (-i)^m Y^{(1)}_{mn}(x;\alpha) \right\}^+_i, \end{aligned} \tag{4.8}$$

since the second solution is the complex conjugate of the first one. Although it is neat and elegant representation, we prefer the ones in the theorem since they will be more useful in finding the algebraic versions of the identities. Hence, multiplying both sides of the last equation by i^m and rearranging we get the first hypergeometric identity in (4.1). The other identities follow from the fact that the four hypergeometric functions in (4.1) are equivalent to one another under Euler's and Pfaff's transformations in (2.12).

On the other hand, the denominator of the right-hand side of (4.1), or equivalently that of (4.2), becomes zero if $\alpha \in \mathcal{S}_{m,n-1}$ and $\alpha \in \mathcal{T}_{m,n-1}$ for even and odd values of n, respectively. But, in this case the hypergeometric functions on the left-hand sides are well-defined. Thus, the numerator of the right-hand side must accordingly be zero for the same values of α, to prevent it from being undefined. This leads to $[0/0]$ indeterminate cases for those α values which can be removed by taking limits as $\alpha \to r \in \mathcal{S}_{m,n-1}$ or $\alpha \to r \in \mathcal{T}_{m,n-1}$ according as n is even or odd, respectively. □

Theorem 3 *The following trigonometric dihedral hypergeometric identities hold for* $\alpha \in \mathbb{C}$, $m, n \in \mathbb{N}_0$ *and* $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$
\begin{aligned}
&(i \sin x)^{m+1} \cos^{-n} x \; {}_2F_1\left(\begin{matrix}\frac{\alpha+m-n+1}{2}, & \frac{-\alpha+m-n+1}{2} \\ m+\frac{3}{2} & \end{matrix}\,\middle|\, \sin^2 x\right) \\
&= (i \sin x)^{m+1} \cos^{n+1} x \; {}_2F_1\left(\begin{matrix}\frac{-\alpha+m+n+2}{2}, & \frac{\alpha+m+n+2}{2} \\ m+\frac{3}{2} & \end{matrix}\,\middle|\, \sin^2 x\right) \\
&= (i \tan x)^{m+1} \cos^{\pm\alpha} x \; {}_2F_1\left(\begin{matrix}\frac{\mp\alpha+m-n+1}{2}, & \frac{\mp\alpha+m+n+2}{2} \\ m+\frac{3}{2} & \end{matrix}\,\middle|\, -\tan^2 x\right) \qquad (4.9) \\
&= C_{mn}(\alpha) \sum_{k=0}^{m} \binom{m+k}{k} \frac{(\alpha-m-n)_{m-k}}{(m-k)!(-2i \sin x)^k} \left\{ e^{i(\alpha-k)x} P_n^{(\alpha-k,k-\alpha)}(-i \tan x) \right\}_x^{\substack{-,\ m-k \text{ even} \\ +,\ m-k \text{ odd}}} \\
&= C_{mn}(\alpha) \sum_{k=0}^{n} \binom{n+k}{k} \frac{(\alpha-m-n)_{n-k}}{(n-k)!(2 \cos x)^k} \left\{ e^{i(\alpha-k)x} P_m^{(\alpha-k,k-\alpha)}(i \cot x) \right\}_{\alpha-k}^{-},
\end{aligned}
$$

in which

$$
\begin{aligned}
C_{mn}(\alpha) &= \frac{m! 2^{m+1} \left(\frac{1}{2}\right)_{m+1}}{2^m \left(\frac{\alpha-m-n+1}{2}\right)_m} \frac{n!}{2^{m+n+1} \left(\frac{\alpha-m-n}{2}\right)_{m+n+1}} \\
&= \frac{m! 2^{m+1} \left(\frac{1}{2}\right)_{m+1}}{(\alpha-m-n)_{2m}} \frac{n!}{2^{n+1} \left(\frac{\alpha+m-n}{2}\right)_{n+1}} \qquad (4.10) \\
&= \frac{m! 2^{m+1} \left(\frac{1}{2}\right)_{m+1}}{(\alpha-m-n)_{2m+1}} \frac{n!}{2^n \left(\frac{\alpha+m-n+2}{2}\right)_n},
\end{aligned}
$$

and one needs to look at the limiting cases when $\alpha \to r \in \mathcal{P}_{m,n} = \{-m \pm 1, -m \pm 3, \ldots, -m \pm n, n, n \pm 1, n \pm 2, \ldots, n \pm (m-1), n+m\}$ *with* $\mathcal{P}_{0,n} = \{\pm 1, \pm 3, \ldots, \pm n\}$ *if* n *is odd and* $\alpha \to r \in \mathcal{Q}_{m,n} = \{-m, -m \pm 2, \ldots, -m \pm n, n, n \pm 1, n \pm 2, \ldots, n \pm (m-1), n+m\}$ *with* $\mathcal{Q}_{0,n} = \{0, \pm 2, \ldots, \pm n\}$ *if* n *is even in order to remove the indeterminacy on the right-hand side.*

Proof The second hypergeometric solution $y_2(x; \alpha)$ in (3.8) can be written as

$$
\begin{aligned}
&\sin^{m+1} x \cos^{-n} x \; {}_2F_1\left(\begin{matrix}\frac{\alpha+m-n+1}{2}, & \frac{-\alpha+m-n+1}{2} \\ m+\frac{3}{2} & \end{matrix}\,\middle|\, \sin^2 x\right) \qquad (4.11) \\
&= C_{mn}^{(1)}(\alpha) Y_{mn}^{(1)}(x; \alpha) + C_{mn}^{(2)}(\alpha) Y_{mn}^{(2)}(x; \alpha),
\end{aligned}
$$

where $C_{mn}^{(1)}(\alpha)$ and $C_{mn}^{(1)}(\alpha)$ are combination constants to be determined. Multiplying both sides by $\sin^m x$ and taking limit as $x \to 0^+$ lead to

$$
C_{mn}^{(1)}(\alpha) + (-1)^m C_{mn}^{(2)}(\alpha) = 0. \qquad (4.12)
$$

Similarly, multiplying both sides of (4.11) by $\cos^n x$ and taking limit as $x \to \frac{\pi}{2}^-$, we obtain

$$(-1)^n e^{i\alpha\pi/2} C_{mn}^{(1)}(\alpha) + e^{-i\alpha\pi/2} C_{mn}^{(2)}(\alpha) = \frac{(-i)^n}{A_{nm}(\alpha)} \, {}_2F_1\left(\begin{matrix} \frac{\alpha+m-n+1}{2}, & \frac{-\alpha+m-n+1}{2} \\ m+\frac{3}{2} & \end{matrix} \middle| 1 \right), \tag{4.13}$$

where $A_{mn}(\alpha)$ is defined in (3.36). Simplifying the right-hand side we arrive at (see Appendix 1)

$$(-1)^n e^{i\alpha\pi/2} C_{mn}^{(1)}(\alpha) + e^{-i\alpha\pi/2} C_{mn}^{(2)}(\alpha) = i^m C_{mn}(\alpha) \begin{cases} \sin\frac{\alpha\pi}{2}, & m+n \ \text{ even} \\ i\cos\frac{\alpha\pi}{2}, & m+n \ \text{ odd} \end{cases}, \tag{4.14}$$

where $C_{mn}(\alpha)$ is given in (4.10). Then, in either case where $m+n$ is even or odd, by solving (4.14) and (4.12) we obtain

$$C_{mn}^{(1)}(\alpha) = \frac{1}{2}(-i)^{m+1} C_{mn}(\alpha), \qquad C_{mn}^{(2)}(\alpha) = \frac{1}{2} i^{m+1} C_{mn}(\alpha). \tag{4.15}$$

Inserting these values in (4.11) and rearranging we get

$$\begin{aligned} &\sin^{m+1} x \cos^{-n} x \, {}_2F_1\left(\begin{matrix} \frac{\alpha+m-n+1}{2}, & \frac{-\alpha+m-n+1}{2} \\ m+\frac{3}{2} & \end{matrix} \middle| \sin^2 x \right) \\ &= C_{mn}(\alpha) \mathrm{Im}\left((-i)^m Y_{mn}^{(1)}(x;\alpha) \right) \\ &= C_{mn}(\alpha) \left\{ (-i)^{m+1} Y_{mn}^{(1)}(x;\alpha) \right\}_i^+ . \end{aligned} \tag{4.16}$$

Finally, multiplying both sides of the last equation by i^m and rearrranging we obtain the first hypergeometric identity in (4.9). The other identities in (4.9) arise from Euler's and Pfaff's transformations in (2.12). As in Theorem 2, indeterminate cases can be removed by taking limits as $\alpha \to r \in \mathcal{P}_{m,n}$ or $\alpha \to r \in \mathcal{Q}_{m,n}$ according as n is odd or even, respectively. □

Clearly the right-hand sides of (4.8) and (4.16) are the real and imaginary parts of the first solution $(-i)^m Y_{mn}^{(1)}(x;\alpha)$, respectively, which can also be written by using Maier's notation defined in (2.18).

Theorem 4 *Let*

$$\begin{aligned} &H_{mn}(x;\alpha) \\ &= D_{mn}(\alpha) \sum_{k=0}^{m} \binom{m+k}{k} \frac{(\alpha-m-n)_{m-k}}{(m-k)!(i\tan x)^k} \left(\frac{1+i\tan x}{2} \right)^{k-\alpha} P_n^{(\alpha-k,k-\alpha)}(i\tan x) \\ &= D_{nm}(\alpha) \sum_{k=0}^{n} \binom{n+k}{k} \frac{(\alpha-m-n)_{n-k}}{(n-k)!} \left(\frac{1+i\tan x}{2} \right)^{k-\alpha} P_m^{(\alpha-k,k-\alpha)}(-i\cot x), \end{aligned} \tag{4.17}$$

where

$$D_{mn}(\alpha)=\frac{m!n!}{(\alpha+1)_{m+n}}\frac{\left(\frac{\alpha+m-n+1}{2}\right)_n}{\left(\frac{\alpha-m-n+1}{2}\right)_n}=D_{nm}(\alpha). \tag{4.18}$$

Then the following trigonometric dihedral hypergeometric identities

$$\begin{aligned}H_{mn}(-x;\alpha)&=\overline{H_{mn}(x;\alpha)}\\&=(-i\tan x)^{n-\alpha}\,{}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2},\ \frac{\alpha+m-n+1}{2}\\1+\alpha\end{matrix}\ \middle|\csc^2 x\right)\\&=(-i\tan x)^{-(n+\alpha+1)}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+m+n+2}{2},\ \frac{\alpha-m+n+1}{2}\\1+\alpha\end{matrix}\ \middle|\csc^2 x\right),\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}H_{mn}(x;\alpha)&=\overline{H_{mn}(-x;\alpha)}\\&=(i\tan x)^{-m}\,{}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2},\ \frac{\alpha-m+n+1}{2}\\1+\alpha\end{matrix}\ \middle|\sec^2 x\right)\\&=(i\tan x)^{m+1}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+m+n+2}{2},\ \frac{\alpha+m-n+1}{2}\\1+\alpha\end{matrix}\ \middle|\sec^2 x\right),\end{aligned} \tag{4.20}$$

hold for $\alpha\in\mathbb{C}\backslash\mathbb{Z}^-$, $m,n\in\mathbb{N}_0$ *and* $x\in\left(0,\frac{\pi}{2}\right)$.

Proof The first solution $y_5(x;\alpha)$ in (3.10) can be written as

$$\begin{aligned}&\sin^{n-\alpha}x\cos^{-n}x\ {}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2},\ \frac{\alpha+m-n+1}{2}\\1+\alpha\end{matrix}\ \middle|\csc^2 x\right)\\&\quad=C_{mn}^{(1)}(\alpha)Y_{mn}^{(1)}(x;\alpha)+C_{mn}^{(2)}(\alpha)Y_{mn}^{(2)}(x;\alpha),\end{aligned} \tag{4.21}$$

which takes the form

$$\begin{aligned}&i^{m+n-\alpha}\sin^{-m}x\cos^{m-\alpha}x\ {}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2},\ \frac{\alpha-m+n+1}{2}\\1+\alpha\end{matrix}\ \middle|\sec^2 x\right)\\&\quad=C_{mn}^{(1)}(\alpha)Y_{mn}^{(1)}(x;\alpha)+C_{mn}^{(2)}(\alpha)Y_{mn}^{(2)}(x;\alpha),\end{aligned} \tag{4.22}$$

after applying the first Pfaff's transformation in (2.12). Multiplying both sides of (4.21) by $\cos^n x$ and then taking limit as $x\to\frac{\pi}{2}^-$ we obtain

$$i^{\alpha-n}C_{mn}^{(1)}(\alpha)+i^{n-\alpha}C_{mn}^{(2)}(\alpha)=\frac{1}{A_{nm}(\alpha)}{}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2},\ \frac{\alpha+m-n+1}{2}\\1+\alpha\end{matrix}\ \middle|1\right), \tag{4.23}$$

and similarly, multiplying both sides of (4.22) by $\sin^m x$ and then taking limit as $x \to 0^+$ we get

$$C_{mn}^{(1)}(\alpha) + (-1)^m C_{mn}^{(2)}(\alpha) = \frac{i^{n-\alpha}}{A_{mn}(\alpha)} {}_2F_1\left(\begin{matrix} \frac{\alpha-m-n}{2}, & \frac{\alpha-m+n+1}{2} \\ 1+\alpha & \end{matrix} \middle| 1 \right). \tag{4.24}$$

Simplifying the right-hand sides of the last two equations we get

$$\begin{aligned} C_{mn}^{(1)}(\alpha) + (-1)^{n-\alpha} C_{mn}^{(2)}(\alpha) &= i^{n-\alpha} 2^\alpha D_{mn}(\alpha) \\ C_{mn}^{(1)}(\alpha) + (-1)^m C_{mn}^{(2)}(\alpha) &= i^{n-\alpha} 2^\alpha D_{mn}(\alpha), \end{aligned} \tag{4.25}$$

where $D_{mn}(\alpha)$ is the constant in (4.18) (see Appendix 1). Then, solving the last two equations we obtain the combination constants as $C_{mn}^{(1)}(\alpha) = i^{n-\alpha} 2^\alpha D_{mn}(\alpha)$ and $C_{mn}^{(2)}(\alpha) = 0$. Finally, inserting these constants in (4.21) and rearranging we obtain the identity for the first hypergeometric function in (4.19). The second hypergeometric identity in (4.19) is due to Euler's transformation. Then, letting $x \to \frac{\pi}{2} - x$ we obtain (4.20) which is also valid on $(0, \frac{\pi}{2})$. An equivalent of (4.20) can be obtained by applying Pfaff's transformations to (4.19), however, they will be valid on $(-\frac{\pi}{2}, 0)$. That is why (4.19) and (4.20) can not be unified in one formula similar to the ones in Theorems 2 and 3. Finally, $H_{mn}(-x;\alpha) = \overline{H_{mn}(x;\alpha)}$ since replacing x by $-x$ is equivalent to replacing i by $-i$. □

General trigonometric identities were first derived by Vidūnas in Sect. 6 of [13] where the right-hand sides of Theorems 2 and 3 were expressed as $P_{m,n}(x)\cos\alpha x + Q_{m,n}(x)\sin\alpha x$ and $P_{m,n}(x)\sin\alpha x - Q_{m,n}(x)\cos\alpha x$, respectively, where $P_{m,n}$ and $Q_{m,n}$ are certain trigonometric finite double sums terminating at m and n. Presumably, the work [13] of Vidūnas avoids the use of complex unit "i" on its right-hand sides. Although, the right-hand sides of our Theorems 2 and 3 are complex in appearance, they are real valued provided α and x are real. However, the formula in Theorem 4 is a complex valued even for real α and x values. This might be the reason that [13] does not present an equivalent trigonometric identity for the ones in Theorem 4 of the present study.

In fact, our trigonometric formulas are closely related to the trigonometric formulas in [13]. However, our way of expressing these formulas differs from [13].

4.2 Algebraic versions of dihedral identities

Theorem 5 *The following algebraic dihedral hypergeometric identities hold for* $\alpha \in \mathbb{C}$, $z \in \mathbb{C}\backslash[1,\infty)$ *and* $m, n \in \mathbb{N}_0$

$$\begin{aligned} &(1-z)^{\frac{1}{2}(m+n)} {}_2F_1\left(\begin{matrix} \frac{\alpha-m-n}{2}, & \frac{-\alpha-m-n}{2} \\ \frac{1}{2}-m & \end{matrix} \middle| \frac{z}{z-1} \right) \\ &= (1-z)^{\frac{1}{2}(m-n-1)} {}_2F_1\left(\begin{matrix} \frac{-\alpha-m+n+1}{2}, & \frac{\alpha-m+n+1}{2} \\ \frac{1}{2}-m & \end{matrix} \middle| \frac{z}{z-1} \right) \end{aligned}$$

$$
\begin{aligned}
&= (1-z)^{\mp\frac{1}{2}\alpha}\ {}_2F_1\left(\begin{matrix}\frac{\mp\alpha-m-n}{2},\ \frac{\mp\alpha-m+n+1}{2}\\ \frac{1}{2}-m\end{matrix}\ \middle|\ z\right)\\
&= \frac{B_{mn}(\alpha)}{\left(\sqrt{z}\right)^{-m}}\sum_{k=0}^{m}\binom{m+k}{k}\frac{(\alpha-m-n)_{m-k}}{(m-k)!(2\sqrt{z})^k}(1-z)^{\frac{k}{2}}\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\frac{\alpha-k}{2}}P_n^{(\alpha-k,k-\alpha)}\left(\sqrt{z}\right)\right\}_{\sqrt{z}}^{\substack{+,\ m-k\ \text{even}\\ -,\ m-k\ \text{odd}}}\\
&= \frac{B_{mn}(\alpha)}{\left(\sqrt{z}\right)^{-m}}\sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-m-n)_{n-k}}{(n-k)!2^k}(1-z)^{\frac{k}{2}}\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\frac{\alpha-k}{2}}P_m^{(\alpha-k,k-\alpha)}\left(\frac{1}{\sqrt{z}}\right)\right\}_{\alpha-k}^{+},
\end{aligned}
\tag{4.26}
$$

where $B_{mn}(\alpha)$ has already defined in (4.2) and the limiting cases in Theorem 2 should be considered in order to remove the indeterminacy on the right-hand side.

Proof Letting $i\tan x = \pm\sqrt{z}$ so that $z = -\tan^2 x$ we obtain the identities of Theorem 5 no matter which sign ($+$ or $-$) we choose for $i\tan x$ since the right-hand sides are reflection symmetric with respect to $\sqrt{z}$. □

Theorem 6 *The following algebraic dihedral hypergeometric identities hold for $\alpha \in \mathbb{C}$, $z \in \mathbb{C}\backslash[1,\infty)$ and $m, n \in \mathbb{N}_0$*

$$
\begin{aligned}
&(1-z)^{-\frac{1}{2}(m-n+1)}\ {}_2F_1\left(\begin{matrix}\frac{\alpha+m-n+1}{2},\ \frac{-\alpha+m-n+1}{2}\\ m+\frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&= (1-z)^{-\frac{1}{2}(m+n+2)}\ {}_2F_1\left(\begin{matrix}\frac{-\alpha+m+n+2}{2},\ \frac{\alpha+m+n+2}{2}\\ m+\frac{3}{2}\end{matrix}\ \middle|\ \frac{z}{z-1}\right)\\
&= (1-z)^{\mp\frac{1}{2}\alpha}\ {}_2F_1\left(\begin{matrix}\frac{\mp\alpha+m-n+1}{2},\ \frac{\mp\alpha+m+n+2}{2}\\ m+\frac{3}{2}\end{matrix}\ \middle|\ z\right)\\
&= \frac{C_{mn}(\alpha)}{\left(-\sqrt{z}\right)^{m+1}}\sum_{k=0}^{m}\binom{m+k}{k}\frac{(\alpha-m-n)_{m-k}}{(m-k)!(2\sqrt{z})^k}(1-z)^{\frac{k}{2}}\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\frac{\alpha-k}{2}}P_n^{(\alpha-k,k-\alpha)}\left(\sqrt{z}\right)\right\}_{\sqrt{z}}^{\substack{-,\ m-k\ \text{even}\\ +,\ m-k\ \text{odd}}}\\
&= \frac{C_{mn}(\alpha)}{\left(-\sqrt{z}\right)^{m+1}}\sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-m-n)_{n-k}}{(n-k)!2^k}(1-z)^{\frac{k}{2}}\left\{\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\frac{\alpha-k}{2}}P_m^{(\alpha-k,k-\alpha)}\left(\frac{1}{\sqrt{z}}\right)\right\}_{\alpha-k}^{-},
\end{aligned}
\tag{4.27}
$$

in which $C_{mn}(\alpha)$ is (4.10) and the limiting cases in Theorem 3 should be considered in order to remove the indeterminacy on the right-hand side.

Proof The proof is essentially the same as that of Theorem 5. □

Theorem 7 *The following algebraic dihedral hypergeometric identities hold for* $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$, $z \in \mathbb{C} \setminus [1, \infty)$ *and* $m, n \in \mathbb{N}_0$

$$
\begin{aligned}
&(1-z)^{\frac{1}{2}(n-\alpha)}\, {}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2}, \frac{\alpha+m-n+1}{2} \\ 1+\alpha\end{matrix}\,\middle|\, \frac{z}{z-1}\right) \\
&= (1-z)^{-\frac{1}{2}(n+\alpha+1)}\, {}_2F_1\left(\begin{matrix}\frac{\alpha+m+n+2}{2}, \frac{\alpha-m+n+1}{2} \\ 1+\alpha\end{matrix}\,\middle|\, \frac{z}{z-1}\right) \\
&= (1-z)^{-\frac{1}{2}m}\, {}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2}, \frac{\alpha-m+n+1}{2} \\ 1+\alpha\end{matrix}\,\middle|\, z\right) \\
&= (1-z)^{\frac{1}{2}(m+1)}\, {}_2F_1\left(\begin{matrix}\frac{\alpha+m+n+2}{2}, \frac{\alpha+m-n+1}{2} \\ 1+\alpha\end{matrix}\,\middle|\, z\right) \qquad (4.28) \\
&= D_{mn}(\alpha) \sum_{k=0}^{m} \binom{m+k}{k} \frac{(\alpha-m-n)_{m-k}}{(m-k)!(1-z)^{\frac{k}{2}}} \left(\frac{1+\sqrt{1-z}}{2}\right)^{k-\alpha} P_n^{(\alpha-k,k-\alpha)}\left(\sqrt{1-z}\right) \\
&= D_{mn}(\alpha) \sum_{k=0}^{n} \binom{n+k}{k} \frac{(\alpha-m-n)_{n-k}}{(n-k)!} \left(\frac{1+\sqrt{1-z}}{2}\right)^{k-\alpha} P_m^{(\alpha-k,k-\alpha)}\left(1/\sqrt{1-z}\right),
\end{aligned}
$$

where $D_{mn}(\alpha)$ *is the constant in* (4.18).

Proof Letting $-i\tan x = +\sqrt{1-z}$ in (4.19), (or $i\tan x = +\sqrt{1-z}$ in (4.20)) of Theorem 4 and then applying Pfaff's transformations to the resulting identities we get the formulas of Theorem 7. □

Remark 4 Note that, for the special case where $m = 0$, the Theorems 5 and 6 reduce to (2.45) and (2.46) while for $n = 0$ they simplify to (2.47) and (2.48), respectively. Similarly, either for $m = 0$ or $n = 0$, Theorem 7 reduces to (2.49). On the other hand, for $m = n = 0$, Theorems 5, 6 and 7 simplify to (2.16), (2.19) and (2.23), respectively. Clearly, the same situation is valid for their trigonometric counterparts, too.

4.3 Generalization of dihedral identities

The identities in Theorems 5–7 can be extended to the case where the characteristic exponent differences are in the set $\{k+\frac{1}{2}, \gamma+\frac{1}{2}, \alpha\}$ up to permutations and sign changes where k is a non negative integer and α and γ are arbitrary complex parameters. This can be accomplished by replacing the non negative integer n (respectively m) by the general complex parameter γ and the related factorials and Pochhammer symbols by the gamma function if the first (respectively second) summation on the right-hand sides is used. In this case, the summand will include the Jacobi functions of the first kind $P_\gamma^{(\alpha,\beta)}(z)$ defined by (3.24) in which factorial is replaced by gamma function and n by the general complex parameter γ. In particular, for $z \in \mathbb{C} \setminus (-\infty, -1]$, the antisymmetric Jacobi functions of the first kind $P_\gamma^{(\alpha,-\alpha)}(z)$ has representations [2, (75)]

$$\begin{aligned} P_{\gamma}^{(\alpha,-\alpha)}(z) =& \frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\alpha} P_{\gamma}^{-\alpha}(z) \\ =& \frac{\Gamma(2\alpha+1)\Gamma(\gamma-\alpha+1)}{2^{\alpha}\Gamma(\gamma+1)\Gamma(\alpha+1)}(z+1)^{\alpha} C_{\gamma-\alpha}^{\alpha+\frac{1}{2}}(z), \end{aligned} \tag{4.29}$$

in terms of the associated Legendre

$$P_{\nu}^{\mu}(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(\begin{matrix} -\nu,\ \nu+1 \\ 1-\mu \end{matrix}\ \middle|\ \frac{1-z}{2}\right), \tag{4.30}$$

and the Gegenbauer

$$C_{\lambda}^{\mu}(z) = \frac{\Gamma(1/2)\Gamma(\lambda+2\mu)}{2^{2\mu-1}\Gamma(\mu)\Gamma(\mu+\frac{1}{2})\Gamma(\lambda+1)} {}_2F_1\left(\begin{matrix} -\lambda,\ \lambda+2\mu \\ \mu+\frac{1}{2} \end{matrix}\ \middle|\ \frac{1-z}{2}\right), \tag{4.31}$$

functions of the first kinds where $\lambda + 2\mu \notin -\mathbb{N}_0$. On the other hand, taking the real segment $(-1, 1)$ as the cut, the Jacobi functions of the first kind on-the-cut defined by $\mathsf{P}_{\gamma}^{(\alpha,\beta)}(z) := P_{\gamma}^{(\alpha,\beta)}(z \pm i0)$ and it is simply an analytic continuation of the Jacobi function of the first kind since it is continuous across the real interval $(-1, 1]$ [2]. Particularly, for $z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, the antisymmetric Jacobi functions of the first kind on-the-cut $\mathsf{P}_{\gamma}^{(\alpha,-\alpha)}(z)$ has representations [2, (74)]

$$\begin{aligned} \mathsf{P}_{\gamma}^{(\alpha,-\alpha)}(z) =& \frac{\Gamma(\gamma+\alpha+1)}{\Gamma(\gamma+1)}\left(\frac{1+z}{1-z}\right)^{\frac{1}{2}\alpha} \mathsf{P}_{\gamma}^{-\alpha}(z) \\ =& \frac{\Gamma(2\alpha+1)\Gamma(\gamma-\alpha+1)}{2^{\alpha}\Gamma(\gamma+1)\Gamma(\alpha+1)}(1+z)^{\alpha} \mathsf{C}_{\gamma-\alpha}^{\alpha+\frac{1}{2}}(z), \end{aligned} \tag{4.32}$$

in terms of the Ferrers function of the first kind

$$\mathsf{P}_{\nu}^{\mu}(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+z}{1-z}\right)^{\mu/2} {}_2F_1\left(\begin{matrix} -\nu,\ \nu+1 \\ 1-\mu \end{matrix}\ \middle|\ \frac{1-z}{2}\right), \tag{4.33}$$

and the Gegenbauer function of the first kind on-the-cut $\mathsf{C}_{\lambda}^{\mu}(z)$ which is identical to the Gegenbauer function of the first kind defined by (4.31), because this function analytically continues to the segment $(-1, 1)$ [2]. The relations in (4.29) and (4.32) could be used to simplify the dihedral identities in Theorems 5-7, but, we have preferred not to do so since Jacobi polynomials seems more familiar than the associated Legendre or Gegenbauer functions. However, if we replace the non negative integers m or n by the complex parameter γ they are no more polynomials and in this case (4.29) and (4.32) can be used to simplify the more general hypergeometric identities. For

example, replacing m by γ in Theorem 5 and using (4.29) we obtain a more general hypergeometric identity

$$\begin{aligned} & z^{-\frac{1}{2}\gamma}(1-z)^{\mp\frac{1}{2}\alpha}\,{}_2F_1\left(\begin{matrix}\frac{\mp\alpha-\gamma-n}{2},\ \frac{\mp\alpha-\gamma+n+1}{2}\\ \frac{1}{2}-\gamma\end{matrix}\,\middle|\,z\right) \\ & \quad = B_n(\alpha,\gamma)\sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-\gamma-n)_{n-k}}{(n-k)!2^k}\frac{\Gamma(\alpha+\gamma-k+1)}{(1-z)^{-\frac{k}{2}}}\left\{P_\gamma^{k-\alpha}(1/\sqrt{z})\right\}_{k-\alpha}^{+}, \end{aligned} \tag{4.34}$$

where $n\in\mathbb{N}_0$, $\alpha,\gamma\in\mathbb{C}$, $z\in(0,1)$ and

$$B_n(\alpha,\gamma)=\frac{\Gamma\left(\frac{1}{2}\right)}{2^\gamma\,\Gamma\left(\gamma+\frac{1}{2}\right)}\frac{n!}{2^n\left(\frac{\alpha-\gamma-n+1}{2}\right)_n}. \tag{4.35}$$

Similarly, replacing n by γ in Theorem 5 and using (4.32) we get the hypergeometric identity

$$\begin{aligned} & z^{-\frac{1}{2}m}(1-z)^{\mp\frac{1}{2}\alpha}\,{}_2F_1\left(\begin{matrix}\frac{\mp\alpha-\gamma-m}{2},\ \frac{\mp\alpha+\gamma-m+1}{2}\\ \frac{1}{2}-m\end{matrix}\,\middle|\,z\right) \\ & \quad = B_m(\alpha,\gamma)\sum_{k=0}^{m}\binom{m+k}{k}\frac{(\alpha-\gamma-m)_{m-k}}{(m-k)!2^k}\frac{\Gamma(\alpha+\gamma-k+1)}{z^{\frac{k}{2}}\,(1-z)^{-\frac{k}{2}}}\left\{\mathsf{P}_\gamma^{k-\alpha}(\sqrt{z})\right\}_{\sqrt{z}}^{\begin{smallmatrix}+,\ m-k\ \text{even}\\ -,\ m-k\ \text{odd}\end{smallmatrix}}, \end{aligned} \tag{4.36}$$

where $m\in\mathbb{N}_0$, $\alpha,\gamma\in\mathbb{C}$, $z\in(0,1)$ and

$$B_m(\alpha,\gamma)=\frac{m!}{2^m\left(\frac{1}{2}\right)_m}\frac{\Gamma\left(\frac{\alpha-\gamma-m+1}{2}\right)}{2^\gamma\,\Gamma\left(\frac{\alpha+\gamma-m+1}{2}\right)}. \tag{4.37}$$

The identities in Theorem 6 can be generalized in the same manner. Replacement of $m\in\mathbb{N}_0$ by $\gamma\in\mathbb{C}$ in Theorem 6 and the use of (4.29) lead to the the hypergeometric identity

$$\begin{aligned} & (1-z)^{\mp\frac{1}{2}\alpha}\,{}_2F_1\left(\begin{matrix}\frac{\mp\alpha+\gamma-n+1}{2},\ \frac{\mp\alpha+\gamma+n+2}{2}\\ \gamma+\frac{3}{2}\end{matrix}\,\middle|\,z\right) \\ & \quad = \frac{C_n(\alpha,\gamma)}{\left(-\sqrt{z}\right)^{\gamma+1}}\sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-\gamma-n)_{n-k}}{(n-k)!2^k}\frac{\Gamma(\alpha+\gamma-k+1)}{(1-z)^{-\frac{k}{2}}}\left\{P_\gamma^{k-\alpha}(1/\sqrt{z})\right\}_{k-\alpha}^{-}, \end{aligned} \tag{4.38}$$

where $n\in\mathbb{N}_0$, $\alpha,\gamma\in\mathbb{C}$, $z\in(0,1)$ and

$$C_n(\alpha,\gamma)=\frac{2^{\gamma+1}\Gamma\left(\gamma+\frac{3}{2}\right)\Gamma(\alpha-\gamma-n)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+\gamma-n)}\frac{n!}{2^{n+1}\left(\frac{\alpha+\gamma-n}{2}\right)_{n+1}}. \tag{4.39}$$

Likewise, replacing $n \in \mathbb{N}_0$ by $\gamma \in \mathbb{C}$ in Theorem 6 and using (4.32) we obtain the identity

$$\begin{aligned}&(1-z)^{\mp\frac{1}{2}\alpha}\ {}_2F_1\left(\begin{matrix}\frac{\mp\alpha-\gamma+m+1}{2}, \frac{\mp\alpha+\gamma+m+2}{2}\\ m+\frac{3}{2}\end{matrix}\ \middle|\ z\right)\\ &= \frac{C_m(\alpha,\gamma)}{\left(-\sqrt{z}\right)^{m+1}}\sum_{k=0}^{m}\binom{m+k}{k}\frac{(\alpha-\gamma-m)_{m-k}}{(m-k)!2^k}\frac{\Gamma(\alpha+\gamma-k+1)}{z^{\frac{k}{2}}(1-z)^{-\frac{k}{2}}}\left\{\mathsf{P}_{\gamma}^{k-\alpha}(\sqrt{z})\right\}_{\sqrt{z}}^{\substack{+,\ m-k\ \text{even}\\ -,\ m-k\ \text{odd}}},\end{aligned} \tag{4.40}$$

where $m \in \mathbb{N}_0$, $\alpha, \gamma \in \mathbb{C}$, $z \in (0,1)$ and

$$C_m(\alpha,\gamma) = \frac{m!2^{m+1}\left(\frac{1}{2}\right)_{m+1}}{(\alpha-\gamma-m)_{2m}}\frac{\Gamma\left(\frac{\alpha-\gamma+m}{2}\right)}{2^{\gamma+1}\Gamma\left(\frac{\alpha+\gamma+m+2}{2}\right)}. \tag{4.41}$$

In Theorem 7, letting $m \in \mathbb{N}_0 \to \gamma \in \mathbb{C}$ and using (4.29) we obtain

$$\begin{aligned}&D_n(\alpha,\gamma)(1-z)^{\frac{1}{2}(\alpha-\gamma)}\ {}_2F_1\left(\begin{matrix}\frac{\alpha-\gamma-n}{2}, \frac{\alpha-\gamma+n+1}{2}\\ 1+\alpha\end{matrix}\ \middle|\ z\right)\\ &=\sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-\gamma-n)_{n-k}}{(n-k)!}\frac{\Gamma(2\alpha-2k+1)\Gamma(\gamma-\alpha+k+1)}{(1-z)^{-\frac{k}{2}}\Gamma(\alpha-k+1)}C_{\gamma-\alpha+k}^{\alpha-k+\frac{1}{2}}\left(1/\sqrt{1-z}\right),\end{aligned} \tag{4.42}$$

where $n \in \mathbb{N}_0$, $\alpha, \gamma \in \mathbb{C}$, $z \in (0,1)$ and

$$D_n(\alpha,\gamma) = \frac{\Gamma(\alpha+\gamma+n+1)}{n!\Gamma(\alpha+1)}\frac{\left(\frac{\alpha-\gamma-n+1}{2}\right)_n}{\left(\frac{\alpha+\gamma-n+1}{2}\right)_n}. \tag{4.43}$$

Finally, letting $n \in \mathbb{N}_0 \to \gamma \in \mathbb{C}$ in Theorem 7 and utilizing (4.32) we get

$$\begin{aligned}&D_m(\alpha,\gamma)(1-z)^{-\frac{1}{2}m}\ {}_2F_1\left(\begin{matrix}\frac{\alpha-\gamma-m}{2}, \frac{\alpha+\gamma-m+1}{2}\\ 1+\alpha\end{matrix}\ \middle|\ z\right)\\ &=\sum_{k=0}^{m}\binom{m+k}{k}\frac{(\alpha-\gamma-m)_{m-k}}{(m-k)!}\frac{\Gamma(2\alpha-2k+1)\Gamma(\gamma-\alpha+k+1)}{(1-z)^{\frac{k}{2}}\Gamma(\alpha-k+1)}\mathsf{C}_{\gamma-\alpha+k}^{\alpha-k+\frac{1}{2}}\left(\sqrt{1-z}\right),\end{aligned} \tag{4.44}$$

where $m \in \mathbb{N}_0$, $\alpha, \gamma \in \mathbb{C}$, $z \in (0,1)$ and

$$D_m(\alpha,\gamma) = \frac{\Gamma(\alpha+\gamma+m+1)}{m!\Gamma(\alpha+1)}\frac{\left(\frac{\alpha-\gamma-m+1}{2}\right)_m}{\left(\frac{\alpha+\gamma-m+1}{2}\right)_m}. \tag{4.45}$$

Remaining identities for the class of Gauss hypergeometric functions where the characteristic exponent differences are in the set $\{k+\frac{1}{2}, \gamma+\frac{1}{2}, \alpha\}$, can be obtained by applying the Euler's and Pfaff's transformations in (2.12) to the identities obtained in this subsection.

5 Some consequences of dihedral identities

5.1 Finite sum relations for dihedral Gauss hypergeometric functions

By using the dihedral identities obtained in the Sect. 2.2 for $m = 0$ (or $n = 0$), on the right-hand sides of the general dihedral identities of the last section, we obtain several formulas relating $n + 2$ (or $m + 2$) dihedral hypergeometric functions. However, in order not to overload the content with similar formulas we present only the ones without pre-factors in front of the hypergeometric functions.

Corollary 1 *The following constant coefficient finite sum formula*

$$(\alpha - m - n)_{2m+1} \frac{2^n \left(\frac{\alpha+m-n+2}{2}\right)_n}{n!} \, {}_2F_1\left(\begin{matrix} \frac{\alpha+m-n+1}{2}, \frac{\alpha+m+n+2}{2} \\ m + \frac{3}{2} \end{matrix} \middle| z\right)$$
$$= \sum_{k=0}^{n} \binom{n+k}{k} \frac{(\alpha - m - n)_{n-k}}{(n-k)! 2^k} (\alpha - k - n)_{2m+1} \, {}_2F_1\left(\begin{matrix} \frac{\alpha-k+m+1}{2}, \frac{\alpha-k+m+2}{2} \\ m + \frac{3}{2} \end{matrix} \middle| z\right), \tag{5.1}$$

of dihedral hypergeometric functions holds for $\alpha \in \mathbb{C}$, $z \in \mathbb{C}\backslash[1, \infty)$ *and* $m, n \in \mathbb{N}_0$.

Proof Use (2.47) on the right hand side of (4.26) with $\alpha \to \alpha - k$ and $n \to m$. □

Corollary 2 *The following constant coefficient finite sum formula*

$$\frac{(\alpha+1)_{m+n}}{n!} \frac{\left(\frac{\alpha-m-n+1}{2}\right)_n}{\left(\frac{\alpha+m-n+1}{2}\right)_n} \, {}_2F_1\left(\begin{matrix} \frac{\alpha-m-n}{2}, \frac{\alpha-m+n+1}{2} \\ 1 + \alpha \end{matrix} \middle| z\right)$$
$$= \sum_{k=0}^{n} \binom{n+k}{k} \frac{(\alpha - m - n)_{n-k}}{(n-k)!} (\alpha - k + 1)_m \, {}_2F_1\left(\begin{matrix} \frac{\alpha-m-k}{2}, \frac{\alpha-m-k+1}{2} \\ 1 + \alpha - k \end{matrix} \middle| z\right), \tag{5.2}$$

of dihedral hypergeometric functions holds for $\alpha \in \mathbb{C}$, $z \in \mathbb{C}\backslash[1, \infty)$ *and* $m, n \in \mathbb{N}_0$.

Proof Use (2.48) on the right hand side of (4.27) with $\alpha \to \alpha - k$ and $n \to m$. □

Corollary 3 *The following constant coefficient finite sum formula*

$$\frac{(\alpha+1)_{m+n}}{n!} \frac{\left(\frac{\alpha-m-n+1}{2}\right)_n}{\left(\frac{\alpha+m-n+1}{2}\right)_n} \, {}_2F_1\left(\begin{matrix} \frac{\alpha-m-n}{2}, \frac{\alpha-m+n+1}{2} \\ 1 + \alpha \end{matrix} \middle| z\right)$$
$$= \sum_{k=0}^{n} \binom{n+k}{k} \frac{(\alpha - m - n)_{n-k}}{(n-k)!} (\alpha - k + 1)_m \, {}_2F_1\left(\begin{matrix} \frac{\alpha-m-k}{2}, \frac{\alpha-m-k+1}{2} \\ 1 + \alpha - k \end{matrix} \middle| z\right), \tag{5.3}$$

of dihedral hypergeometric functions holds for $\alpha \in \mathbb{C} \setminus \mathbb{Z}^-$, $z \in \mathbb{C}\backslash[1, \infty)$ *and* $m, n \in \mathbb{N}_0$.

Proof Use (2.49) on the right hand side of (4.28) with $\alpha \to \alpha - k$ and $n \to m$. □

Remark 5 For $z = 0$, Corollaries 1, 2 and 3 respectively lead to the Pochhammer symbol summation identities

$$\frac{2^n\left(\frac{\alpha-m-n+1}{2}\right)_n}{n!} = \sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-m-n)_{n-k}}{(n-k)!2^k} \tag{5.4}$$

$$(\alpha-m-n)_{2m+1}\frac{2^n\left(\frac{\alpha+m-n+2}{2}\right)_n}{n!} = \sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-m-n)_{n-k}}{(n-k)!2^k}(\alpha-k-n)_{2m+1} \tag{5.5}$$

$$\frac{(\alpha+1)_{m+n}}{n!}\frac{\left(\frac{\alpha-m-n+1}{2}\right)_n}{\left(\frac{\alpha+m-n+1}{2}\right)_n} = \sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-m-n)_{n-k}}{(n-k)!}(\alpha-k+1)_m, \tag{5.6}$$

which hold for $\alpha \in \mathbb{C}$ and $m, n \in \mathbb{N}_0$.

5.2 An interesting Jacobi polynomial identity

As stated at the end of Sect. 2, we have the following Jacobi polynomial identity of both trigonometric and algebraic arguments. It is interesting in the sense that the degree n is fixed but the parameter α varies.

Corollary 4 *The finite sum relation*

$$P_n^{(\frac{\alpha}{2},-\frac{\alpha}{2})}(i\cot 2x) = \frac{n!}{2^{2n}\left(\frac{\alpha+1-2n}{2}\right)_n}\sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-2n)_{n-k}}{(n-k)!}\left(\frac{1-i\tan x}{2}\right)^k P_n^{(\alpha-k,k-\alpha)}(i\cot x), \tag{5.7}$$

where $x \in \left(0, \frac{\pi}{2}\right)$ *or its algebraic equivalent*

$$\begin{aligned} &P_n^{(\frac{\alpha}{2},-\frac{\alpha}{2})}\left(\frac{1+z^2}{2z}\right) \\ &= \frac{n!}{2^{2n}\left(\frac{\alpha+1-2n}{2}\right)_n}\sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-2n)_{n-k}}{(n-k)!}\left(\frac{1+z}{2z}\right)^k P_n^{(\alpha-k,k-\alpha)}(z) \\ &= \frac{n!}{2^{2n}\left(\frac{\alpha+1-2n}{2}\right)_n}\sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-2n)_{n-k}}{(n-k)!}\left(\frac{1+z}{2}\right)^k P_n^{(\alpha-k,k-\alpha)}(1/z), \end{aligned} \tag{5.8}$$

for $z \in \mathbb{C}\setminus\{0\}$ *holds for the Jacobi polynomials of degree n with varying parameters of the form* $(a, -a)$.

Proof Take $m = n$ in Theorem 2 (resp. 3 or 4) and compare with (2.53) (resp. (2.54) or (2.59)). For the second part, just replace $z = i\cot x$ in (5.7) or let $m = n$ in Theorem 5 (resp. 6 or 7) and compare with (2.63) (resp. (2.64) or (2.65)). □

6 Some comments and conclusion

Firstly, as mentioned in Introduction, we present explicit representations

$$\begin{aligned}\mathsf{P}_{m,n} = \overline{\mathsf{Q}_{m,n}} &= \sum_{k=0}^{n}\binom{n+k}{k}\frac{(\alpha-m-n)_{n-k}}{(n-k)!}\left(\frac{1-i\tan x}{2}\right)^k P_m^{(\alpha-k,k-\alpha)}(i\cot x)\\ &= \sum_{k=0}^{m}\binom{m+k}{k}\frac{(\alpha-m-n)_{m-k}}{(m-k)!}\left(\frac{1+i\cot x}{2}\right)^k P_n^{(\alpha-k,k-\alpha)}(-i\tan x),\end{aligned} \tag{6.1}$$

for the polynomials $\mathsf{P}_{m,n}$ and $\mathsf{Q}_{m,n}$ appearing in Darboux's representation (1.5) for the general solution of (1.2). These representations directly follow from the solutions in (3.30) and (3.31), respectively. Moreover, it is pointed out in [7, p. 113] that $\mathsf{P}_{m,n}$ and $\mathsf{Q}_{m,n}$ can be represented as rational functions of forms $\mathsf{F}(\cot x; i\alpha)$ and $\mathsf{F}(-\cot x; i\alpha)$ (or $\mathsf{F}(i\cot x; \alpha)$ and $\mathsf{F}(i\cot x; -\alpha)$) respectively, where

$$\mathsf{F}(u;a) = c_m(a)u^m + c_{m-1}(a)u^{m-1} + \ldots + c_0(a) + \ldots + c_{1-n}(a)u^{1-n} + c_{-n}(a)u^{-n}, \tag{6.2}$$

and hence, of types $\mathsf{G}(i\tan x; \alpha)$ and $\mathsf{G}(i\tan x; -\alpha)$ where

$$\mathsf{G}(u;a) = c_{-n}(a)u^n + c_{1-n}(a)u^{n-1} + \ldots + c_0(a) + \ldots + c_{m-1}(a)u^{1-m} + c_m(a)u^{-m}, \tag{6.3}$$

since $\cot x = 1/\tan x$. These properties can readily be observed from (6.1) by expressing each pre-factor in terms of the variable of the Jacobi polynomial, i.e., $i\tan x$ in terms of $i\cot x$ and vice versa.

Secondly, we have derived explicit trigonometric and algebraic identities for dihedral Gauss hypergeometric functions in terms of the Jacobi polynomials with some pre-factors by expressing dihedral hypergeometric solutions in terms of elementary ones. Dihedral formulas are first obtained in [13] in which the algebraic ones are in terms of terminating Appell's F_2 or F_3 double sums and the trigonometric ones are in terms of certain related trigonometric finite double sums. We would like to draw attention to a small detail here. It seems that the formula (6.22) of [13]

$${}_2F_1\left(\begin{matrix}\frac{3+\lambda}{2}, \frac{3-\lambda}{2}\\ \frac{5}{2}\end{matrix}\,\middle|\,\sin^2 x\right) = 3\frac{\cos 2x\sin\lambda x - \frac{1}{2}\lambda\sin 2x\cos\lambda x}{\lambda(\lambda-2)(\lambda+2)\sin^3 x},$$

has typos since, for example, when $\lambda = 3$ the left-hand side assumes the value 1 whereas the right-hand side simplifies to $(2\cos 2x + 3)/5$. The correct version

$$ {}_2F_1\left(\begin{matrix}\frac{3+\alpha}{2}, & \frac{3-\alpha}{2} \\ & \frac{5}{2}\end{matrix}\,\middle|\,\sin^2 x\right) = 3\frac{\cos x \sin\alpha x - \alpha \sin x \cos\alpha x}{\alpha(\alpha-1)(\alpha+1)\sin^3 x \cos x}, \tag{6.4} $$

follows from (4.9) by taking $m = 1$ and $n = 0$ or from (2.38) by taking $n = 1$. Note that, for $\alpha \in \mathcal{U}_1 = \{0, \pm 1\}$ the right hand side of the equation in (6.4) is undetermined which can be removed by a limiting process as $\alpha \to 0, \pm 1$ leading to the so called logarithmic solutions of the Gauss hypergeometric equation. However, for larger values of n it becomes extremely cumbersome since both the set $\mathcal{U}_n$ and the right-hand side get crowded. Hence, the undetermined cases need closed form expressions, too. Furthermore, the explicit dihedral identities obtained in this study have some other potential applications. Studies on these topics are ongoing and the results will be reported in due course.

Appendix A

Claim 1 *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ we have the equality*

$$ \frac{i^n}{A_{nm}(\alpha)}\,{}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2}, & \frac{-\alpha-m-n}{2} \\ & \frac{1}{2}-m\end{matrix}\,\middle|\,1\right) = \frac{(-i)^m}{A_{mn}(\alpha)}\begin{cases}\cos\frac{\alpha\pi}{2}, & m+n \ \text{ even} \\ i\sin\frac{\alpha\pi}{2}, & m+n \ \text{ odd}\end{cases}, \tag{A.1} $$

where $A_{mn}(\alpha)$ is defined in (3.40).

Proof Using Gauss's theorem [4, (15.4.20)]

$$ {}_2F_1\,(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c) > \mathrm{Re}(a+b), \tag{A.2} $$

and the definition of $A_{mn}(\alpha)$ in (3.40) we write

$$ \begin{aligned} &\frac{i^n}{A_{nm}(\alpha)}\,{}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2}, & \frac{-\alpha-m-n}{2} \\ & \frac{1}{2}-m\end{matrix}\,\middle|\,1\right) \\ &= i^n \frac{n!}{2^n\left(\frac{1}{2}\right)_n}\frac{m!}{2^m\left(\frac{\alpha-m-n+1}{2}\right)_m}\frac{\Gamma\left(\frac{1}{2}-m\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{-\alpha-m+n+1}{2}\right)\Gamma\left(\frac{\alpha-m+n+1}{2}\right)}. \end{aligned} \tag{A.3} $$

On the right-hand side, we have

$$ \frac{\Gamma\left(\frac{1}{2}-m\right)\Gamma\left(n+\frac{1}{2}\right)}{\left(\frac{1}{2}\right)_n} = \frac{\Gamma\left(\frac{1}{2}-m\right)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} = \frac{(-1)^m\pi\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} = \frac{(-1)^m\pi}{\left(\frac{1}{2}\right)_m}, \tag{A.4} $$

upon using the definition $(a)_n = \Gamma(a+n)/\Gamma(a)$ of the Pochhammer's symbol with $a = 1/2$ (see (2.36)) and a version of the Euler's reflection formula

$$\Gamma\left(\tfrac{1}{2}-z\right)\Gamma\left(\tfrac{1}{2}+z\right) = \frac{\pi}{\cos \pi z}, \qquad z \neq n+\tfrac{1}{2}, \quad n \in \mathbb{Z}, \tag{A.5}$$

with $z = m$. Moreover, the remaining Pochhammer and gamma function terms simplifies to

$$\begin{aligned}\frac{1}{\left(\frac{\alpha-m-n+1}{2}\right)_m \Gamma\left(\frac{-\alpha-m+n+1}{2}\right)\Gamma\left(\frac{\alpha-m+n+1}{2}\right)} &= \frac{\Gamma\left(\frac{\alpha-m-n+1}{2}\right)}{\Gamma\left(\frac{\alpha+m-n+1}{2}\right)\Gamma\left(\frac{-\alpha-m+n+1}{2}\right)\Gamma\left(\frac{\alpha-m+n+1}{2}\right)} \\ &= \frac{\cos(\alpha+m-n)\frac{\pi}{2}}{\pi\left(\frac{\alpha-m-n+1}{2}\right)_n} \\ &= \frac{i^{m-n}}{\pi\left(\frac{\alpha-m-n+1}{2}\right)_n}\begin{cases}\cos\frac{\alpha\pi}{2}, & m+n \text{ even} \\ i\sin\frac{\alpha\pi}{2}, & m+n \text{ odd}\end{cases},\end{aligned} \tag{A.6}$$

upon the use of (A.5) with $z = (\alpha+m-n)/2$ and summation formula for cosine in the last step. Finally, inserting (A.6) and (A.4) in (A.3) we obtain (A.1). □

Claim 2 *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ we have the equality*

$$\frac{(-i)^n}{A_{nm}(\alpha)}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+m-n+1}{2}, & \frac{-\alpha+m-n+1}{2} \\ m+\frac{3}{2} & \end{matrix}\,\middle|\,1\right) = i^m C_{mn}(\alpha)\begin{cases}\sin\frac{\alpha\pi}{2}, & m+n \text{ even} \\ i\cos\frac{\alpha\pi}{2}, & m+n \text{ odd}\end{cases}, \tag{A.7}$$

where $A_{mn}(\alpha)$ and $C_{mn}(\alpha)$ are introduced in (3.40) and (4.10), respectively.

Proof Using (3.40) and (A.2) we have

$$\begin{aligned}&\frac{(-i)^n}{A_{nm}(\alpha)}\,{}_2F_1\left(\begin{matrix}\frac{\alpha+m-n+1}{2}, & \frac{-\alpha+m-n+1}{2} \\ m+\frac{3}{2} & \end{matrix}\,\middle|\,1\right) \\ &\qquad = (-i)^n \frac{n!}{2^n\left(\frac{1}{2}\right)_n}\frac{m!}{2^m\left(\frac{\alpha-m-n+1}{2}\right)_m}\frac{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{-\alpha+m+n+2}{2}\right)\Gamma\left(\frac{\alpha+m+n+2}{2}\right)}.\end{aligned} \tag{A.8}$$

Now, we have for

$$\frac{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\left(\frac{1}{2}\right)_n} = \frac{\Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)_{m+1}\Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n} = \pi\left(\frac{1}{2}\right)_{m+1}, \tag{A.9}$$

and

$$\begin{aligned}\frac{1}{\Gamma\left(\frac{-\alpha+m+n+2}{2}\right)\Gamma\left(\frac{\alpha+m+n+2}{2}\right)} &= \frac{1}{\Gamma\left(\frac{-\alpha+m+n+2}{2}\right)\Gamma\left(\frac{\alpha-m-n}{2}\right)}\frac{\Gamma\left(\frac{\alpha-m-n}{2}\right)}{\Gamma\left(\frac{\alpha+m+n+2}{2}\right)}\\ &= \frac{\sin(\alpha-m-n)\frac{\pi}{2}}{\pi}\frac{1}{\left(\frac{\alpha-m-n}{2}\right)_{m+n+1}}\\ &= \frac{i^{m+n}}{\pi\left(\frac{\alpha-m-n}{2}\right)_{m+n+1}}\begin{cases}\sin\frac{\alpha\pi}{2}, & m+n \ \text{ even}\\ i\cos\frac{\alpha\pi}{2}, & m+n \ \text{ odd}\end{cases},\end{aligned} \tag{A.10}$$

where, after multiplying and dividing by $\Gamma\left(\frac{\alpha-m-n}{2}\right)$, we have used Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z}, \tag{A.11}$$

with $z = (\alpha - m - n)/2$ and definition (2.36) of the Pochhammer's symbol with $a = (\alpha - m - n)/2$ and $k = m + n + 1$. The last step makes use of the summation formula for the sine function. Finally, plugging (A.10) and (A.9) in (A.8) we obtain (A.7). On the other hand, equivalent representations of the constant $C_{mn}(\alpha)$ in (4.10) can be obtained by employing the Pochhammer symbol identities

$$(a)_{k+l} = (a)_k(a+k)_l, \tag{A.12}$$

and

$$(a)_{2k} = 2^{2k}\left(\frac{a}{2}\right)_k\left(\frac{a+1}{2}\right)_k, \qquad (a)_{2k+1} = 2^{2k+1}\left(\frac{a}{2}\right)_{k+1}\left(\frac{a+1}{2}\right)_k, \tag{A.13}$$

where k and l are non-negative integers and a is an arbitrary complex number. In fact, the second representation in (4.10) can be obtained as

$$\begin{aligned}C_{mn}(\alpha) &= \frac{m!2^{m+1}\left(\frac{1}{2}\right)_{m+1}}{2^m\left(\frac{\alpha-m-n+1}{2}\right)_m}\frac{n!}{2^{m+n+1}\left(\frac{\alpha-m-n}{2}\right)_{m+n+1}}\\ &= \frac{m!n!}{2^{m+n}}\frac{\left(\frac{1}{2}\right)_{m+1}}{\left(\frac{\alpha-m-n+1}{2}\right)_m\left(\frac{\alpha-m-n}{2}\right)_m\left(\frac{\alpha+m-n}{2}\right)_{n+1}}\\ &= \frac{m!2^{m+1}\left(\frac{1}{2}\right)_{m+1}}{(\alpha-m-n)_{2m}}\frac{n!}{2^{n+1}\left(\frac{\alpha+m-n}{2}\right)_{n+1}},\end{aligned} \tag{A.14}$$

where we have used (A.12) and the first identity in (A.13), respectively. The third representation can be obtained in a similar manner where now (A.12) and the second identity in (A.13) are necessary. The second and the third representations in (4.10) are useful for the special cases where $(m = 0,\ n \geq 0)$ and $(n = 0, m \geq 0)$, respectively. □

Claim 3 *For $m, n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ we have the equality*

$$\frac{1}{A_{nm}(\alpha)}\,{}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2},\ \frac{\alpha+m-n+1}{2}\\ 1+\alpha\end{matrix}\,\middle|\,1\right) = 2^{\alpha} D_{mn}(\alpha), \tag{A.15}$$

where $D_{mn}(\alpha)$ is the constant in (4.18).

Proof Using Gauss's theorem in (A.2) and $A_{mn}(\alpha)$ in (3.40) we write

$$\begin{aligned}&\frac{1}{A_{nm}(\alpha)}\,{}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2},\ \frac{\alpha+m-n+1}{2}\\ 1+\alpha\end{matrix}\,\middle|\,1\right)\\ &\quad= \frac{n!}{2^n\left(\frac{1}{2}\right)_n}\frac{m!}{2^m\left(\frac{\alpha-m-n+1}{2}\right)_m}\frac{\Gamma(\alpha+1)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha-m+n+1}{2}\right)\Gamma\left(\frac{\alpha+m+n+2}{2}\right)}.\end{aligned} \tag{A.16}$$

We can simplify the right-hand side terms as

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\left(\frac{1}{2}\right)_n} = \Gamma\left(\tfrac{1}{2}\right), \tag{A.17}$$

and

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha-m+n+1}{2}\right)\Gamma\left(\frac{\alpha+m+n+2}{2}\right)} &= \frac{\Gamma\left(\frac{\alpha+m+n+1}{2}\right)}{\Gamma\left(\frac{\alpha-m+n+1}{2}\right)}\frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha+m+n+1}{2}\right)\Gamma\left(\frac{\alpha+m+n+2}{2}\right)}\\ &= \frac{\left(\frac{\alpha-m+n+1}{2}\right)_m 2^{\alpha+m+n}\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+m+n+1)}\\ &= \frac{2^{\alpha+m+n}\left(\frac{\alpha-m+n+1}{2}\right)_m}{\Gamma\left(\frac{1}{2}\right)(\alpha+1)_{m+n}},\end{aligned} \tag{A.18}$$

by using the Legendre duplication formula [4, (5.5.5)]

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = \Gamma\left(\tfrac{1}{2}\right)2^{1-2z}\Gamma(2z), \qquad 2z \neq 0, -1, -2, \ldots, \tag{A.19}$$

with $z = (\alpha+m+n+1)/2$ and the definition (2.36) of the Pochhammer's symbol with $a = (\alpha-m+n+1)/2$ and $k = m$ in the second and $a = \alpha+1$ and $k = m+n$ in the last step. Then, inserting (A.18) and (A.17) in (A.16) we obtain

$$\frac{1}{A_{nm}(\alpha)}\,{}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2},\ \frac{\alpha+m-n+1}{2}\\ 1+\alpha\end{matrix}\,\middle|\,1\right) = 2^{\alpha} D_{nm}(\alpha) = 2^{\alpha} D_{mn}(\alpha), \tag{A.20}$$

since

$$\begin{aligned} D_{nm}(\alpha) &= \frac{n!m!}{(\alpha+1)_{n+m}} \frac{\left(\frac{\alpha-m+n+1}{2}\right)_m}{\left(\frac{\alpha-m-n+1}{2}\right)_m} \\ &= \frac{m!n!}{(\alpha+1)_{m+n}} \frac{\Gamma\left(\frac{\alpha+m+n+1}{2}\right)\Gamma\left(\frac{\alpha-m-n+1}{2}\right)}{\Gamma\left(\frac{\alpha-m+n+1}{2}\right)\Gamma\left(\frac{\alpha+m-n+1}{2}\right)} \\ &= \frac{m!n!}{(\alpha+1)_{m+n}} \frac{\left(\frac{\alpha+m-n+1}{2}\right)_n}{\left(\frac{\alpha-m-n+1}{2}\right)_n} = D_{mn}(\alpha). \end{aligned} \tag{A.21}$$

Finally, (4.24) can be evaluated as

$$\frac{i^{n-\alpha}}{A_{mn}(\alpha)} {}_2F_1\left(\begin{matrix}\frac{\alpha-m-n}{2}, \frac{\alpha-m+n+1}{2} \\ 1+\alpha\end{matrix}\,\middle|\, 1\right) = i^{n-\alpha}2^{\alpha} D_{mn}(\alpha), \tag{A.22}$$

by interchanging the roles of m and n in (A.20). □

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