Modern Physics Letters A, Vol. 15, No. 13 (2000) 833–839
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DIRAC GAMMA MATRICES AS REPRESENTATIONS OF THE ALGEBRA OF LITTLE GROUPS

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Received 7 March 2000

The algebra of O(3, 3) is contracted to that of E(2)-like groups in the infinite-momentum limit. It is shown through this process that there are 18 different possibilities for the representations of E(2)-like algebra if they are to be expressed in terms of Dirac γ-matrices. It is also pointed out that the generators of this algebra are independent of the particular representation of the γ-matrices. Possible implications of this analysis is discussed.

1. Introduction

In 1939, Wigner published his fundamental paper on little groups, where they were defined to be the maximal subgroup of the Poincaré group whose transformations leave the four-momentum of a relativistic particle invariant. He observed that little groups are locally isomorphic to the rotation and Euclidean groups for massive and massless particles, respectively.[1] Later, Inönü and Wigner introduced the concept of group contraction and showed that the rotation group O(3) can be contracted to the Euclidean group E(2).[2]

The rotation group has been long-known, however the applications of the groups that are locally isomorphic to the Euclidean group have relatively been of a more recent interest. Applications extend from classical to quantum optics, as well as to free electromagnetic fields. They come into view in discussing the symmetries of generalized two-mode coherent states,[3] or of photon polarization vectors.[4] Stokes parameters, which are like the components of a four-vector can describe partially coherent light waves. The transformation matrices obtained from the translation-like generators of this group preserve coherence when applied to Stokes vectors.[5] While for some features of a problem E(2)-like transformations exhibit invariances, for some others they are the means of generating significant states. For instance, E(2)-like symmetry for the four-vector potential presents itself as gauge transformations.[6–8] On the other hand, the state vectors for the electromagnetic field accommodate only those spinors that are invariant under

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such transformations.[9,10] It can be observed that four-dimensional representations are essential in most of these applications. Therefore, we believe that a deeper understanding of the representations of the E(2)-like groups in this dimension is important.

Contractions have also been found useful in discussing the infinite-momentum limits of massive particles and thereof representing the kinematical aspects of massless particles.[11] The algebra of the O(3, 1) Lorentz group can be contracted to the algebra of E(2)-like groups by means of a similar limiting procedure. The generators of its rotation subgroup are boosted to represent the little group for a massive particle in relativistic motion. For massless particles the little group is obtained in the infinite limit of the boost parameter. Likewise, larger Lorentz groups can be contacted to various E(2)-like groups. In this letter, we shall elaborate on this issue and exploit its results for a systematic exposition of all possible representations of the algebra of E(2)-like groups in terms of Dirac γ-matrices.

2. Dirac γ-Matrices as a Representation of the Algebra of O(3, 3)

Dirac gave "A remarkable representation for the $3 + 2$ de Sitter group" in terms of his γ-matrices.[12] It is possible to extend this representation for the algebra of O(3, 3) by defining its generators M^{ab} as[13]:

$$M^{ab} = \frac{i}{4}[\gamma^a, \gamma^b], \qquad a, b = 1, \ldots, 5 \tag{1}$$

and $M^{6a} = \frac{1}{2}\gamma^a$, with $M^{ab} = -M^{ba}$. The generators of the O(3, 3) group satisfy the following commutation relations:

$$[M^{ab}, M^{cd}] = i(g^{ad}M^{bc} - g^{ac}M^{bd} + g^{bc}M^{ad} - g^{bd}M^{ac}), \tag{2}$$

where $g^{ab} = \mathrm{diag}(-1, -1, -1, 1, 1, 1)$. This group has two sets of rotations, one for the three space coordinates, another for the three time coordinates and has three sets of boosts, one for each of the time coordinates. The corresponding generators are respectively, written as:

$$J_i = \frac{1}{2}\varepsilon_{ijk}M^{jk}, \qquad T_i = \frac{1}{2}\varepsilon_{ijk}M^{k+3\,j+3}, \qquad L_{ij} = M^{i\,j+3}, \tag{3}$$

where indices $i, j, \ldots$ run from 1 to 3. They satisfy the commutation relations:

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \qquad [T_i, T_j] = i\varepsilon_{ijk}T_k, \qquad [T_i, J_j] = 0, \tag{4}$$

$$[T_i, L_{jk}] = i\varepsilon_{ikl}L_{jl}, \qquad [J_i, L_{jk}] = i\varepsilon_{ijl}L_{lk}, \tag{5}$$

$$[L_{ij}, L_{kl}] = -i\delta_{ik}\varepsilon_{jlm}T_m - i\delta_{jl}\varepsilon_{ikm}J_m. \tag{6}$$

In the following sections, the O(3, 3) algebra together with the contraction process will constitute the basic machinery for the construction of E(2)-like algebras in terms of γ-matrices.

3. Contractions of the O(3, 1) Algebra to E(2)-Like Algebras

A contraction of a group is a continuous degenerate transformation on its generators which alters the structure of its algebra. The simplest and well-known example is the contraction of the rotation group to the Euclidean group. Geometrically, it is a flat surface approximation of the sphere to a tangent plane.[14] Here, contractions will be achieved through a sequence of operations on the generators of the O(3, 1) Lorentz algebra: First, by boosting its rotation generators in one of the three space-like directions:

$$J_i' = B(\eta) J_i B(\eta)^{-1}\,, \qquad i = 1, 2, 3\,, \tag{7}$$

where

$$B(\eta) = e^{-i\eta\, K_j} \tag{8}$$

with K_j being a fixed boost generator and the parameter η is associated with the speed of the boost. Then the boosted generators are normalized and finally the generators of the new algebra are obtained in the infinite limit of the parameter.

A particular representation of the γ-matrices is of convenience to demonstrate the procedure and to find an algorithm as to how to obtain the E(2)-like group generators from larger Lorentz groups. In the Weyl representation the γ-matrices are:

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{9}$$

with the replacement of the superscript 4 with 0 in (1) and M^{6a}, where $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. The basic advantage of this representation is that the generators of rotations and boosts take block diagonal form:

$$J_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \qquad K_i = \frac{1}{2}\begin{pmatrix} i\sigma_i & 0 \\ 0 & -i\sigma_i \end{pmatrix}. \tag{10}$$

After this preparation, the transformations (7), with say $j = 3$, explicitly read as:

$$J_1' = \frac{1}{2}\left\{\sinh\eta \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} + \cosh\eta \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}\right\}, \tag{11}$$

$$J_2' = \frac{1}{2}\left\{\sinh\eta \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} + \cosh\eta \begin{pmatrix} \sigma_2 & 0 \\ 0 & i\sigma_1 \end{pmatrix}\right\}, \tag{12}$$

$$J_3' = J_3\,. \tag{13}$$

Now, the generators of the resultant algebra are expressed as:

$$N_1 = \lim_{\eta\to\infty} (\cosh\eta)^{-1} J_1' = J_1 + K_2\,, \tag{14}$$

$$N_2 = \lim_{\eta\to\infty} (\cosh\eta)^{-1} J_2' = J_2 - K_1\,. \tag{15}$$

The operators N_1, N_2 and $J = J_3$ satisfy the same commutation relations with that of E(2):

$$[J, N_1] = i\, N_2\,, \qquad [J, N_2] = -i\, N_1\,, \qquad [N_1, N_2] = 0\,. \tag{16}$$

If one goes through the same process with K_1 and K_2 in (7) and (8), then there will be two more sets to generate the same algebra, which are, respectively:

$$J = J_1\,, \qquad N_1 = J_2 + K_3\,, \qquad N_2 = J_3 - K_2\,, \tag{17}$$

$$J = J_2\,, \qquad N_1 = J_3 + K_1\,, \qquad N_2 = J_1 - K_3\,. \tag{18}$$

It can easily be observed that in the various constructions of the generators, cyclic permutations play a crucial role. In the following section this will be more apparent.

4. Dirac γ-Matrices and Representations of E(2)-Like Algebra

Since we are interested in representing the E(2)-like algebra in terms of γ-matrices and since there are 15 independent matrices formed from their products,[15] it is obvious that the maximal extension of $O(3,1)$ which will serve for our purposes is the 15 parameter Lorentz group $O(3,3)$. For each rotation generator there are three sets of representations for the E(2)-like algebra. Given this and the fact that $O(3,3)$ has six rotation generators, there are 18 possibilities for the representations of E(2)-like algebra from γ-matrices.

The particular combinations of the generators of $O(3,1)$ to construct the generators of an E(2)-like algebra which is presented in the preceding section can be read as a special case of a general rule when there are more than one time-like coordinates for a larger Lorentz group and consequently more boost and rotation generators. One can certainly go through the straightforward calculations for contractions to act all boost-like transformations on the rotation generators J_i and T_i to obtain the same result. We, therefore, have the following sets for the generators. One assemblage is labeled as $\mathcal{S}$:

$$J = J_1\,, \qquad N_1 = J_2 + L_{3i}\,, \qquad N_2 = J_3 - L_{2i}\,, \tag{19}$$

$$J = J_2\,, \qquad N_1 = J_3 + L_{1i}\,, \qquad N_2 = J_1 - L_{3i}\,, \tag{20}$$

$$J = J_3\,, \qquad N_1 = J_1 + L_{2i}\,, \qquad N_2 = J_2 - L_{1i}\,, \tag{21}$$

where i runs from 1 to 3. The role of the cyclic permutations can be seen more clearly from Fig. 1. If the rotation generator J belongs to the time-like-space, then the generators for translations are linear combinations of boosts and rotations in that time-like-space; a fact which is independent of the algorithm we have introduced and is due to the commutation relations given in (4)–(6). Similar arguments were

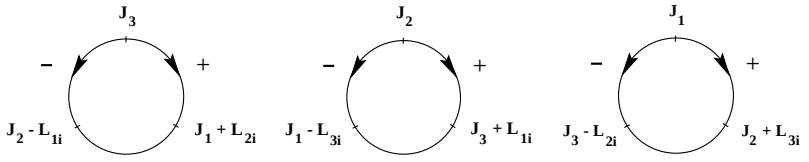


Fig. 1. The generators of the E(2)-like group can be constructed by referring to the circles. Translation-like generators are linear combinations of J_k and L_{ij}. Constituent rotation generators follow a cyclic permutation, while the first index in L_{ij} alternate with that of J_k. The arrows also indicate signs of boosts.

also valid for the sets in $\mathcal{S}$. Therefore, we classify the following generator sets as $\mathcal{T}$:

$$J = T_1\,, \qquad N_1 = T_2 + L_{i3}\,, \qquad N_2 = T_3 - L_{i2}\,, \tag{22}$$

$$J = T_2\,, \qquad N_1 = T_3 + L_{i1}\,, \qquad N_2 = T_1 - L_{i3}\,, \tag{23}$$

$$J = T_3\,, \qquad N_1 = T_1 + L_{i2}\,, \qquad N_2 = T_2 - L_{i1}\,, \tag{24}$$

where $i = 1, 2, 3$.

With Sec. 2, it is straightforward to express these generators in terms of the products of γ-matrices. For an immediate use, we give the explicit expressions which are independent of any particular representation. The sets in $\mathcal{S}$ read as:

$$J = \frac{i}{2}\gamma^2\gamma^3\,, \qquad N_1 = \frac{i}{2}\gamma^3(\gamma^1 - \gamma^0)\,, \qquad N_2 = -\frac{i}{2}\gamma^2(\gamma^1 + \gamma^0)\,, \tag{25}$$

$$J = \frac{i}{2}\gamma^2\gamma^3\,, \qquad N_1 = \frac{i}{2}\gamma^3(\gamma^1 - \gamma^5)\,, \qquad N_2 = -\frac{i}{2}\gamma^2(\gamma^1 + \gamma^5)\,, \tag{26}$$

$$J = \frac{i}{2}\gamma^2\gamma^3\,, \qquad N_1 = \frac{i}{2}\gamma^3(\gamma^1 + iI)\,, \qquad N_2 = -\frac{i}{2}\gamma^2(\gamma^1 + iI)\,, \tag{27}$$

$$J = \frac{i}{2}\gamma^3\gamma^1\,, \qquad N_1 = \frac{i}{2}\gamma^1(\gamma^2 + \gamma^0)\,, \qquad N_2 = -\frac{i}{2}\gamma^3(\gamma^2 + \gamma^0)\,, \tag{28}$$

$$J = \frac{i}{2}\gamma^3\gamma^1\,, \qquad N_1 = \frac{i}{2}\gamma^1(\gamma^2 + \gamma^5)\,, \qquad N_2 = -\frac{i}{2}\gamma^3(\gamma^2 + \gamma^5)\,, \tag{29}$$

$$J = \frac{i}{2}\gamma^3\gamma^1\,, \qquad N_1 = \frac{i}{2}\gamma^1(\gamma^2 + iI)\,, \qquad N_2 = -\frac{i}{2}\gamma^3(\gamma^2 + iI)\,, \tag{30}$$

$$J = \frac{i}{2}\gamma^1\gamma^2\,, \qquad N_1 = \frac{i}{2}\gamma^2(\gamma^3 + \gamma^0)\,, \qquad N_2 = -\frac{i}{2}\gamma^1(\gamma^3 + \gamma^0)\,, \tag{31}$$

$$J = \frac{i}{2}\gamma^1\gamma^2\,, \qquad N_1 = \frac{i}{2}\gamma^2(\gamma^3 + \gamma^5)\,, \qquad N_2 = -\frac{i}{2}\gamma^1(\gamma^3 + \gamma^5)\,, \tag{32}$$

$$J = \frac{i}{2}\gamma^1\gamma^2\,, \qquad N_1 = \frac{i}{2}\gamma^2(\gamma^3 + iI)\,, \qquad N_2 = -\frac{i}{2}\gamma^1(\gamma^3 + iI)\,. \tag{33}$$

The generator sets in $\mathcal{T}$ read as follows:

$$J=\frac{1}{2}\gamma^5\,,\qquad N_1=-\frac{1}{2}(\gamma^0+\gamma^1)\,,\qquad N_2=\frac{i}{2}\gamma^5(\gamma^0+\gamma^1)\,, \tag{34}$$

$$J=\frac{1}{2}\gamma^5\,,\qquad N_1=-\frac{1}{2}(\gamma^0+\gamma^2)\,,\qquad N_2=\frac{i}{2}\gamma^5(\gamma^0+\gamma^2)\,, \tag{35}$$

$$J=\frac{1}{2}\gamma^5\,,\qquad N_1=-\frac{1}{2}(\gamma^0+\gamma^3)\,,\qquad N_2=\frac{i}{2}\gamma^5(\gamma^0+\gamma^3)\,, \tag{36}$$

$$J=-\frac{1}{2}\gamma^0\,,\qquad N_1=-\frac{i}{2}\gamma^0(\gamma^5+\gamma^1)\,,\qquad N_2=\frac{1}{2}(\gamma^5+\gamma^1)\,, \tag{37}$$

$$J=-\frac{1}{2}\gamma^0\,,\qquad N_1=-\frac{i}{2}\gamma^0(\gamma^5+\gamma^2)\,,\qquad N_2=\frac{1}{2}(\gamma^5+\gamma^2)\,, \tag{38}$$

$$J=-\frac{1}{2}\gamma^0\,,\qquad N_1=-\frac{i}{2}\gamma^0(\gamma^5+\gamma^3)\,,\qquad N_2=\frac{1}{2}(\gamma^5+\gamma^3)\,, \tag{39}$$

$$J=\frac{i}{2}\gamma^5\gamma^0\,,\qquad N_1=\frac{1}{2}\gamma^5(I-i\gamma^1)\,,\qquad N_2=-\frac{1}{2}\gamma^0(I-i\gamma^1)\,, \tag{40}$$

$$J=\frac{i}{2}\gamma^5\gamma^0\,,\qquad N_1=\frac{1}{2}\gamma^5(I-i\gamma^2)\,,\qquad N_2=-\frac{1}{2}\gamma^0(I-i\gamma^2)\,, \tag{41}$$

$$J=\frac{i}{2}\gamma^5\gamma^0\,,\qquad N_1=\frac{1}{2}\gamma^5(I-i\gamma^3)\,,\qquad N_2=-\frac{1}{2}\gamma^0(I-i\gamma^3)\,. \tag{42}$$

Incidentally, one may consider the double valuedness of the Lorentz group, and change the sign of the boost operator, which does not affect its commutation relations. This is sometimes referred as the "dotted" representation. The commutation relations of the E(2)-like algebra inherit such a double valuedness. Accordingly, the sign of L_{ij} can be changed in the sets $\mathcal{S}$ and $\mathcal{T}$, thus the plus, minus signs in the circles.

It follows from the properties of γ-matrices that all translation-like generators are nilpotent (i.e. $N_1^2=N_2^2=0$). Certainly, for physical applications, one should refer to the explicit forms of the transformation matrices by using a specific representation. While we leave possible applications for future studies, we note that, since the generators are nilponent, the transformation matrices will include only linear parameters yielding to less cumbersome transformations. For instance, the basic symmetry of the two-mode sheared states are governed by E(2)-like groups,[3] which is generated by the "dotted" set of (29) in Majorana representation.

In this work, we have given a rule for a systematic construction of the Lie algebra of E(2)-like groups from the generators the O(3, 3) Lorentz group which can easily be generalized to larger Lorentz groups. We have particularly chosen O(3, 3) in order to be able to represent them in terms of γ-matrices. In the past, many attempts have been made to provide a geometric interpretation to the Dirac γ-matrices, and this letter is not likely to be the last one.

Acknowledgments

I am grateful to Y. S. Kim for giving me some of his works in this area, along with many related references which I would otherwise not have known.

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