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A comment on metric vs metric-affine gravity

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ABSTRACT

We consider the sum of the Einstein-Hilbert action and a Pontryagin density (PD) in arbitrary even dimension $D \geq 4$. All curvatures are functions of independent affine (torsionless) connections only. In arbitrary even dimension, not only in $D = 4n$, these first order PD terms are shown to be covariant divergences of "Chern-Simons" currents. The field equation for the connection leads to it being Levi-Civita, and to the metric and affine field equations being equivalent to the second order metric theory. This result is a counterexample to the theorem stating that purely metric and metric-affine models can only be equivalent for Lovelock theories.

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The action we consider in $D = 2n$ dimensions reads

$$S = \int d^D x \left(\sqrt{|g|} g^{ab} R_{ab}(\Gamma) + \tfrac{1}{n} \theta \epsilon^{a_1 a_2 \ldots a_D} R^{i_1}{}_{i_2 a_1 a_2} R^{i_2}{}_{i_3 a_3 a_4} \ldots R^{i_n}{}_{i_1 a_{D-1} a_D}(\Gamma) \right) . \tag{1}$$

In the metric formulation the connection is the Levi-Civita connection and the second term vanishes for odd n due to the additional symmetries of the curvature tensor. The topological nature of the second term becomes manifest if written as $\partial_a K^a$ where,[1] schematically,

$$K^{a_1} \approx \theta \epsilon^{a_1 a_2 \ldots a_D} \Gamma^{i_1}{}_{a_2 i_2} \left(R^{i_2}{}_{i_3 a_3 a_4} \ldots R^{i_n}{}_{i_1 a_{D-1} a_D} + \Gamma^2 R^{n-2} + \cdots + \Gamma^{D-2} \right) . \tag{2}$$

The $D = 4$ expression reads [1,2]

$$K^a = \theta \epsilon^{abcd} \Gamma^i{}_{bj} \left(R^j{}_{icd} - \frac{2}{3} \Gamma^j{}_{ck} \Gamma^k{}_{di} \right) , \tag{3}$$

while in $D = 6$ we have

$$K^a = \frac{2}{3} \theta \epsilon^{abcdef} \Gamma^i{}_{bj} \left(R^j{}_{kcd} R^k{}_{ief} - R^j{}_{kcd} \Gamma^k{}_{er} \Gamma^r{}_{fi} + \frac{2}{5} \Gamma^j{}_{ck} \Gamma^k{}_{dp} \Gamma^p{}_{eq} \Gamma^q{}_{fi} \right) .$$

The field equations that follow from varying the connection in (1) read

$$-2 \nabla_a \left(\theta \epsilon^{aba_3 \ldots a_D} R^d{}_{ca_3 a_4} \ldots R^{i_n}{}_{i_1 a_{D-1} a_D} \right) + \nabla_c (\sqrt{|g|} g^{db}) - \delta^d{}_c \nabla_a (\sqrt{|g|} g^{ab}) = 0 . \tag{4}$$

The first term vanishes due to Bianchi identities and the remaining terms may be massaged to give

$$\nabla_c (\sqrt{|g|} g^{db}) = 0 , \tag{5}$$

which implies that the connection is the Levi-Civita connection. This means that the action (1) is completely equivalent to its purely metric form where the connection is Levi-Civita from the outset.

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[1] Here we assume that θ is constant. In the Chern-Simons modification of General Relativity (GR) [1], it is taken to be a scalar field, much in the spirit of the Fradkin-Tseytlin term in string theory. This leads to interesting deviations from GR. It follows from the results in the present paper that promoting θ to a scalar field will give modifications of GR also for odd n, as in, e.g., six dimensions. Doing so will alter the solution for the connection, which will now depend on θ.

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In [3] it is argued that the only case when the metric and metric-affine formulations of gravity are equivalent is Lovelock gravity [4]. However, gravity amended with a Pontryagin term as described above is not a Lovelock gravity and thus constitutes a counterexample. It must be noted though that the discussion in [3] is carried out at the level of the equations of motion, whereas the statement in this note is at the level of the action.

Note added

While we were writing these results up, a paper, [5], appeared on the net which contains the metric-affine description of the $4D$ version of (1). Similar results also appeared in [6].

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Ulf Lindstrom reports financial support was provided by Scientific and Technological Research Council of Turkey. Ulf Lindstrom reports a relationship with Scientific and Technological Research Council of Turkey that includes: funding grants.

Data availability

No data was used for the research described in the article.

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