Energy of a Stringy Charged Black Hole in the Teleparallel Gravity*†

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We use the teleparallel geometry analog of the Møller energy-momentum complex to calculate the energy distribution (due to matter plus field including gravity) of a charged black hole solution in heterotic string theory. We find the same energy distribution as obtained by Gad who investigated the same problem by using the Møller energy-momentum complex in general relativity. The total energy depends on the black hole mass M and charge Q. The energy obtained is also independent of the teleparallel dimensionless coupling constant, which means that it is valid not only in the teleparallel equivalent of general relativity, but also in any teleparallel model. Furthermore, our results also sustains (a) the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given spacetime and (b) the viewpoint of Lessner that the Møller energy-momentum complex is a powerful concept of energy and momentum.

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I. INTRODUCTION

Einstein [1] defined the energy and momentum conservation law in the form

$$\partial_\mu[\sqrt{-g}(T^\mu_\nu + t^\mu_\nu)] = 0 \tag{1}$$

where $\mu, \nu = 0, 1, 2, 3$ and T^μ_ν is the stress energy density of matter. Einstein identified t^μ_ν as representing the stress energy density of gravitational fields. He also noted that t^μ_ν was not a tensor, but concluded that the equations given above hold good in all coordinate systems since they were directly obtained from the principle of general relativity. In literature, after Einstein's expression for the energy and momentum distributions of the gravitational field, many attempts have been proposed to resolve the gravitational energy-momentum problem [2, 3, 4, 5, 6, 7, 8, 9, 10]. The Møller energy-momentum prescription does not necessitate carrying out calculation in "Cartesian" coordinates, while the others do. Therefore, we can calculate the energy density in any coordinate system. Lessner [11] argued that the Møller prescription is a powerful concept of energy-momentum in general relativity. Teleparallel version of this complex was obtained by Mikhail *et al* [9].

In general relativity and teleparallel gravity, several examples of particular space-times have been investigated and different energy-momentum pseudo-tensor are known to give the same energy distribution for a given space-time [12, 13, 14, 15, 16, 17, 18, 19]. Virbhadra[13], using the energy and momentum complexes of Einstein, Landau-Lifshitz, Papapetrou and Weinberg for a general non-static spherically symmetric metric of the Kerr-Schild class, showed that all of these energy-momentum formulations give the same energy distribution as in the Penrose energy-momentum formulation. Recently, The problem of energy-momentum localization was also considered in alternative gravitation theory, namely teleparallel gravity [10, 21]. The authors found that energy-momentum also localize in this alternative theory, and their results agree with some previous papers which were studied in the general theory of relativity. Vargas [10] using the definitions of Einstein and Landau-Lifshitz in teleparallel gravity, found that the total energy is zero in Friedmann-Robertson-Walker space-times and his result is the same as calculated in general relativity [22].

In this paper, we evaluate the energy of a stringy charged black hole by using the Møller definition in teleparallel gravity, and then we compare the results with the ones obtained by Gad [19].

Notations and conventions: $c = h = 1$, metric signature $(+, -, -, -)$, Greek indices and Latin ones run from 0 to 3. Throughout this paper, Latin indices (i, j, ...) represent the vector number, and Greek indices (μ, ν,...) represent the vector components. Latin indices are never raised.

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II. MØLLER'S ENERGY IN TELEPARALLEL GEOMETRY

Møller modified general relativity by constructing a new field theory in teleparallel space. The aim of this theory was to overcome the problem of the energy-momentum complex that appears in Riemannian Space [23]. The field equations in this new theory were derived from a Lagrangian which is not invariant under local tetrad rotation. Saez [24] generalized Møller theory into a scalar tetrad theory of gravitation. Meyer [25] showed that Møller theory is a special case of Poincare gauge theory [26, 27].

In teleparallel gravity, the super-potential of Møller is given by Mikhail *et al.*[9] as

$$U_{\mu}^{\nu\beta}=\frac{(-g)^{1/2}}{2\kappa}P_{\chi\rho\sigma}^{\tau\nu\beta}[\Phi^{\rho}g^{\sigma\chi}g_{\mu\tau}-\lambda g_{\tau\mu}\xi^{\chi\rho\sigma}-(1-2\lambda)g_{\tau\mu}\xi^{\sigma\rho\chi}] \tag{2}$$

where $\xi_{\alpha\beta\mu}=h_{i\alpha}h_{i\beta;\mu}$ is the con-torsion tensor and $h_i{}^{\mu}$ is the tetrad field and defined uniquely by $g^{\alpha\beta}=h_i^{\alpha}h_j^{\beta}$. κ is the Einstein constant and λ is free-dimensionless coupling parameter of teleparallel gravity. For the teleparallel equivalent of general relativity, there is a specific choice of this constant.

Φ_{ρ} is the basic vector field given by

$$\Phi_{\mu}=\xi^{\rho}{}_{\mu\rho} \tag{3}$$

and $P_{\chi\rho\sigma}^{\tau\nu\beta}$ can be found by

$$P_{\chi\rho\sigma}^{\tau\nu\beta}=\delta_{\chi}^{\tau}g_{\rho\sigma}^{\nu\beta}+\delta_{\rho}^{\tau}g_{\sigma\chi}^{\nu\beta}-\delta_{\sigma}^{\tau}g_{\chi\sigma}^{\nu\beta} \tag{4}$$

with $g_{\rho\sigma}^{\nu\beta}$ being a tensor defined by

$$g_{\rho\sigma}^{\nu\beta}=\delta_{\rho}^{\nu}\delta_{\sigma}^{\beta}-\delta_{\sigma}^{\nu}\delta_{\rho}^{\beta}. \tag{5}$$

The energy-momentum density is defined by

$$\Xi_{\alpha}^{\beta}=U_{\alpha,\lambda}^{\beta\lambda} \tag{6}$$

where comma denotes ordinary differentiation. The energy is expressed by the surface integral;

$$E=\lim_{r\to\infty}\int_{r=constant}U_0^{0\zeta}\eta_{\zeta}dS \tag{7}$$

where η_{ζ} is the unit three-vector normal to surface element dS.

III. CALCULATIONS

The line-element represented a static spherically symmetric stringy charged black hole is given by [28]

$$ds^2=(1-\frac{2M}{r})dt^2-(1-\frac{2M}{r})^{-1}dr^2-(1-\frac{\alpha}{r})r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{8}$$

where

$$\alpha=\frac{Q^2}{M}e^{-2\Phi_0} \tag{9}$$

M and Q are the mass and charge parameters, respectively, Φ_0 is the asymptotic value of dilation field. This metric almost identical to the Schwarzschild space-time. The only difference is that the areas of the spheres of constant r and t depend on the charge Q [28].

The matrice of the $g_{\mu\nu}$ is given by

$$\begin{pmatrix}(1-\frac{2M}{r}) & 0 & 0 & 0\\ 0 & -(1-\frac{2M}{r})^{-1} & 0 & 0\\ 0 & 0 & -(1-\frac{\alpha}{r})R^2 & 0\\ 0 & 0 & 0 & -(1-\frac{\alpha}{r})R^2\sin^2\theta\end{pmatrix} \tag{10}$$

and its inverse $g^{\mu\nu}$ defined by

$$\begin{pmatrix} (1-\frac{2M}{r})^{-1} & 0 & 0 & 0 \\ 0 & -(1-\frac{2M}{r}) & 0 & 0 \\ 0 & 0 & -(1-\frac{\alpha}{r})^{-1}\frac{1}{R^2} & 0 \\ 0 & 0 & 0 & -(1-\frac{\alpha}{r})^{-1}\frac{1}{R^2\sin^2\theta} \end{pmatrix}. \tag{11}$$

The general form of the tetrad, h_i^{μ}, having spherical symmetry, was given by Robertson [28]. In the Cartesian form it can be written as

$$h_0^0 = i\Upsilon, \qquad h_a^0 = \kappa x^a, \qquad h_0^{\alpha} = i\Pi x^{\alpha}, \qquad h_a^{\alpha} = \zeta\delta_a^{\alpha} + \Psi x^a x^{\alpha} + \epsilon_{a\alpha\beta}\Delta x^{\beta} \tag{12}$$

where $\Upsilon, \zeta, \kappa, \Pi, \Psi$, and Δ are functions of t and $r = \sqrt{x^a x^a}$, and the zeroth vector e_0^{μ} has the factor $i = \sqrt{-1}$ to preserve Lorentz signature. We impose the boundary condition that in the case of $r \to \infty$ the tetrad given above approaches the tetrad of Minkowski space-time, $h_a^{\mu} = \mathtt{diag}(i, \delta_a^{\mu})$ (where $a = 1, 2, 3$).

Using the general coordinate transformation

$$h_{a\mu} = \frac{\partial \mathbf{X}^{\nu'}}{\partial \mathbf{X}^{\mu}} h_{a\nu} \tag{13}$$

where $\{\mathbf{X}^{\mu}\}$ and $\left\{\mathbf{X}^{\nu'}\right\}$ are, respectively, the isotropic and Schwarzschild coordinates (t, r, θ, ϕ). In the spherical, static and isotropic coordinate system $\mathbf{X}^1 = r\sin\theta\cos\phi$, $\mathbf{X}^2 = r\sin\theta\sin\phi$, $\mathbf{X}^3 = r\cos\theta$. We obtain the tetrad components of $h_a{}^{\mu}$ as

$$\begin{pmatrix} \frac{i}{\sqrt{1-\frac{2M}{r}}} & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi\sqrt{1-\frac{2M}{r}} & \frac{\cos\theta\cos\phi}{r\sqrt{1-\frac{\alpha}{r}}} & -\frac{B\sin\phi}{r\sin\theta\sqrt{1-\frac{\alpha}{r}}} \\ 0 & \sin\theta\sin\phi\sqrt{1-\frac{2M}{r}} & \frac{\cos\theta\sin\phi}{r\sqrt{1-\frac{\alpha}{r}}} & \frac{B\cos\phi}{r\sin\theta\sqrt{1-\frac{\alpha}{r}}} \\ 0 & \cos\theta\sqrt{1-\frac{2M}{r}} & -\frac{\sin\theta}{r\sqrt{1-\frac{\alpha}{r}}} & 0 \end{pmatrix} \tag{14}$$

where $i^2 = -1$. The required non-vanishing component of $U_{\mu}^{\nu\beta}$ is [30, 31]

$$U_0^{01} = \frac{2M}{r\kappa}(r-\alpha)\sin\theta \tag{15}$$

From this point of view, the energy of a stringy charged black hole is

$$E(r) = M(1-\frac{\alpha}{r}), \tag{16}$$

at the large distances ($r \to \infty$)

$$\lim_{r\to\infty} E(r) = M. \tag{17}$$

IV. DISCUSSION

In literature, there are several calculations on the energy(due to matter plus fields) distribution of charged black holes. The energy distribution was found that it depends on the mass M and the charge Q. For example; Chamorro-Virbhadra [14] and Xulu[16] showed, considering the general relativity analogs of Einstein and Møller's definitions, that the energy of a charged dilaton black hole depends on the value ω which controls the coupling between the dilaton and the Maxwell fields.

$$E_{Einstein} = M - \frac{Q^2}{2R}(1-\omega^2), \qquad E_{Moller} = M - \frac{Q^2}{R}(1-\omega^2) \tag{18}$$

Also, Virbhadra [12, 13] and Xulu [16] obtained that the energy distribution in the sense of Einstein and Møller disagree in general. Next, Lessner [11] showed that the Møller energy-momentum complex is a powerful concept of

energy and momentum. Furthermore, according to the Cooperstock hypothesis [32], the energy is confined to the region of non-vanishing energy-momentum tensor of matter and all non-gravitational fields.

This paper is devoted to calculate the energy distribution (due to matter plus fields including gravitation) associated with a stringy charged black hole. Therefore, we investigated the Møller energy-momentum definition in teleparallel gravity. We found the same energy obtained by Gad in general relativity. The total energy depends on the black hole mass M and charge Q,

$$E(r) = M(1 - \frac{\alpha}{r}) \tag{19}$$

without charge ($Q = 0$ this means $\alpha = 0$), the energy is obtained as

$$E(r) = M. \tag{20}$$

at the large distances ($r \to \infty$), we also find the same energy as in the case $Q = 0$.

Finally, this paper sustains (a) the importance of the energy-momentum definitions in the evaluation of the energy distribution of a given space-time and (b) the viewpoint of Lessner that the Møller energy-momentum complex is a powerful concept of energy and momentum and (c) the results by Virbhadra and Xulu. Furthermore, our result is also independent of the teleparallel dimensionless coupling constant, which means that it is valid not only in the teleparallel equivalent of general relativity, but also in any teleparallel model.

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