SOME CONSTRUCTIONS OF DIFFUSE INVARIANT RANDOM SUBGROUPS

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ABSTRACT

SOME CONSTRUCTIONS OF DIFFUSE INVARIANT RANDOM SUBGROUPS

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Invariant random subgroups (abbreviated IRSs), which are conjugation invariant Borel probability measures on the space of subgroups of a countable group, emerged as a probabilistic generalization of normal subgroups and lattices. An active area of research is finding and classifying IRSs of a given countable group. In order to understand arbitrary IRSs, it is sufficient to understand ergodic IRSs since the ergodic decomposition theorem allows us to write arbitrary IRSs in terms of ergodic IRSs. Among ergodic IRSs, Thomas recently distinguished those that do not assign full measure to an isomorphism class, namely, the diffuse IRSs. So far, there have not been many examples of diffuse IRSs. This thesis is intended to give a survey of two different IRS constructions that satisfy diffuseness and related properties.

Keywords: invariant, random subgroup, diffuse, group

ÖZ

BAZI YAYINIK DEĞİŞMEZ RASSAL ALTGRUP İNŞALARI

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Sayılabilir bir grubun altgrupları uzayında, konjügasyon altında sabit kalan Borel olasılık ölçüleri olarak tanımlanan Değişmez Rassal Altgruplar (kısaca DRA), normal altgrupların ve latislerin olasılıksal bir genelleştirmesi olarak ortaya çıktılar. Aktif bir çalışma alanı, verilen sayılabilir bir grubun DRA'larını bulup sınıflandırmaktır. Keyfi bir DRA'yı anlamak için ergodik DRA'ları anlamak yeterlidir çünkü ergodik ayrışım teoremine göre herhangi bir DRA'yı ergodik DRA'lara bağlı olarak yazabiliriz. Ergodik DRA'lar içinde, Thomas tek bir izomorfizm sınıfına tam ölçü atamayan DRA'lara yayınık DRA ismini verdi. Şimdiye dek çok fazla yayınık DRA örneği bulunamadı. Bu tez yayınık DRA ve benzer özelliklere sahip DRA'lara yönelik bir tarama çalışması olarak yazılmıştır.

Anahtar Kelimeler: değişmez, rassal altgrup, yayınık, grup

To those who support me almost everywhere

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LIST OF ABBREVIATIONS

IRS	Invariant Random Subgroup
p.m.p action	probability measure preserving action

CHAPTER 1

INTRODUCTION

"Resistance to change is proportional to how much the future might be altered by any given act." said Stephen King in his book $11.22.63$. It is well known to mathematicians that a small change can alter many things. This is why throughout history the study of change has been an attraction to them. Upon studying change, it is also convenient to study what remains unaltered with change, in other words what remains invariant.

In this thesis our 'given act' will be the conjugation action of countable groups on their Chabauty space of subgroups, and the invariant to study will be conjugation invariant Borel probability measures on this space, this is what we call an Invariant Random Subgroup.

The study Invariant Random Subgroups, abbreviated IRSs, is a recently emerged field in the intersection of measurable group theory and descriptive set theory.

IRSs can be seen as a probabilistic generalization of normal subgroups since for every normal subgroup N of a countable group Γ, the Dirac measure δ_N concentrated on N is an IRS. Similarly, for every lattice H of Γ, i.e every discrete subgroup $H \leq \Gamma$ which admits a Γ-invariant probability measure μ on Γ/H, we can push the measure μ to Sub_Γ obtain an IRS. So IRSs can be treated as a generalization of lattices as well.

In literature, the notion of an invariant random subgroup was first defined by Abert, Glasner, Virag [1], they defined an invariant random subgroup to be a random subgroup of Γ, i.e. a measurable map $H : X \to Sub_\Gamma$ where X is a probability space, whose probability distribution $\mu : Sub_\Gamma \to [0, 1]$ is a conjugation invariant Borel probability measure on Sub_Γ. Later this definition was abandoned and now only the

distribution μ is called an IRS instead. They proved that a probability measure preserving action on a Borel space naturally give rise to an IRS and moreover every IRS can be realized as the stabilizer distribution of a Borel probability measure preserving action.

Later work on the subject was classifying the IRSs of groups. Bowen, Grigorchuk and others studied many groups to show what their invariant random subgroups looked like. For example IRSs of the free group [5], lamplighter groups [6], groups acting on hyperbolic spaces [13] and groups acting on rooted trees [3] were examined.

Ergodic IRSs gained importance after Creutz and Peterson [8] showed that ergodic p.m.p actions give rise to ergodic IRSs. Until recently it was believed that every ergodic IRS had the property that the measure focuses on a fixed isomorphism type of subgroups. Thomas [21] showed that this need not be the case and introduced the concept of diffuse IRSs.

A diffuse IRS ν of a countable group Γ is an ergodic IRS of Γ such that for any subgroup $H \leq \Gamma$, the set of subgroups isomorphic to H has 0 measure with respect to ν. Thomas constructed a diffuse IRS in his paper using tools from the theory of linear groups and left the question whether there are more 'natural' examples of diffuse IRSs open.

Raimbault [17], in search of a natural example of a diffuse IRS, used Fuchsian groups, that is, the discrete groups of even isometries, to provide an ergodic IRS with a possibly weaker property than diffuseness which we call the $\bigstar$ property, we say that an ergodic IRS ν has this property if there are uncountably many non-isomorphic subgroups in the support of ν. Raimbault used reflection subgroups in the hyperbolic plane generated by the triangle T_4 with all three angles equal to $\pi/4$ to construct an ergodic IRS with the $\bigstar$ property.

This thesis is intended to be a survey of the work on diffuse IRSs and related concepts. In Chapter 2, first we will give the necessary background on invariant and ergodic measures, hyperbolic polygons and the space of subgroups. Then we will introduce IRSs, give some basic examples of IRSs. Then we will show that every p.m.p action gives rise to an IRS and every IRS is the stabilizer distribution of a p.m.p action.

In Chapter 3 we will introduce ergodic and diffuse IRSs. The remaining of the chapter will be devoted to construct a diffuse IRS following Thomas' approach. We will find ourselves in the realm of linear groups and see how they give rise to a diffuse IRS.

Chapter 4 will follow Raimbault's approach where Fuchsian groups will come into play, we will construct hyperbolic polygons tiled by the triangle T_4 and use IRS supported on reflection subgroups associated with these polygons to obtain an ergodic IRS with the $\bigstar$ property.

CHAPTER 2

PRELIMINARIES

In this thesis we will investigate invariant random subgroups. The word 'subgroup' in the name might suggest that they are algebraic objects, however this is not entirely true. An invariant random subgroup will mean a probability measure satisfying certain conditions on the space of subgroups of a countable group.

In this section we provide the necessary background on invariant and ergodic measures, hyperbolic polygons and the space of subgroups of countable groups. Some preliminary results which will be useful later will be provided in this section, then we will define what an invariant random subgroup is. We will finish this section by giving some basic examples of invariant random subgroups and proving basic facts about them.

2.1 Invariant and Ergodic Measures

We begin by defining invariant and ergodic measures. Invariant measures are important for our work as invariant random subgroups are themselves invariant measures. The importance of ergodic measures is that due to the Ergodic Decomposition Theorem, we can write invariant measures as a convex combination of ergodic measures. Also in the study of invariant random subgroups ergodic ones are studied quite often.

Definition 1. *Let $(X, \mathbb{B}, \mu)$ be a probability space and $T : X \to X$ be a measurable function. T is called **measure preserving** if for all $A \in \mathbb{B}$, $\mu(T^{-1}(A)) = \mu(A)$. If this is the case, μ is called **invariant** under T (or μ is an invariant measure for T) and $(X, \mathbb{B}, \mu, T)$ is called a **measure preserving dynamical system**.*

We will be working with groups. So for a given group Γ, we will be interested in Γ-invariant measures and in particular conjugation invariant measures.

Definition 2. *Let Γ be a group acting on a measurable space X. Then a measure μ on X is called Γ-invariant if for every measurable subset $A \subseteq X$, $\mu(A) = \mu(g^{-1} \cdot A)$ for every $g \in \Gamma$. If the action of Γ is conjugation action, i.e, $g \cdot A = gAg^{-1}$, μ is called conjugation invariant.*

Now we will define a type of measure which will be in the center of our discussions, namely ergodic measures.

Definition 3. *Let $(X, \mathbb{B}, \mu, T)$ be a measure preserving dynamical system. μ is called an **ergodic measure** (for T) if for all $A \in \mathbb{B}$, $T^{-1}(A) = A$ implies $\mu(A) \in \{0, 1\}$.*

Ergodic measures have importance in the space of invariant measures because provided that certain conditions are satisfied, an invariant measure can be written as a convex combination of ergodic measures as we shall soon see.

Let X be a compact metric space and $T : X \to X$ be a continuous map. We denote the set of T-invariant Borel probability measures on X by $\mathcal{M}^T(X)$.

Theorem 1 (Krylov-Bogolyubov)**.** *[12] Let X be a compact metric space, and let $T : X \to X$ be a continuous map. Then there exists at least one T-invariant Borel probability measure μ on X, such that:*

$$\mu(T^{-1}(A)) = \mu(A) \quad \textit{for all Borel sets } A \subseteq X.$$

This theorem secures that there is at least one T-invariant probability measure on X, thus $\mathcal{M}^T(X)$ is non-empty. We can say more about $\mathcal{M}^T(X)$, but we need a couple theorems before that.

Theorem 2 (Banach-Alaoglu)**.** *[2] If X is a normed space then the closed unit ball in the continuous dual space X' (endowed with its usual operator norm) is compact with respect to the weak-* topology.*

Theorem 3 (Prokhorov's Theorem)**.** *[15] Let $\{\mu_n\}$ be a sequence of probability measures on a Polish space S (i.e., a complete separable metric space). Then the following are equivalent:*

1. *The sequence $\{\mu_n\}$ is tight, meaning that for every $\epsilon > 0$, there exists a compact set $K \subset S$ such that*

$$\mu_n(K) \geq 1 - \epsilon \quad \textit{for all } n.$$

2. *The sequence $\{\mu_n\}$ is relatively compact in the space of probability measures equipped with the weak topology, meaning that every subsequence has a further subsequence that converges weakly to some probability measure μ.*

By Banach–Alaoglu Theorem[2] and Prokhorov's Theorem[15], $\mathcal{M}^T(X)$ is a closed compact metric space.

Recall that a point $\mu \in \mathcal{M}^T(X)$ is an **extreme point** if for all $\eta_1, \eta_2 \in \mathcal{M}^T(X)$ and all $t \in [0,1]$ with $\mu = t.\eta_1 + (1-t).\eta_2$, we have $\mu = \eta_1$ or $\mu = \eta_2$.

Theorem 4. *[10, Theorem 4.4] A measure $\mu \in \mathcal{M}^T(X)$ is an extreme point of $\mathcal{M}^T(X)$ if and only if it is an ergodic measure.*

Proof. Let μ be a T-invariant probability measure that is not ergodic. Then there exists a measurable set A such that $T^{-1}(A) = A$ and $\mu(A) \in (0,1)$. Let μ_1 and μ_2 be defined as $\mu_1(B) = \frac{\mu(B \cap A)}{\mu(A)}$ and $\mu_2(B) = \frac{\mu(B \setminus A)}{1-\mu(A)}$. So $\mu_1 \neq \mu$ and

$$\mu = \mu(A)\mu_1 + (1 - \mu(A))\mu_2$$

moreover both μ_1, μ_2 are T-invariant probability measures. So μ can be expressed as a strict convex combination of μ_1 and μ_2. So it is not an extreme point. Thus, if an invariant probability measure is an extreme point then it has to be ergodic.

Conversely, let μ be an ergodic measure. Suppose that μ can be written as a strict convex combination of the invariant measure ν_1 and ν_2 as $\mu = s\nu_1 + (1-s)\nu_2$. By the strictness of this combination, $s > 0$, so $\nu_1 \ll \mu$, i.e. ν_1 is absolutely continuous with respect to μ. Thus, there exists some positive function f with finite integral so that $\nu_1(A) = \int_A f d\mu$ for any measurable A, namely, f is the Radon-Nikodym derivative.

Consider the set $B = \{x \in X | f(x) < 1\}$. Since f is a measurable function, B is measurable and so:

$$\begin{aligned}\int_{B\cap T^{-1}(B)} f d\mu + \int_{B\setminus T^{-1}(B)} f d\mu &= \nu_1(B) \\ &= \nu_1(T{-}1(B)) \\ &= \int_{B\cap T^{-1}(B)} f d\mu + \int_{T^{-1}(B)\setminus B} f d\mu\end{aligned}$$

which yields that

$$\int_{B\setminus T^{-1}(B)} f d\mu = \int_{T^{-1}(B)\setminus B} f d\mu.$$

By definition of f, $f(x) < 1$ when $x \in B \setminus T^{-1}(B)$ and $f(x) \geq 1$ when $x \in T^{-1}(B) \setminus B$. Also we have,

$$\begin{aligned}\mu(T^{-1}(B) \setminus B) &= \mu(T^{-1}(B)) - \mu(T^{-1}(B) \cap B) \\ &= \mu(B) - \mu(T^{-1}(B) \cap B) \\ &= \mu(B \setminus T^{-1}(B)).\end{aligned}$$

It follows that,

$$\mu(B \setminus T^{-1}(B)) = 0 = \mu(T^{-1}(B) \setminus B).$$

So we have:

$$\mu((T^{-1}(B))\Delta B) = 0.$$

Since μ is ergodic we must have $\mu(B) \in \{0, 1\}$. Assume that $\mu(B) = 1$, then,

$$\begin{aligned}\nu_1(X) &= \int_X f d\mu \\ &< \mu(B) \\ &= 1\end{aligned}$$

Which is a contradiction. Thus $\mu(B) = 0$.

One can similarly prove $\mu(\{x \in X \mid f(x) > 1\}) = 0$, which means that $\mu(\{x \in X \mid f(x) = 1\}) = 1$. This concludes that $\nu_1 = \mu$ and hence μ is an extreme point of $\mathcal{M}^T(X)$. □

Let $\mathcal{E}^T(X)$ denote the set of extreme points of $\mathcal{M}^T(X)$. By the last proposition it is exactly the set of ergodic measures. Now we will state a theorem that tells us how we can use ergodic measures.

Theorem 5 (Ergodic Decomposition Theorem)**.** *[10, Theorem 4.8] Let X be a compact metric space and $T : X \to X$ be a continuous map. Then for any $\mu \in \mathcal{M}^T(X)$ there is a unique probability measure λ defined on the Borel subsets of the compact metric space $\mathcal{M}^T(X)$ with the properties that:*

(1) $\lambda(\mathcal{E}^T(X)) = 1$ and

(2) $\displaystyle\int_X f d\mu = \int_{\mathcal{E}^T(X)} \left(\int_X f d\nu \right) d\lambda(\nu)$ *for any continuous* $f : X \to \mathbb{R}$.

We will not give a proof for this theorem here. The proof uses Choquet's theorem and the fact that ergodic measures are the extreme points of $\mathcal{M}^T(X)$.

The importance of the Ergodic Decomposition Theorem for us is that when we define IRSs this theorem will enable us to write an IRS as a convex combination of ergodic ones, where the "convex combination" is to be interpreted as possibly an infinite sum given by the relevant integral.

2.2 Hyperbolic Polygons

Many natural examples of groups arise from geometry. Reflections and symmetries of polytopes provide infinite sources for discovering new groups. In this thesis we will make use of some reflection subgroups of the group of isometries in the hyperbolic plane $\mathbb{H}^2$.

There are several models of hyperbolic geometry, in this thesis we will use the Poincaré disk model, which is the disk $\{z = x + yi : |z| < 1\}$ with the Poincaré

metric given by the formula

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$

We will not prove that this formula gives a metric.

We will make extensive use of hyperbolic polygons in our proofs, so first we will define what we mean by a hyperbolic polygon. In the Poincaré disk model, a **hyperbolic line** is either a circular arc contained within the disk which is orthogonal to the unit circle or a diameter of the unit circle. To define a hyperbolic polygon we need to define the hyperbolic half-planes first.

Definition 4. *Given a hyperbolic line* l *in* $\mathbb{H}^2$*, the complement of* l *has two components called the two* ***open half-planes*** *determined by* l*. A* ***closed half-plane*** *determined by* l *is the union of* l *with one of the open half-planes determined by* l*.*

Although the hyperbolic geometry shares mostly the same foundations with our classical Euclidean geometry, the parallel postulate in Euclidean axioms is replaced with the statement that for any given line l and point p not on l, in the plane containing both l and p there are at least two distinct lines through p that do not intersect l.

This change in the axiomatic system is the reason why our hyperbolic lines look different than what we are used to. So many geometric concepts also seem different to the eye although their definitions don't change much.

Definition 5. *A subset* $S \subseteq \mathbb{H}^2$ *is called* ***convex*** *if for any two points* $p, q \in S$ *the line segment joining* p *and* q *lie entirely in* S*.*

So some of the shapes in the hyperbolic plane that do not seem to be convex at first might actually turn out to be convex.

Definition 6. *A* ***hyperbolic polygon*** *is a closed convex set in* $\mathbb{H}^2$ *that can be expressed as the intersection of a locally finite collection (that is, each point in the space has a neighbourhood that intersects only finitely many of the sets in the collection) of closed half-planes.*

Due to the idiosyncratic nature of hyperbolic polygons, the groups generated by the reflections in their faces show some interesting features as well, in later sections we will prove a bunch of them.

In Chapter 4 of the thesis we will benefit from a special type of hyperbolic polygons, namely Coxeter polygons.

Definition 7. *[17] A hyperbolic polygon is called* ***Coxeter*** *if its internal angles are each of the form π/m for some natural numbers $m \geq 2$.*

In Chapter 4 we will examine reflection subgroups generated by Coxeter polygons and we will construct a group with an invariant random subgroup with a special property using those groups. Thus Coxeter polygons will be quite essential for our work. We will continue with defining angle sequences, another tool that will be relevant later.

Definition 8. *[17] Let P be a convex polygon in the hyperbolic plane and set a vertex $x_0 \in P$ to be its root. Then the* ***angle sequence*** *of P is the sequence $[0, \pi)^{\mathbb{Z}}$ obtained by listing the angles at its vertices in the counter-clockwise direction starting from the angle at x_0.*

Now we will define a topology that will function as the basis of our work throughout the thesis, namely the Chabauty topology.

Definition 9. *[4] Let X be a proper metric space (i.e a metric space in which every closed ball is compact) and let $C(M)$ be the space of closed subsets of X. The Chabauty topology on $C(M)$ is generated by the subsets*

$$\{C \in C(M) \mid C \cap K = \emptyset\}, \{C \in C(M) \mid C \cap U \neq \emptyset\}$$

where $K \subseteq X$ is compact and $U \subseteq X$ is open.

Without proving we remark that the hyperbolic plane with the Poincaré metric above is a proper metric space, so the Chabauty topology can be defined for $\mathbb{H}^2$.

The Chabauty topology is actually metrizable, we will give a metric to it. To do this we should first define the Hausdorff distance on M.

Definition 10. *Let C_1 and C_2 be closed subsets of M. Then the Hausdorff distance d_{haus} is defined as $d_{Haus}(C_1, C_2) = \inf\{\epsilon \mid C_1 \subseteq B(C_2, \epsilon)$ and $C_2 \subseteq B(C_1, \epsilon)\}$.*

Lemma 1. *[4] Fix a point $p \in M$. If $A \subseteq M$ is closed and $R > 0$, set $A_R = A \cap \overline{B(p,R)}$. Now let $d_R(A,B) = \min\{1, d_{Haus}(A_R, B_R)\}$. The Chabauty topology on $C(M)$ is induced by the metric $d(A,B) = \int_0^\infty e^{-R} d_R(A,B) d(R)$.*

Now we will prove a lemma about the angle sequences of hyperbolic polygons, our aim is to show that if a sequence (P_n) of hyperbolic polygons converge to another polygon P, then the angle sequences of (P_n) also converge to the angle sequence of P. This result will be useful for our work later.

Lemma 2. *If P is a Chabauty limit of a sequence (P_n) of polygons such that the edge lengths of the P_n are bounded then the angle sequence of P is the limit of the angle sequences of the P_n.*

Proof. Let P_n be a convergent Chabauty sequence of polygons with bounded edge lengths and let P be their limit. Consider the angle sequences (α_n) of P_n. We will show that (α_n) is convergent and their limit (α) is the angle sequence of P.
Assume towards a contradiction that (α_n) is not convergent. Fix $p_0 \in P$ and let p_n be a sequence of vertices, where each $p_n \in P_n$, converging to p_0. If the sequence of the angles at p_n is not convergent, then look at the adjacent vertices of p_n. For each $n \in \mathbb{N}$ denote the vertices with one less and one more index than p_n by p_n^- and p_n^+ respectively. p_n^+ (or p_n^-) do not make a convergent sequence as the angles at p_n do not converge.
We get a contradiction as P_n is a convergent sequence. Since p_0 was arbitrary we conclude that for each sequence $\{p_n\}$ with $p_n \in P_n$ and $lim_n p_n \in P$, the angles at p_n make a convergent sequence.
Now we will show that if a sequence of vertices $\{p_n\}$ converge to p_0, than the sequence of their angles converge to that of p_0. Again fix a vertex p_0 and let $\{p_n\}$ converge to p_0. So $\{p_n^+\}$ converges to p_0^+ since P_n converges to P. If the limit of the sequence of angles at p_n converge to some other value that the angle at p_0, there are two different angles between p_0 and p_0^+, the angle at p_0 and the limit of the angles of p_n, which is a contradiction.
Our fixed vertices were arbitrary, so at each vertex the angle at that vertex is the same as limit of the angles of vertices converging to that vertex. Therefore, the angle sequence of P is the limit of angle sequences of P_n. □

2.3 Space of Subgroups and Invariant Random Subgroups

Our study of invariant random subgroups will be based on the study of the space of subgroups of countable groups. We will first start with a more general setting, we define the space of subgroups and then invariant random subgroups for locally compact groups and construct our first examples in this setting.

Later we will restrict ourselves to work with countable discrete groups as they will be sufficient for our concern, diffuse IRSs will be defined for these types of groups.

Definition 11. *A group Γ is called a **locally compact group** if Γ is equipped with a topology which is locally compact and Hausdorff.*

For a locally compact group Γ we denote the space of closed subgroups of Γ by Sub_Γ.

We can examine some basic examples of familiar additive groups.

Example 2.1. *Consider the additive group $\mathbb{Z}$, which is locally compact. Since $\mathbb{Z}$ has the discrete topology all of its subgroups are closed. The subgroups of $\mathbb{Z}$ has the form $n\mathbb{Z}$ for $n \in \mathbb{N}$. Thus, $Sub_\mathbb{Z} = \{n\mathbb{Z} \mid n \in \mathbb{N}\}$.*

Example 2.2. *Now consider the additive group $\mathbb{R}$ which is also locally compact. The closed subgroups of $\mathbb{R}$ are either itself or subgroups of the form $\alpha\mathbb{Z}$ for $\alpha \in \mathbb{R}$. Thus, $Sub_\mathbb{R} = \{\mathbb{R}\} \cup \{\alpha\mathbb{Z} \mid \alpha \in \mathbb{R}\}$.*

Note that Γ acts on Sub_Γ with the conjugation action, that is, for any $g \in \Gamma$ and $H \in Sub_\Gamma$ we have $g \cdot H = gHg^{-1}$.

Now we will define the Chabauty topology with minor changes again, this time for the space of subgroups and so we will have a topological space of subgroups to work with. Then using the open sets of this topology, we will define the Borel σ-algebra on this space and this will give us a measurable space.

Invariant random subgroups will be defined as the probability measures on this measurable space that is invariant under the action of conjugation.

We define the Chabauty topology for the space of subgroups of Γ as follows:

Definition 12. *[7] Consider the following two types of sets:*

- $O_1(K) = \{H \in Sub_\Gamma \mid H \cap K = \emptyset\}$ *where* $K \subseteq \Gamma$ *is compact*
- $O_2(U) = \{H \in Sub_\Gamma \mid H \cap U \neq \emptyset\}$ *where* $U \subseteq \Gamma$ *is open.*

The Chabauty topology on Sub_Γ is defined as the topology having the two types of sets above as the sub-bases.

If Γ is a discrete group, the Chabauty topology on Γ is the same as the subspace topology inherited from the product topology on $\{0,1\}^\Gamma$. In general Sub_Γ is a compact space. We will show this result for countable discrete Γ.

Note that for a discrete group Γ the space Sub_Γ of closed subgroups of Γ is the same as the space of all subgroups of Γ as in the discrete topology any subset is closed. So for discrete Γ instead of defining Sub_Γ as the space of closed subgroups, we can define it as the space of subgroups of Γ.

Proposition 2.1. *[22] Let* Γ *be a discrete group. Then,* Sub_Γ *is compact.*

Proof. Firstly, $\mathcal{P}(\Gamma) = \{0,1\}^\Gamma$ is a compact space as $\{0,1\}$ is a compact space and the product of compact spaces is compact by Tychonoff's Theorem. It is enough to show that Sub_Γ is closed in $\mathcal{P}(\Gamma)$ as closed subspaces of compact spaces are compact.

Sub_Γ is a closed subset of $\mathcal{P}(\Gamma)$ if and only if any convergent sequence of subgroups converge to another subgroup of Γ. Let $\{H_n\}_{n\in\omega}$ be a convergent sequence of subgroups. Let $H = lim_n H_n$. Clearly the identity element of Γ, say e, is in H as $e \in H_n$ for all n.

Now let $g, h \in H$, so there exist $M, N \in \omega$ so that $g \in H_m$ for all $m \geq M$ and $h \in H_n$ for all $n \geq N$. Let $K = max\{M, N\}$. Then for all $k \geq K$ we have both $g \in H_k$ and $h \in H_k$. Since all H_n are subgroups, it means that $g * h \in H_k$ for all $k \geq K$. Thus $g * h \in H$.

Now let $\gamma \in H$. So there exists $\Lambda \in \omega$ so that $\gamma \in H_\lambda$ for all $\lambda \geq \Lambda$. By the same argument, $\gamma^{-1} \in H_\lambda$ as well, so $\gamma^{-1} \in H$. This concludes that $H \in Sub_\Gamma$.

Thus, when we take a convergent sequence of subgroups in Sub_Γ, the limit of the sequence is also a subgroup of Γ. Hence, Sub_Γ is a closed subspace of the compact space $\mathcal{P}(\Gamma)$, making it also compact. □

Another way to define the Chabauty topology on discrete groups is by defining the convergent sequences of this topology, a sequence (H_n) of subgroups of Γ converge if and only if for each element g of the group Γ, the 01-sequence (a_n) obtained by letting $a_n = 1$ if $g \in H_n$ and $a_n = 0$ if $g \notin H_n$ eventually stabilizes.

Now that there is a topology defined on Sub_Γ, one can study the Borel structure of Sub_Γ arising from the open sets of this topology. We equip the space Sub_Γ with the Borel σ-algebra arising from the Chabauty topology to obtain a measurable space.

Our main interest in this thesis is to study invariant random subgroups, a special type of Borel probability measures on Sub_Γ. Note that Γ acts on Sub_Γ by conjugation, that is, $g \cdot H = H^g = gHg^{-1}$.

Definition 13. *Let Γ be a countable group. An Invariant Random Subgroup of Γ is a conjugation invariant Borel probability measure on Sub_Γ.*

So in contrast to what the name suggests, an invariant random subgroup is actually a probability measure. In the historical context the probability distribution of a random subgroup was called an invariant random subgroup but later this was dropped.

From now on we will abbreviate invariant random subgroup as **IRS**.

First examples of IRSs arise from normal subgroups and subgroups having finite conjugacy classes. For that reason IRSs can be seen as a generalization of normal subgroups.

Example 2.3. *Let Γ be a countable group and let $N \trianglelefteq \Gamma$ be normal subgroup of Γ. Then the Dirac measure δ_N concentrated on N defined by:*

$$\delta_N(A) = \begin{cases} 1, & N \in A \\ 0, & N \notin A \end{cases}$$

is an IRS of Γ.

Proof. Let $g \in \Gamma$. Since $N \trianglelefteq \Gamma$ is a normal subgroup of Γ we have $g^{-1}Ng = N$ and so for any measurable $A \subseteq Sub_\Gamma$ $N \in A$ if and only if $g^{-1}Ng \in A$. It follows that $N \in gAg^{-1}$ only when $N \in A$ and so $\delta_N(A) = \delta_N(gAg^{-1})$ for all $g \in \Gamma$. Hence δ_N is an IRS. □

$\delta_{\mathbf{1}}$ and $\delta_{\mathbf{\Gamma}}$ are called the trivial IRSs of Γ. For any $N \leq \Gamma$, δ_N are also examples of what we call ergodic IRSs, i.e. an IRS that is also an ergodic measure. Ergodic IRSs will be important for our discussions later.

Example 2.4. *Suppose that $H \leq \Gamma$ has finite conjugacy class, say $|\{gHg^{-1} : g \in \Gamma\}| = n$ for some $n < \omega$. Then $\nu = \frac{1}{n}\sum_{g\in\Gamma} \delta_{H^g}$ is an IRS of Γ.*

Proof. Assume that H has finite conjugacy class, consisting of $H^{g_1}, H^{g_2}, \ldots, H^{g_n}$. Let $A \subseteq Sub_\Gamma$ be measurable and assume that only for $g_{k_1}, g_{k_2}, ..., g_{k_m}$ we have $H^{g_{k_i}} \in A$. Note that for any $h \in \Gamma$ we have $H^g \in A$ if and only if $hH^gh^{-1} \in hAh^{-1}$. So we have

$$\begin{aligned}
\nu(A) &= \frac{1}{n}\sum_{g\in\Gamma} \delta_{H^g}(A) \\
&= \frac{1}{n}\sum_{gh\in\Gamma} \delta_{hH^gh^{-1}}(hAh^{-1}) \\
&= \frac{1}{n}\sum_{g\in\Gamma} \delta_{H^g}(hAh^{-1}) \\
&= \nu(hAh^{-1})
\end{aligned}$$

The second to last line follows from the above one because taking the sum on $gh \in \Gamma$ and $g \in \Gamma$ will both give all the subgroups in the conjugacy class of H. Hence this measure is conjugation invariant. Finally, since for any H^{g_i} in the conjugacy class of H we have $H^{g_i} \in Sub_\Gamma$, it follows that

$$\nu(Sub_\Gamma) = \frac{1}{n}\sum_{h\in\Gamma} \delta_{H^g}(Sub_\Gamma) = \frac{1}{n}.n = 1$$

Thus ν is also a probability measure, hence it is an IRS. □

Definition 14. *Let Γ be a locally finite group. A discrete subgroup $G \leq \Gamma$ is called a* ***lattice*** *if Γ/G admits a Γ-invariant probability measure.*

Example 2.5. *Every lattice gives rise to an IRS.*

Proof. Let $H \leq \Gamma$ be a lattice. Then Γ/H admits a Γ-invariant probability measure, say μ. Consider $f : \Gamma/H \to Sub_\Gamma$, $f(gH) = gHg^{-1}$. Then the push forward of μ to Sub_Γ is an IRS. □

So IRSs can be seen as a generalization of lattices as well. In fact, recently people have been studying IRSs as generalizations of lattices.

Now we have a general understanding of the space of subgroups and invariant random subgroups, we can restrict ourselves to countable groups, since diffuse IRSs will be defined for them. In the rest of the thesis we will work with countable groups.

Invariant random subgroups caught attention when Abert, Glasner and Virag [1] proved that they are in close relationship with probability measure preserving actions. Next we will see the relationship between them.

Definition 15. *Let Γ be a countable group. A* ***probability measure preserving action*** *(p.m.p action for short) $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ by Γ on a probability space $(X, \mathcal{A}, \mu)$ is a map $\cdot : \Gamma \times X \to X$ with the following properties:*

1. *For each $\gamma \in \Gamma$, $x \mapsto \gamma \cdot x$ preserves μ.*
2. *For each $x \in X$, $e \cdot x = x$.*
3. *For each $x \in X$ and $\gamma, \eta \in \Gamma$ we have $\gamma \cdot (\eta \cdot x) = (\gamma\eta) \cdot x$.*

Now we will give another important definition, namely the stabilizer distribution, that is, the probability distribution of the stabilizer map. Then we will see that stabilizer distributions give rise to IRSs.

Definition 16. *Let Γ be a group, and let $(Z, \mathcal{A}, \mu)$ be a Borel probability space and $\Gamma \curvearrowright (Z, \mathcal{A}, \mu)$ be a measure-preserving action. Let $\Gamma_z = \{g \in \Gamma \mid g.z = z\}$ be the stabilizer of z in Γ and $f : Z \to Sub_\Gamma$ be the stabilizer map, that is, $f(z) = \Gamma_z$. Then*

*the **stabilizer distribution** of the action $\Gamma \curvearrowright (Z, \mathcal{A}, \mu)$ is the map $f_*\mu : Sub_\Gamma \to [0,1]$ defined as $f_*\mu(X) = \mu(f^{-1}(X))$.*

IRSs are in correspondence with p.m.p actions on Borel probability spaces. The stabilizer distribution of a probability measure preserving action on a Borel space gives rise to an IRS and every IRS actually arise this way.

Proposition 2.2. *[1] Let $\Gamma \curvearrowright (Z, \mathcal{A}, \mu)$ be a p.m.p action. Let $f : Z \to Sub_\Gamma$ be the stabilizer map. Then, $\nu = f_*\mu$ is an IRS of Γ.*

Proof. Assume that $A \subseteq Sub_\Gamma$ is a Borel subset and $\gamma \in \Gamma$. Then,

$$\begin{aligned} x \in f^{-1}(A) &\Leftrightarrow \Gamma_x \in A \\ &\Leftrightarrow \Gamma_{\gamma \cdot x} = \gamma \Gamma_x \gamma^{-1} \in \gamma A \gamma^{-1} \\ &\Leftrightarrow \gamma \cdot x \in f^{-1}(\gamma A \gamma^{-1}) \end{aligned}$$

Since the action of Γ on Z is measure-preserving, it follows that

$$\begin{aligned} \nu(A) &= f_*\mu(A) \\ &= \mu(f^{-1}(A)) \\ &= \mu(f^{-1}(\gamma A \gamma^{-1})) \\ &= f_*\mu(\gamma A \gamma^{-1}) \\ &= \nu(\gamma A \gamma^{-1}) \end{aligned}$$

Hence, ν is an IRS of Γ. □

So every p.m.p action gives rise to an IRS. Now we'll show that every IRS actually arises this way. This result was first shown by Abert, Glasner and Virag [1], the proof we provide is based on seminar notes of Simon Thomas, provided to us by Burak Kaya.

Theorem 6. *[1] If ν is an IRS of Γ, then there is a probability space (X, μ) and a measure-preserving action $\Gamma \curvearrowright (X, \mu)$ such that ν is the stabilizer distribution of this action.*

Proof. For each $H \in Sub_\Gamma$, let $\Gamma/H = \{kH \mid k \in \Gamma\}$. Then we define

$$\begin{aligned} X &= \bigsqcup_{H \in Sub_\Gamma} [0,1]^{\Gamma/H} \\ &= \{(H, f) \in Sub_\Gamma \times [0,1]^\Gamma \mid \forall k \in \Gamma, \forall h \in H, f(kh) = f(k)\}. \end{aligned}$$

The product metric on X arising from the usual metric on $[0,1]$ makes X a complete and separable metric space and the metric defines the Borel σ-algebra on X. Thus X is a standard Borel space.

Now we define an action of Γ on $\bigsqcup_{H \in Sub_\Gamma} \Gamma/H$ by:

$$g \cdot kH = kHg^{-1} = kg^{-1}(gHg^{-1})$$

So g takes a left coset of H and takes it to a left coset of gHg^{-1}. We shall prove that this formula defines a group action.

Let e denote the identity of Γ.

1. $e \cdot kH = kHe^{-1} = kH$
2. $g \cdot (h \cdot kH) = g \cdot (kHh^{-1}) = kHh^{-1}g^{-1} = (gh) \cdot kH$ for all $g, h \in \Gamma$.

So it indeed defines an action.

This action on $\bigsqcup_{H \in Sub_\Gamma} \Gamma/H$ in return induces an action on X by:

$$\begin{aligned} f \in [0,1]^{\Gamma/H} &\xrightarrow{g} g \bullet f \in [0,1]^{\Gamma/g^{-1}Hg} \\ (g \bullet f)(kg^{-1}Hg) &= f(g \cdot kg^{-1}Hg) \\ &= f(kg^{-1}Hgg^{-1}) \\ &= f(kg^{-1}H) \end{aligned}$$

So g takes a function defined on the left cosets of H and returns a function defined on the left cosets of $g^{-1}Hg$. We also need to prove this new formula defines a group action as well.

1. $(e \bullet f)(kH) = f(e \cdot kH) = f(kH)$

2. Let $g, h \in \Gamma$.

$$\begin{aligned}(g \bullet (h \bullet f))(kh^{-1}g^{-1}Hgh) &= g \bullet f(h \cdot kh^{-1}g^{-1}Hgh) \\ &= g \bullet f(kh^{-1}g^{-1}Hg) \\ &= f(g \cdot kh^{-1}g^{-1}Hg) \\ &= f(kh^{-1}g^{-1}H) \\ &= f(k(h^{-1}g^{-1}Hgh)h^{-1}g^{-1}) \\ &= f(gh \cdot kh^{-1}g^{-1}Hgh) \\ &= (gh) \bullet f(kh^{-1}g^{-1}Hgh)\end{aligned}$$

So this is indeed an action on X.

Note that if $f \in [0,1]^{\Gamma/H}$ and $h \in H$ then,

$$\begin{aligned}(h \bullet f)(kH) &= f(h \cdot kH) \\ &= f(kHh^{-1}) \\ &= f(kH)\end{aligned}$$

So we have $h \bullet f = f$.

Next suppose that $f : \Gamma/H \to [0,1]$ is injective and that $g \in \Gamma$ satisfies $g \bullet f = f$. Then clearly $g \in N_\Gamma(H)$. Also,

$$\begin{aligned}f(H) &= (g \bullet f)(H) \\ &= f(g \cdot H) \\ &= f(Hg^{-1})\end{aligned}$$

Since f is injective this means that $g \in H$. Thus $\Gamma_f = H$.

For each $H \in Sub_\Gamma$ let λ_H be the usual product measure on $[0,1]^{\Gamma/H}$. Given an IRS ν of Γ we define a Borel probability measure μ on X by:

$$\mu(Z) = \int_{Sub_\Gamma} \lambda_H(Z \cap [0,1]^{\Gamma/H}) d\nu(H)$$

Note that for $H \in Sub_\Gamma$, $\lambda_H(\{f \in [0,1]^{\Gamma/H} \mid f \text{ is } 1-1\}) = 1$.

Thus $\mu(\{f \in X \mid f \text{ is } 1-1\}) = 1$. Hence if $Z \subseteq Sub_\Gamma$ is a Borel subset and s is the stabilizer function, then

$$s^{-1}(Z) \cap \{f \in X \mid f \text{ is } 1-1\} = \{f \in X \mid \exists H \in Z, f \in [0,1]^{\Gamma/H} \text{ and } f \text{ is } 1-1\}$$

and so $\mu(s^{-1}(Z)) = \nu(Z)$. Thus, $\nu = s_*\mu$. □

This proves that every IRS is the stabilizer distribution of a probability measure preserving action.

CHAPTER 3

DIFFUSE INVARIANT RANDOM SUBGROUPS AND A CONSTRUCTION

3.1 Ergodic Invariant Random Subgroups

In the study of IRSs, ergodic IRSs have a central part as invariant measures can be written as a convex combination of ergodic ones. An IRS ν is ergodic when ν is an ergodic measure.

Example 3.1. *For any countable group Γ and a normal subgroup $N \trianglelefteq \Gamma$, δ_N is an ergodic IRS.*

Proof. We already showed that δ_N are IRSs for any normal subgroup $N \trianglelefteq \Gamma$. Since any measurable subset of Sub_Γ has measure either 0 or 1, δ_N is automatically ergodic. Hence it is an ergodic IRS. □

Example 3.2. *If $H \leq \Gamma$ has finite conjugacy class, then $\nu = \frac{1}{n}\sum_{g\in\Gamma} \delta_{H^g}$ is an ergodic IRS of Γ.*

Proof. By definition, conjugacy classes are invariant under the action of conjugation and any measurable subset containing the conjugacy class of H has full measure while the rest have zero measure. Thus ν is an ergodic IRS. □

Now that we know it is not empty, let $\mathcal{E}(\Gamma)$ denote the set of ergodic IRSs of Γ.

How many ergodic IRSs does a given group have is an interesting question on its own. Likewise, whether they can be classified or not is another important question. Clearly for any countable group Γ, δ_1 and δ_Γ are ergodic IRSs. If Γ has no other ergodic IRSs, it is called **strongly simple**.

Note that if Γ had some nontrivial normal subgroup $N \trianglelefteq \Gamma$, δ_N would be an ergodic IRS, so Γ has no nontrivial normal subgroups. Hence strongly simple groups are simple.

An example of strongly simple groups is due to Peterson and Thom.

Theorem 7. *[14] If K is a countably infinite field and let $n \geq 2$. Then, $PSL(n, K)$ (def. 21) is strongly simple.*

Strongly simple groups have special importance in the study of ergodic IRSs and there are open problems on their classification.

We now turn our attention to groups with more ergodic IRSs. This leads us to ergodic actions.

Definition 17. *Let $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ be a p.m.p action on a probability space. $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ is called an **ergodic action** if $(g^{-1} \cdot A = A) \Rightarrow \mu(A) \in \{0, 1\}$ for all $g \in \Gamma$.*

Just as p.m.p actions give rise to IRSs, ergodic actions give rise to ergodic IRSs.

Theorem 8. *[8] If ν is an ergodic IRS of Γ, then ν is the stabilizer distribution of an ergodic action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$.*

For our next result, we should introduce the concept of atoms in measure spaces.

Definition 18. *Let (X, Σ, μ) be a measure space. A subset $A \in \Sigma$ is called an **atom** of X if $\mu(A) > 0$ and for all measurable $B \subseteq A$ either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. If no such A exists μ is called a **non-atomic** measure.*

Another important result on the classification of ergodic IRSs is due to Stuck and Zimmer. They showed that every non-atomic ergodic IRS concentrates on a lattice.

Theorem 9. *[20] Let ν be a non-atomic ergodic IRS of Γ. Then ν is of the form μ_H for some lattice $H \leq \Gamma$.*

An important remark about ergodic IRSs is that if ν is an ergodic IRS of a countable group Γ, then for every group-theoretic property Φ, the set of subgroups satisfying the property Φ is ν-measurable and thus $\nu(\{H \in Sub_\Gamma \mid H \text{ satisfies } \Phi\}) \in \{0, 1\}$.

In particular, if ν is an ergodic IRS of Γ, then $\nu(\{H \in Sub_\Gamma \mid H \cong K\}) \in \{0, 1\}$ for any subgroup $K \leq \Gamma$.

3.1.1 Diffuse Invariant Random Subgroups

In many examples of ergodic IRSs, the measure concentrates on the subgroups of a fixed isomorphism type. For example if Γ is the free group, there is a subgroup $K_\nu \leq \Gamma$ such that $\nu(\{H \leq \Gamma \mid H \cong K_\nu\}) = 1$.

Until recently it wasn't clear whether this must always be the case. Thomas [21] constructed a group where the measure does not concentrate on any isomorphism class of subgroups. Diffuse IRSs are the ergodic IRSs such that the measure does not concentrate on any isomorphism classes.

Definition 19. *[21] Let ν be an ergodic IRS of Γ. Then ν is called **diffuse** if*

$$\nu(\{H \in Sub_\Gamma \mid H \cong K\}) = 0$$

for every subgroup $K \leq \Gamma$.

The word diffuse comes from the fact that no collection of subgroups of the group Γ are accumulated around a subgroup, they are all spreaded within Γ.

3.2 A Semidirect Product with a Diffuse Invariant Random Subgroup

Before we go on to prove the existence of diffuse IRSs, we will dive a little more into group theory and prove some lemmas which will be useful for us. We begin with a definition.

Definition 20. *A group G is called **centerless** if its center $Z(G)$ is the trivial subgroup.*

Next lemma will tell us that given a simple non-abelian group S and two groups of automorphisms of S, if these groups are isomorphic then they can be obtained by each other by conjugation with a single automorphism of S.

Lemma 3. *[21] Let S be a simple non-abelian group and let G, H be groups such that $Inn(S) \leq G, H \leq Aut(S)$. If $\pi : G \to H$ is an isomorphism, then there exists $\phi \in Aut(S)$ so that $\pi(g) = \phi g \phi^{-1}$ for all $g \in G$.*

Proof. We will first show that $Inn(S)$ is the unique minimal non-trivial subgroup of both G and H. For each $s \in S$, let i_s denote the inner automorphism taking x to sxs^{-1}. For any $\psi \in Aut(S)$,

$$\begin{aligned}(\psi i_s \psi^{-1})(x) &= \psi(s\psi^{-1}(x)s^{-1}) \\ &= \psi(s)x\psi(s^{-1}) \\ &= \psi(s)x\psi(s)^{-1} \\ &= i_{\psi(s)}(x)\end{aligned}$$

This shows that $Inn(S) \trianglelefteq Aut(S)$. In particular, say ψ commutes i_s, then we have $i_{\psi(s)} = \psi i_s \psi^{-1} = i_s \psi \psi^{-1} = i_s$, which follows that $\psi = 1$.

Thus, $C_{Aut(S)}(Inn(S)) = 1$.

Now suppose that G is a group such that $Inn(S) \leq G \leq Aut(S)$. Since $Inn(S) \trianglelefteq Aut(S)$, it follows that $Inn(S) \trianglelefteq G$. Let $N \trianglelefteq G$ be a non-trivial normal subgroup of G.

Then both being normal subgroups of G, we have $[Inn(S), N] \leq Inn(S) \cap N$ and since $C_{Aut(S)}(Inn(S)) = 1$, we have $[Inn(S), N] \neq 1$ as if it were equal to 1 we'd have $N \leq C_{Aut(S)}(Inn(S)) = 1$, a contradiction as N is non-trivial.

So $Inn(S) \cap N$ is a non-trivial normal subgroup of $Inn(S)$.

Since S is simple and non-abelian it is centerless, and centerless groups are isomorphic to their inner automorphism groups. So we conclude $Inn(S)$ is simple and thus $Inn(S) \leq N$. Hence $Inn(S)$ is the unique minimal non-trivial normal subgroup of G.

Now let G, H be so that $Inn(S) \leq G, H \leq Aut(S)$. So for any isomorphism $\pi : G \to H$, $\pi(Inn(S)) = Inn(S)$ as $Inn(S)$ is the unique minimal non-trivial normal subgroup of both G and H. Hence there exists $\phi \in Aut(S)$ such that for all

$c \in Inn(S)$, $\pi(c) = \phi c \phi^{-1}$. Now take any $g \in G$. Then for all $c \in Inn(S)$,

$$\begin{aligned}(\phi g)c(\phi g)^{-1} &= \phi(gcg^{-1})\phi^{-1} \\ &= \pi(gcg^{-1}) \\ &= \pi(g)\phi c \phi^{-1}\pi(g)^{-1} \\ &= (\pi(g)\phi)c(\pi(g)\phi)^{-1}\end{aligned}$$

Since $C_{Aut(S)}(Inn(S)) = 1$, it follows that $\phi g = \pi(g)\phi$ and so $\pi(g) = \phi g \phi^{-1}$.

□

Before we continue, we need to introduce some important groups, namely the projective linear group and the projective special linear group. These groups will help us construct a group with a diffuse IRS.

Definition 21. *Let F be a field and V be the 2-dimensional vector space over F. The **projective linear group** $PGL(2, F)$ is the quotient group $GL(2, F)/Z(2, F)$ where $GL(2, F)$ is the group of 2×2 invertible matrices over V and $Z(2, F)$ is the center of $GL(2, F)$.*

*Similarly we define the **projective special linear group** $PSL(2, F)$ to be the quotient group $SL(2, F)/SZ(2, F)$ where $SL(2, F)$ is the group of 2×2 invertible matrices over V with determinant 1 and $SZ(2, F)$ is the center of $SL(2, F)$.*

An important operation on groups is the semi-direct product of two groups. If we have two groups, K and H where $K \curvearrowright H$, we can define an operation on the Cartesian product of the sets H and K such that $H \times K$ becomes a group with this operation and H is then a normal subgroup of it.

Definition 22. *Let H and K be groups such that $K \curvearrowright H$. We define the **semi-direct product** of H and K, denoted $H \rtimes K$ to be the group consisting of the set $H \times K$ and the group operation defined by $(h_1, k_1) \cdot (h_2, k_2) = (h_1(k_1 \cdot^K h_2), k_1 k_2)$.*

We will make use of semi-direct products shortly. First we will give some lemmas about automorphism groups.

Lemma 4. *[11] If Γ is a countable group, then there is a countable field F such that $Aut(F) = \Gamma$.*

If Γ is a countable group that is the automorphism group of a countable field F, then the automorphism group of the projective special linear group $PSL(2, F)$ is isomorphic to the semi-direct product of the projective linear group $PGL(2, F)$ and Γ. This result is due to Schreier and van der Waerden.

Lemma 5. *[19] Let Γ be a countable group, F be a countable field such that $Aut(F) = \Gamma$. Then, $Aut(PSL(2, F)) = PGL(2, F) \rtimes \Gamma$.*

To be able to use this result in our proof, we also need $PSL(2, F)$ to be simple, which is indeed the case for if F has more than 3 elements.

Lemma 6. *[18] If F is a field such that $|F| > 3$, then $PSL(2, F)$ is simple.*

This next lemma which is due to Thomas is the final tool for our first construction of a diffuse IRS.

Lemma 7. *[21] If Γ is a countable group, then there exist a countable group N and a semi-direct product $H = N \rtimes \Gamma$ such that for all $K_1, K_2 \in Sub_\Gamma$,*

$$N \rtimes K_1 \cong N \rtimes K_2 \Leftrightarrow (\exists g \in \Gamma) g K_1 g^{-1} = K_2.$$

Proof. Let Γ be a given countable group and let F be a field whose automorphism group is Γ. Let $N = PGL(2, F) \rtimes \Gamma$. Suppose that $K_1, K_2 \in Sub_\Gamma$. Clearly if K_1 and K_2 are conjugate subgroups of Γ, then $PGL(2, F) \rtimes K_1$ and $PGL(2, F) \rtimes K_2$ are conjugate subgroups of $PGL(2, F) \rtimes \Gamma$, and so $PGL(2, F) \rtimes K_1 \cong PGL(2, F) \rtimes K_2$.

Conversely, suppose that $\pi : PGL(2, F) \rtimes K_1 \to PGL(2, F) \rtimes K_2$ is an isomorphism. As $PSL(2, F)$ is simple by Lemma 6 and $Aut(PSL(2, F)) = PGL(2, F) \rtimes \Gamma$ by Lemma 5, there exists an element $h \in PGL(2, F) \rtimes \Gamma$ such that $h(PGL(2, F) \rtimes K_1)h^{-1} = PGL(2, F) \rtimes K_2$ by Lemma 3.

After factoring by $PGL(2, F)$, we conclude that K_1 and K_2 are conjugate subgroups of Γ. □

We are now ready to state and prove our main theorem.

Theorem 10. *[21] There exist countable groups with diffuse IRSs.*

Proof. Let Γ be a countable group with an ergodic IRS μ not concentrating on a single conjugacy class of subgroups of Γ. For example the finitary symmetric group $Fin(\mathbb{N})$ or a diagonal limit of finite alternating groups are such groups.

Let N and $H = N \rtimes \Gamma$ be countable groups as in Lemma 7 and let $j : Sub_\Gamma \to Sub_H$ be defined by $j(K) = N \rtimes K$ and let $\nu = j_*\mu$. We'll show that ν is an IRS of H. ν is clearly a probability measure as $\nu(Sub_H) = \mu(j^{-1}(Sub_H)) = \mu(Sub_\Gamma) = 1$.

Let $A \in Sub_H$. Observe that for any $g \in \Gamma$,

$$\begin{aligned}
A^{(e,g)} &= (e,g)A(e,g)^{-1} \\
&= (e,g)A(g^{-1}e^{-1}g, g^{-1}) \\
&= (e,g)A(e,g^{-1}) \\
&= \{(e,g)(n,h)(e,g^{-1}) \mid (n,h) \in A\} \\
&= \{(egng^{-1}, gh)(e,g^{-1}) \mid (n,h) \in A\} \\
&= \{(egng^{-1}gheh^{-1}g^{-1}) \mid (n,h) \in A\} \\
&= \{(gng^{-1}, ghg^{-1}) \mid (n,h) \in A\} \\
&= \{(m, ghg^{-1}) \mid (m,h) \in A\}
\end{aligned}$$

So taking a conjugate of an element $A \in Sub_H$ with an element $g \in \Gamma$ (by identifying g as (e,g) in A) corresponds to taking conjugates index-wise on the second index. It follows that for any Borel subset $\Lambda \subseteq H$,

$$\nu(\Lambda^{(e,g)}) = j_*\mu(\Lambda^{(e,g)}) = \mu(j^{-1}(\Lambda^{(e,g)})) = \mu((j^{-1}(\Lambda))^g) = \mu(j^{-1}(\Lambda)) = \nu(\Lambda).$$

So ν is Γ-invariant, moreover ν is ergodic.

Now let $N \rtimes G \in j(Sub_\Gamma)$. Consider $(N \rtimes G)^{(m,e)}$.

$$\begin{aligned}
(N \rtimes G)^{(m,e)} &= (m,e)(N \rtimes G)(e^{-1}m^{-1}e, e^{-1}) \\
&= \{(m,e)(n,g)(m^{-1},e) | (n,g) \in N \rtimes G\} \\
&= \{(mene^{-1}, eg)(m^{-1},e) | (n,g) \in N \rtimes G\} \\
&= \{(mn,g)(m^{-1},e) | (n,g) \in N \rtimes G\} \\
&= \{(mngm^{-1}g^{-1}, g) | (n,g) \in N \rtimes G\} \\
&= N \rtimes G
\end{aligned}$$

So N acts trivially on $j(Sub_\Gamma)$, which means that ν is H-invariant. So ν is an ergodic IRS of H. Finally, the isomorphism classes on $j(Sub_\Gamma)$ correspond to the conjugacy classes on Sub_Γ, either ν is diffuse or μ concentrates on a single conjugacy class of Sub_Γ, which is not the case. Therefore, ν is a diffuse IRS of H. □

Hence we constructed a group with a diffuse IRS.

CHAPTER 4

HYPERBOLIC GROUPS AND ERGODIC INVARIANT RANDOM SUBGROUPS

After Thomas constructed the diffuse IRS in the last chapter, he left the question whether there was a more natural example of a diffuse IRS open. Although this question is still open, the most progress on the subject came from Raimbault. He constructed a group with a property we will call ★, which is possibly weaker than diffuseness.

First we introduce an important concept about measures, namely the support of a measure.

Definition 23. *Let X be a metric space and μ be a Borel probability measure on X. The **support** of μ is the minimal closed set $C \subseteq X$ such that $\mu(C) = 1$. The support of μ is denoted as $supp(\mu)$.*

Now we can define the ★ property.

Definition 24. *Let μ be an ergodic IRS. We say that μ satisfies the ★ property if there are uncountably many pairwise non-isomorphic subgroups in the support of μ.*

Lemma 8. *Let μ be a diffuse IRS. Then μ satisfies the ★ property.*

Proof. Assume that μ is a diffuse IRS of Γ. Assume towards a contradiction that $supp(\mu)$ has subgroups from only countably many isomorphism classes, say from $\{[H_i]\}_{i\in\omega}$. But then, since μ is diffuse $\mu([H_i]) = 0$ for all i, and so $\mu([H_i] \cap supp(\mu)) = 0$ as well. Since the sets $[H_i] \cap supp(\mu)$ are pairwise disjoint for different $[H_i]$, we get:

$$
\begin{aligned}
0 &= \sum_{i\in\omega} \mu([H_i] \cap supp(\mu)) \\
&= \mu(\bigcup_{i\in\omega}([H_i] \cap supp(\mu))) \\
&= \mu(supp(\mu)) \\
&= 1
\end{aligned}
$$

which is a contradiction. □

The converse need not hold in general as there might be uncountably many isomorphism types in the support of an ergodic IRS which concentrates on a single one of them.

Raimbault [17] constructed a group with the ★ property using hyperbolic reflection groups, and in this chapter we shall follow his approach for our second construction.

Let T_4 be the unique triangle in the hyperbolic plane all of whose angles equals $\pi/4$. Let Γ denote the subgroup of isometries whose elements are generated by reflections in the faces of T_4. This group of isometries is discrete and when we take the translations of T_4 by elements of Γ we can tile the hyperbolic plane.

This group can be given by the presentation $\Gamma = \langle s_1, s_2, s_3 | (s_i s_{i+1})^4, s_i^2 \rangle$ where the indices are taken cyclically over i = 1,2,3.

The curious reader might check [9] for details on how the presentations of groups can be obtained.

From now on whenever we mention the group Γ we will not specify that indices are taken cyclically, but that will still be the case. We can realize s_i as the reflections in the faces of T_4.

By the end of this chapter, we will have proven that Γ admits an IRS satisfying the ★ property. We will show that it has an ergodic IRS whose support contains subgroups from uncountably many isomorphism classes.

4.1 Coxeter Polygons and Angle Sequences

To prove that Γ has an ergodic IRS with the $\bigstar$ property, we are going to need a sequence of preliminary results.

We will need results on angle sequences of Coxeter polygons, but as it is easier to work with 01-sequences, we will changer our angle sequences into $0-1$ sequences. First we will define a metric on $\{0,1\}^{\mathbb{Z}}$ that is compatible with its product topology.

Lemma 9. *Let $X = \{0,1\}^{\mathbb{Z}}$ and let $d : X \times X \to \mathbb{R}$ be defined by*

$$d(\alpha,\beta) = \begin{cases} \dfrac{1}{min\{|k| : \alpha_k \neq \beta_k\}+1}, & \alpha \neq \beta \\ 0, & \alpha = \beta \end{cases}$$

Then d is a complete metric on X.

Proof. Clearly for all α,$\beta \in X$, $d(\alpha,\beta) \geq 0$, $d(\alpha,\beta) = 0$ if and only if $\alpha = \beta$ and $d(\alpha,\beta) = d(\beta,\alpha)$.

Now let $\alpha,\beta,\gamma \in X$ where no two are the same sequence. Let $|k|,|l|,|m|$ be the smallest numbers where $\alpha_k \neq \beta_k$, $\beta_l \neq \gamma_l$ and $\alpha_m \neq \gamma_m$ respectively. If $|k| \leq |l|$, then we must have $|k| = |m|$ since there are only two options for an entry. Similarly if $|l| \leq |k|$ we must have $|l| = |m|$. Without loss of generality say $|k| \leq |l|$. Then,

$d(\alpha,\gamma) = \frac{1}{|m|+1} = \frac{1}{|k|+1} \leq \frac{1}{|k|+1} + \frac{1}{|l|+1} = d(\alpha,\beta) + d(\beta,\gamma)$.

If any two of α,β, or γ are equal then the triangle equality holds trivially. Thus d is a metric on X.

Now let (α^n) be a Cauchy sequence in X. Assume that there is $k \in \mathbb{Z}$ such that for no $N \in \mathbb{N}$ it is the case that "for all $n \geq N$, α^n_k stabilizes on 0 or 1". Then, for $\epsilon = \frac{1}{|k|+2}$, there is no $N \in \mathbb{N}$ so that for all $m,n \geq N$, $d(\alpha^n,\beta^n) \leq \epsilon$. This contradicts (α^n) being a Cauchy sequence. Thus, if α^n is a Cauchy sequence in X, then for all $k \in \mathbb{Z}$, α^n_k stabilizes. It follows that (α^n) is convergent.

So every Cauchy sequence in X is convergent, thus (X,d) is complete. □

This proof also shows that our metric d is compatible with the product topology on X.

Now we will prove some properties about Coxeter polygons and their angle sequences.

Proposition 4.3. *[17] There exists a sequence of Coxeter polygons P_n in the hyperbolic plane all of which are tiled by the triangle T_4 such that their edge lengths are bounded and the closure of their angle sequences in $\{\pi/4, \pi/2\}^{\mathbb{Z}}$ contains uncountably many sequences.*

Although Proposition 4.3 appeared in the manuscript [17], due to some gaps in the original proof, we shall give a different proof based on Raimbault's ideas.

Proof. We will construct the sequence by hand. We will begin with the triangle T_4, construct some polygons tiled by T_4 and glue these polygons together to obtain a sequence satisfying the proposition.

For the sake of readability, we will work with 01-sequences instead of $\frac{\pi}{2}\frac{\pi}{4}$-sequences. So for the rest of the proof a 0 will denote an angle of $\pi/2$ and a 1 will denote an angle of $\pi/4$.

We take the triangle T_4 all of whose edge lengths are l, and we rotate it to get a regular octagon O, whose edge lengths are still l. This is shown in Figure 4.1. Note that the angle sequence of O is the constant sequence $(\dots[\underline{0}0000000]\dots)$ where the underlined 0 is at the 0^{th} index and at a vertex p_0 which is specified as the root. The edge in red is the edge where a polygon will be glued to O.

Let A denote the polygon obtained by gluing another octagon O and two of half of the octagon O as in 4.2. The edge lengths of A is bounded by $3l$. If A is to be glued to another polygon, this will be done at the edge that is painted blue. If another polygon will be glued to A, this will be done at the edge painted red.

Let B denote the polygon obtained by gluing another copy of O at the end of O as in 4.3. The edge lengths of B is bounded by $2l$. Similarly, if B is to be glued to another

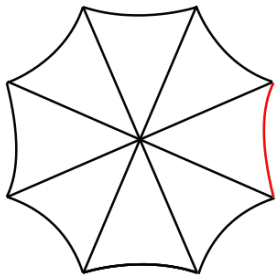


Figure 4.1: Octagon O Tiled by T_4

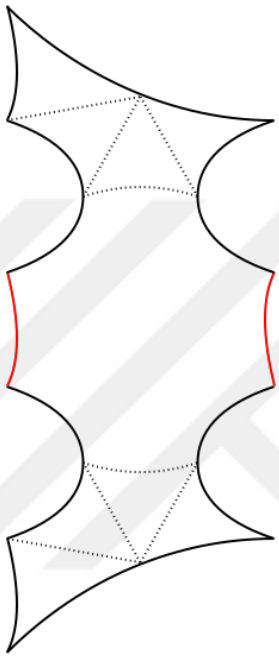


Figure 4.2: Polygon A, An Octagon with Two Half Octagons Glued

polygon, this will be done at the edge painted blue and if another polygon will be glued to B, this will be done at the edge painted red.

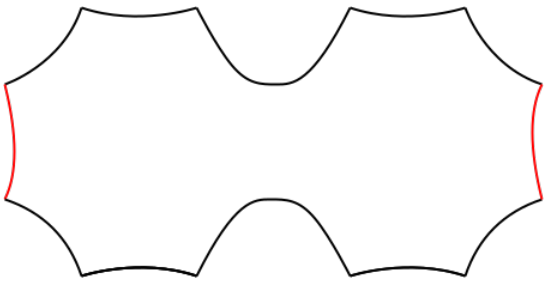


Figure 4.3: Polygon B, Two Octagons Glued Together Along a Common Edge

Note that with the gluing described above, $3l$ acts as an upper bound to the edge lengths of any polygon obtained by gluing A, B and O to each other.

Observe that the angle sequence of A is $[\mathbf{00}01100\mathbf{00}00110]$ and the angle sequence of B is $[\mathbf{00}00000\mathbf{00}00000]$ where the bold fonted 0's are the angles to be deleted in the next gluings (the ones in the left are to be deleted when A or B are glued to a polygon, and the middle ones are to be deleted when another polygon is to be glued to A or B).

Let $P_0 = O$. Let γ denote the angle sequence of P_0. Then we define $a(\gamma)$ to be the angle sequence of the polygon P_a obtained by gluing A to O as described above. Similarly we define $b(\gamma)$ to be the angle sequence of the polygon P_b obtained by gluing B to O. Thus,

$$a(\gamma) = ([\underline{0}00001100\mathbf{00}0011000])$$

$$b(\gamma) = ([\underline{0}00000000\mathbf{00}000000]).$$

Now we recursively extend the definitions of a and b as follows:

Let $\sigma \in \{a, b\}^n$. Assume that $\sigma(\gamma)$ is well defined. We define $a(\sigma(\gamma))$ to be the angle sequence of the polygon obtained by gluing an A to the polygon whose angle sequence is $\sigma(\gamma)$ according to the gluing rules described above, similarly we define $b(\sigma(\gamma))$ to be the angle sequence of the polygon obtained by gluing a B to the polygon whose angle sequence is $\sigma(\gamma)$.

In general, let P be a polygon obtained by successive applications of a and b to O. Then the angle sequence of P looks like $\sigma(\gamma) = ([S00\mathbf{00}00T])$ where the 0's exist because of the octagonal part in A or B and the bold fonted 0's will disappear in the next gluing. Then,

$$a(\sigma(\gamma)) = ([S0001100\mathbf{00}0011000T]),$$
$$b(\sigma(\gamma)) = ([S0000000\mathbf{00}0000000T]).$$

So, applying a to $\sigma(\gamma)$ deletes the bold fonted 0's and adds a block of $01100\mathbf{00}00110$, and applying b to $\sigma(\gamma)$ deletes the bold fonted 0's and adds a block of $0000\mathbf{00}0000$.

We'll now show that even if $\sigma \in \{a, b\}^{\mathbb{N}}$, $\sigma(\gamma)$ still makes sense. To show this we need to define the initial segment of σ.

Definition 25. *Let $\sigma \in \{a, b\}^{\mathbb{N}}$. The initial segment of σ of length k, denoted $\sigma|_k$ is the finite sequence obtained by restricting the domain of σ to k.*

We can now formally define $\sigma(\gamma)$ for $\sigma \in \{a, b\}^{\mathbb{N}}$.

Definition 26. *Let $\sigma \in \{a, b\}^{\mathbb{N}}$. We define $\sigma(\gamma)$ to be $\lim_{k\to\infty} \sigma|_k(\gamma)$ if the limit exists.*

So if σ is an infinite sequence of a and b's, $\sigma(\gamma)$ makes sense only if the defining limit exists. We'll show that it always exists.

Lemma 10. *Let $\sigma \in \{a, b\}^{\mathbb{N}}$. Then $(\sigma|_k(\gamma))_{k\in\mathbb{N}}$ is a convergent sequence.*

Proof. We will prove that $(\sigma|_k(\gamma))_{k\in\mathbb{N}}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. We will show that there exists some index $n \in \mathbb{N}$ such that $d(\sigma|_r(\gamma), \sigma|_s(\gamma)) \leq \epsilon$ for all $r, s \geq n$. Let $m \in \mathbb{Z}^+$ be such that $\frac{1}{m} \leq \epsilon$. Consider $\sigma|_m(\gamma)$.

Note that at each step the angle sequence is of the form $\alpha = ([S00\mathbf{00}00T])$. So if $len(S) = s$ and $len(T) = t$, $d(\alpha, \gamma(\alpha)) \leq \frac{1}{min\{s,t\}+3}$. Also note that at each step a changes the sequence to $[(S'00\mathbf{00}00T'])$ where $len(S') = len(S) + 5$ and $len(T') = len(T) + 5$, similarly b changes the sequence to $([S''00\mathbf{00}00T''])$ where $len(S'') = len(S) + 4$ and $len(T'') = len(T) + 4$.

So if $\alpha = \sigma|_k(\gamma) = ([S'00\mathbf{00}00T'])$, then $len(T') = k \times len(T) + 4$.

Thus, $d(\sigma|_r(\gamma), \sigma|_s(\gamma)) \leq \frac{1}{4m+4} \leq \frac{1}{m} \leq \epsilon$.

Thus, $(\sigma|_k(\gamma))_{k\in\mathbb{N}}$ is a Cauchy sequence. Since $\{0, 1\}^{\mathbb{Z}}$ is complete any Cauchy sequence, and in particular $(\sigma|_k(\gamma))_{k\in\mathbb{N}}$, is convergent. □

So $\sigma(\gamma)$ is well defined for any $\sigma \in \{a, b\}^{\mathbb{N}}$.

Next we will show that for any two different $\sigma_1, \sigma_2 \in \{a, b\}^{\mathbb{N}}$, applying σ_1 and σ_2 to γ will give us different sequences.

Lemma 11. *Let $\sigma_1, \sigma_2 \in \{\alpha, \beta\}^{\mathbb{N}}$. If $\sigma_1 \neq \sigma_2$, then $\sigma_1(\gamma) \neq \sigma_2(\gamma)$.*

Proof. We will look that at the first index, say m, where $\sigma_1|_m$ and $\sigma_2|_m$ differ. We will show that for all $n \geq m$ the sequences $\sigma_1|_n(\gamma)$ and $\sigma_2|_n(\gamma)$ will be different and they do not converge to the same sequence. Say at the m^{th} step σ_1 applies a and σ_2 applies b, call the resulting polygons P^a and P^b, respectively.

Let P_{n-1} be the polygon whose angle sequence is the periodic sequence consisting of the block $\sigma|_{n-1}(\gamma)$.

Then the angle sequence of P_{n-1} looks like $[S00\mathbf{00}00T]$ where the 0's come from the octagon in the most recently glued polygon and the bold fonted 0's are the ones to be deleted in the next gluing. Consider the angle sequences $a(\sigma|_{n-1}(\gamma))$ and $b(\sigma|_{n-1}(\gamma))$.

Since a has the effect of deleting the bold fonted 0's and adding $[01100\mathbf{00}00110]$ there, the angle sequence of P^a is $[S0001100\mathbf{00}0011000T]$. Notice that no new gluing will change the first 1 after the block S. So if the length of the block S is k, then for every polygon obtained from P^a by applying a sequence of a and b's, $(k+4)^{th}$ angle is always a 1.

Now, b has the effect of deleting the bold fonted 0's from the angle sequence of P_{n-1} and adding ten 0's instead, so the angle sequence of P^b is $[S0000000\mathbf{00}000000T]$. By the same argument, any polygon that can be obtained from P^b has a 0 at the $(k+4)^{th}$ angle.

Since taking limits won't affect the $(k+4)^{th}$ angle, we conclude that $\sigma_1(\gamma) \neq \sigma_2(\gamma)$. □

Since $\{a,b\}^{\mathbb{N}}$, the closure of $\{a,b\}^{<\mathbb{N}}$ is uncountable and any two elements of it applied to γ gives us two different sequences, $\{\sigma(\gamma) \mid \sigma \in \{a,b\}^{\mathbb{N}}\}$ is an uncountable set.

Now for each $n \in \mathbb{N}$, let Q_n denote the set $\{\sigma(\gamma) \mid \sigma \in \{a,b\}^{\leq n}\}$ with the order induced from the lexicographic order on $\{a,b\}^{\leq n}$. For each $n \in \mathbb{N}$, Q_n is a finite set. Then, $\bigcup_{n\in\mathbb{N}} Q_n$ is a countable set whose closure is the set $\{\sigma(\gamma) \mid \sigma \in \{a,b\}^{\mathbb{N}}\}$. Without changing the order we can renumerate the elements of $\bigcup_{n\in\mathbb{N}} Q_n$ to have a

sequence P_n.

Thus, P_n is a sequence of Coxeter polygons with bounded edge lengths closure of whose angle sequences contains uncountably many sequences. This is the result we desired. □

Let P be a Coxeter polygon in $\mathbb{H}^2$. We denote the subgroup of $Isom(\mathbb{H}^2)$ generated by reflections in the faces of P by Γ_P. If we have a sequence P_i of Coxeter polygons, we use the notation Γ_i instead of Γ_{P_i}.

Lemma 12. *Let Γ_n be the reflection group associated with the polygon P_n in the proof of Prop. 4.3. Then, $|[\Gamma : \Gamma_n]| < \infty$.*

Proof. Let k be the smallest number such that k-many copies of T_4 tile P_n. Since Γ_n is generated by reflections in the faces of P_n, P_n acts as a fundamental domain for Γ_n, that is it contains exactly one point from the orbit of any point in $\mathbb{H}^2$. Since T_4 tile P_n with k copies, $[\Gamma : \Gamma_n] = k$. □

Since Γ_n have finite index in Γ, Γ_n have finite conjugacy class and so there exist ergodic invariant random subgroups μ_n of Γ concentrated on Γ_n for all n.

4.2 A Reflection Group with an Ergodic Invariant Random Subgroup

We will construct our ergodic IRS with the ★ property as the limit of IRSs supported on reflection subgroups, and the next lemma which is due to Raimbault guarantees that an IRS constructed this way is supported on the subgroups generated by reflections.

Let us first define the notion of weak convergence of measures.

Definition 27. *Let X be a metric space and let $C_B(X)$ denote the class of continuous bounded functions on X. We say that a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to a measure μ if for all $f \in C_B(X)$, $\int_X f d\mu_n \to \int_X f d_\mu$.*

Lemma 13. *[16, Theorem 3] If μ belongs to the closure (in topology of weak convergence of measures on Sub_Γ) of the set of all μ_p for P a bounded Coxeter polygon in*

the hyperbolic plane, then μ-almost every subgroup is non-trivial and generated by reflections.

Now we can turn our attention to reflection groups again. We need a few lemmas left to state before proving our main result.

Lemma 14. *[17] Let Γ_1, Γ_2 be reflection groups associated with polygons in $\mathbb{H}^2$ all of whose angles are $\pi/2$ or $\pi/4$. They are isomorphic to each other if and only if their angle sequences are shifted or mirrored from each other.*

Proof. Let (α_i) and (β_i) denote the angle sequences of the polygons and let $\Gamma_1 = \langle s_i | (s_i s_{i+1})^{\pi/\alpha_i}, s_i^2 \rangle$ and $\Gamma_2 = \langle t_i | (t_i t_{i+1})^{\pi/\beta_i}, t_i^2 \rangle$ be the presentations of the reflection groups these polygons are associated with. Let $\Psi : \Gamma_1 \to \Gamma_2$ be an isomorphism. We will show that for all $i \in \mathbb{Z}$, $\Psi(s)$ is conjugated to $t_{\epsilon i+k}$ for some $\epsilon = \pm 1$ and $k \in \mathbb{Z}$.

Since the composition of two involutions is an involution if and only if they commute, and two reflections commute only when they have orthogonal axes, an involution in Γ_1 is either a single reflection or the product of two reflections with orthogonal axes, say r and t, and if this is the case there exist a pair s_i, s_{i+1} such that $r, t \in \langle s_i, s_{i+1} \rangle$.

If α_i, angle between the edges associated with s_i and s_{i+1}, is $\pi/2$ then we can shift the sequence so that α_i is now α_0 and $\langle s_0, s_1 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$. Since any subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ are also contained in $\langle t_k, t_{k+1} \rangle$ for some k, we have $\Psi(s_0), \Psi(s_1) \in \langle t_k, t_{k+1} \rangle$.

Since there is no $p \in \Gamma_1$ with the property that $p^2 = s_0 s_1$ we have $\beta_k = \pi/2$. Thus, the involutions in $\langle t_k, t_{k+1} \rangle$ are only $t_k, t_{k+1}, t_k t_{k+1}$. Now, if it were the case that $\Psi(s_o) = t_k t_{k+1}$ then $\Psi(s_{-1})$ and $\Psi(s_0)$ couldn't be generated by consecutive generators of Γ_2, so $\langle \Psi(s_{-1}), \Psi(s_0) \rangle$ would form an infinite subgroup, which can't be the case as $\langle s_{-1}, s_0 \rangle$ is a finite subgroup of Γ_1.

So we have $\Psi(s_0) = t_k$ and $\Psi(s_1) = t_{k+1}$ or vice versa but we can assume without loss of generality that this is the case, we can reverse the order the sequence (β_j) if needed.

Now consider α_1. If $\alpha_1 = \pi/2$ then the same argument as above shows that

$\Psi(s_2) = t_{k+2}$. If $\alpha_1 = \pi/4$ then $\Psi(s_2) \in \langle t_k, t_{k+1} \rangle$. Since $(\Psi(s_1 s_2))^4 = 1$ and $(\Psi(s_1 s_2))^2 \neq 1$, we must have either $\Psi(s_2) = t_{k+2}$ or $\Psi(s_2) = t_{k+1} t_{k+2} t_{k+1}$.

Induction on α_i shows that each s_i is conjugated to $t_{\epsilon i+k}$ for some $k \in \mathbb{Z}$ and $\epsilon = \pm 1$.

Thus, such an isomorphism Ψ exists if and only if the angle sequences are the mirrored and/or shifted versions of one another. □

Now we can return to the group $\Gamma = \langle s_1, s_2, s_3 | (s_i s_{i+1})^4, s_i^2 \rangle$. We are almost ready to prove that this group admits an ergodic IRS with the ★ property. First, we must show that the reflection subgroups of Γ having finite conjugacy orbit are not too many in size, namely there can't be uncountably many of them.

Lemma 15. *[17] At most countably many reflection subgroups of* Γ *have a finite conjugacy orbit.*

Proof. Let $m < \omega$ and H be a subgroup of Γ with the property that the conjugacy class of H consists of m subgroups. Let $N_\Gamma(H)$ denote the normalizer of H in Γ. For $i = 1, 2, 3$; let R_i denote the set of conjugates of s_i, the generators of Γ.

We can write the set of all reflections in Γ as $R = R_1 \cup R_2 \cup R_3$. Any R_i is itself a conjugacy class in Γ and under conjugacy by $N_\Gamma(H)$ we get at most m conjugacy classes from each R_i. So we get at most 2^{3m} subsets of R which are left the same after conjugation by $N_\Gamma(H)$. Thus $N_\Gamma(H)$ normalizes at most 2^{3m} reflection subgroups of Γ. Since Γ has finitely many subgroups of index m, it has only finitely many subgroups whose conjugacy class consists of m elements.

There are countably many $m < \omega$ and for each of them there are finitely many reflection subgroups whose conjugacy orbit is of cardinality m, therefore Γ has at most countably many reflection subgroups with finite conjugacy orbit. □

Now that we have all the tools we need, we are ready to prove the main theorem of this chapter.

Theorem 11. *[17] The group* $\Gamma = \langle s_1, s_2, s_3 | (s_i s_{i+1})^4, s_i^2 \rangle$ *admits an IRS satisfying the* ★ *property.*

Proof. We will construct an ergodic IRS of Γ whose support contains uncountably many isomorphism types of subgroups.

We showed in Proposition 4.3 that there exists a sequence of Coxeter polygons in $\mathbb{H}^2$ with bounded edge lengths, say P_n, all tiled by T_4 such that when we take the closure of their angle sequences we get uncountably many sequences.

Now for all $n \in \mathbb{N}$, let Γ_n denote the subgroup of reflections in the faces of P_n. Since all Γ_n have finite index in Γ, there is an IRS μ_n of Γ supported on the conjugacy class of Γ_n. Let μ be a limit point of μ_n. By Lemma 13, μ is supported on the set of subgroups of Γ generated by reflections.

By Lemma 2, the angle sequences of the polygons corresponding to subgroups generated by reflections are exactly the limit points of the angle sequences of P_n. Since there are uncountably many angle sequences in the closure, $supp(\mu)$ consists of uncountably many subgroups.

Now, by Lemma 14, any two reflection subgroup associated with Coxeter polygons in $\mathbb{H}^2$ whose angle sequences consist of $\pi/2$ and $\pi/4$ are isomorphic to each other if and only if the corresponding angle sequences are obtained from each other by shifting and/or mirroring.

It follows that there are uncountably many isomorphism types in $supp(\mu)$. If μ is ergodic we are done as it is an ergodic IRS whose support contains uncountably many isomorphism types of subgroups, because then μ satisfies the $\bigstar$ property.

Assume that μ is not ergodic. Since $supp(\mu)$ is uncountable and only countably many subgroups of Γ have finite conjugacy orbit by Lemma 15, there are subgroups with infinite conjugacy classes in $supp(\mu)$.

μ is a weak limit point of IRSs supported on the conjugacy class of reflection subgroups, so there can't be a finite conjugacy class to which μ attains positive measure, otherwise it wouldn't be a limit point. Thus, there exists a measurable set $B \in supp(\mu)$ such that subgroups in B have infinite conjugacy classes and $\mu(B) > 0$.

We can use Theorem 5 to write μ as an integral combination $\mu = \int_{\mathcal{E}(\Gamma)} \mu_\alpha \lambda(\mu_\alpha)$ of ergodic IRS's $\mu_\alpha \in \mathcal{E}(\Gamma)$, each of which is supported on reflection subgroups.

So there exists some ergodic μ_γ in the decomposition where $\mu_\gamma(B) > 0$. Since μ_γ is an IRS, it is conjugation invariant. Thus any subgroup conjugate to a subgroup in B is in $supp(\mu_\gamma)$.

The subgroups in B don't have a finite conjugacy class, so $supp(\mu_\gamma)$ is infinite and since μ_γ is ergodic its support can not have exactly countably many subgroups, hence $supp(\mu_\gamma)$ is uncountable. Again, only countably many subgroups in $supp(\mu_\gamma)$ can be isomorphic to on another, it follows that $supp(\mu_\gamma)$ contains uncountably many isomorphism types.

Thus μ_γ is an ergodic IRS whose support contains uncountably many isomorphism types of subgroups, so μ_γ satisfies the ★ property. □

This completes our construction of a group with an ergodic IRS satisfying the ★ property.

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