OPTIMAL LIQUIDATION WITH CONDITIONS ON MINIMUM PRICE

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ABSTRACT

OPTIMAL LIQUIDATION WITH CONDITIONS ON MINIMUM PRICE

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The classical optimal trading problem is the closure of an initial position q_0 in a financial asset over a time interval $[0T]$; the trader tries to maximize an expected utility under the constraint $q_T = 0$, which is the liquidation constraint. Given that the trading takes place in a stochastic environment, the constraint $q_T = 0$ may be too restrictive; the trader may want to relax this constraint or slow down/stop trading depending on price behavior. The goal of this thesis is the formulation and a study of these types of modified liquidation orders. We introduce two new parameters to the stochastic optimal control formulation of this problem: a process I taking values in $\{0, 1\}$ and a measurable set $\boldsymbol{S}$. The set $\boldsymbol{S}$ prescribes when full liquidation is required and I prescribes when trading takes place. We give four examples for $\boldsymbol{S}$ and I which are all based on a lower bound specified for the price process. We show that the minimal supersolution of a related backward stochastic differential equation (BSDE) with a singular terminal value and with a convex driver term gives both the value function and the optimal control of the modified stochastic optimal control problem. The novelties of the BSDE arising from the modified control problem are as follows: the relaxation of the constraint $q_T = 0$ implies that the terminal value of the BSDE can take negative values; this and the convexity of the driver imply that the driver is no longer monotone and results from the currently available literature giving the minimal supersolution of this type of BSDE are not directly applicable. The same aspects of the problem imply that the BSDE can explode to $-\infty$ backward in time.

To tackle these we introduce an assumption that balances the market volume process and the permanent price impact in the model over the trading horizon. The BSDEs reduce to PDE for Markovian price processes; we also present an analysis of these PDE for a Markovian price process involving stochastic volatility.

We quantify the financial performance of our models by the percantage difference between the initial stock price and the average price at which the position is (partially) closed in the time interval $[0, T]$. We note that this difference can be divided into three pieces: one corresponding to permanent price impact (A_1), one corresponding to random fluctuations in the price (A_2) and one corresponding to transaction/bid-ask spread costs (A_3). A_1 turns out to be a linear function of $1 - q_T/q_0$, the portion of the portfolio that is closed; therefore, its distribution is fully determined by that of q_T/q_0. We provide a numerical study of the distribution of q_T/q_0 and the conditional distributions of A_2 and A_3 given q_T/q_0 under the assumption that the price process is Brownian for a range of choices of I and $\boldsymbol{S}$.

Keywords: Optimal Liquidation, BSDE, Minimum Price

ÖZ

MİNİMUM FİYAT KISITI ALTINDA OPTİMAL LİKİDİTASYON

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Optimal likidasyon veya pozisyon kapatma, belirli bir $[0, T]$ zaman aralığında finansal bir varlıkta bulunan bir q_0 pozisyonunun kapatılması sorusudur. Yatırımcının amacı bu zaman aralığında pozisyonu tam kapatarak işlemden gerçekleşecek kazancın beklenen faydasını maksimize etmektir. Likidasyon işlemleri stokastik ortamlarda gerçekleştiği için pozisyonu tam kapatma kısıtı çok bağlayıcı olabilir. Yatırımcı fiyattaki değişimleri baz alarak tam likidasyon kısıtını gevşetmek veya alım satım işlemlerini yavaşlatmak/durdurmak isteyebilir. Bu tezin amacı bu esnekliklere sahip optimal pozisyon kapatma emirlerinin formülasyonu ve bunların matematiksel olarak çalışılmasıdır. Bu esneklikte emirler üretmek için ilgili stokastik optimal kontrol problemine iki yeni parametre eklenmiştir: $\{0, 1\}$ değerlerini alan bir I süreci ve ölçülebilir bir $\boldsymbol{S}$ olayı. I ne zaman işlem yapılabildiğini belirlerken $\boldsymbol{S}$ kümesi tam pozisyon kapatma işleminin koşullarını belirler. Fiyat süreci için yatırımcının belirleyeceği bir alt limiti baz alan dört farklı $\boldsymbol{S}$ ve I örneği verilmiştir. Önerilen yeni stokastik optimal kontrol sorusuna karşılık gelen geriye dönük stokastik denklemi (BSDE) belirlenmiş; bu denklemin minimal üstçözümlerinin stokastik optimal kontrol probleminin hem değer fonksiyonunu hem de optimal kontrolunu verdiği gösterilmiştir. Fiyat sürecini Markov olduğunda BSDE'ler kısmı differansiyel denklemlere dönüşmektedirler (PDE). Stokastik volatiliteli Markovian fiyat süreçleri için bu PDE'lerin analizleri de verilmiştir.

Önerilen algoritmanın finansal performansı, pozisyonun (kısmen) kapatıldığı ortalama fiyatın varlığın başlangıç fiyatından yüzde sapması ile ölçülmüştür. Bu sapma üç parçaya ayrılabilir: kalıcı fiyat etkisine bağlı bir parça (A_1), fiyattaki stokastik dalgalanmalarla ilgili bir parça (A_2) ve alış-satış fiyat farkı maliyeti ve işlem ücretleri ile ilgili bir parça (A_3). A_1, $1 - q_T/q_0$'nin doğrusal bir fonksiyonu olduğu ve bu sebeple dağılımının tamamen q_T/q_0 (pozisyonun kapatılmayan kısmının başlangıçtaki büyüklüğüne oranı)'ın dağılımı tarafından belirlendiği gözlenmiştir. Fiyat sürecinin Brownian olduğu varsayımı altında I ve $\boldsymbol{S}$'nin dört farklı değeri için q_T/q_0'ın dağılımı ve A_2 ve A_3'ün q_T/q_0'a göre şartlı dağılımları numerik olarak hesaplanmış ve bunların model parametreleriyle nasıl değiştiği numerik olarak gösterilmiştir.

Anahtar Kelimeler: Optimal Likiditasyon, BSDE, Minimum Fiyat

To my beloved parents, Hatice and Sultan.

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CHAPTER 1

INTRODUCTION

There is a range of order types available to an investor to construct or liquidate a position on an asset; the book [14] presents the following: implementation shortfall (IS), target close (TC) and volume weighted average price (VWAP). Given a trading horizon $[0, T]$, all of these order types are constrained to attain a target position at terminal time T. Mathematically this is expressed as the constraint $q_T = 0$, where q_t denotes the position of the investor at time t (we'll always assume $q_0 > 0$, i.e., an initial long position, for a more brief presentation; everything below applies to a short position $q_0 < 0$ by multiplying the relevant quantities by -1). Given that the order is being executed in a stochastic environment, this constraint can be too restrictive. For example, in IS orders the goal is to close an initial position near the initial price S_0; it may happen that the price drops substantially during the trading interval and the investor holding the position may no longer wish to be strict about closing the position. The goal of the present work is to offer algorithms that offer this type of flexibility in execution. We focus on the IS type orders targeting the initial price S_0 but similar ideas can be considered for other types of orders. Firstly we gave a brief summary of literature Chapter 2. Later we review the stochastic optimal control formulation of the standard IS order in Chapter 3.

The value function of this control problem has a backward stochastic differential equation (BSDE) representation where the constraint $q_T = 0$ is represented by the singular terminal condition $Y_T = \infty$; this connection was first observed in [4] for a filtration generated by a Brownian motion and a quadratic driver function; it was generalized to more general filtration and driver terms in [17] and other works (see

the references in Chapter 2). The BSDE representation will be our primary tool for the analysis of the control problems we propose. Chapter 4 reviews the BSDE formulation of the stochastic optimal control problem for the standard IS order. In section 3.1 we propose two ways the IS order can be modified to delay/stop liquidation depending on price behavior 1) by relaxing the full liquidation constraint if the price is too low 2) stopping/pausing trade if the price is too low. The first modification is represented as a terminal cost for the stochastic optimal control problem while the second modification is represented by setting market volume to 0 when the price is too low (see (3.10), (3.11) and (3.12)). These two modifications are parameterized by a measurable set $\boldsymbol{S}$ and a process I taking values $\{0, 1\}$. The set $\boldsymbol{S}$ prescribes when full liquidation is required and I prescribes when trading takes place. We give four examples for $\boldsymbol{S}$ and I in section 3.1 which are all based on a lower bound L specified for the price process. It is explained in the same section that both of these modifications have natural representations in the BSDE framework: the first modification corresponds using $\boldsymbol{S}$ to modify the terminal value of the BSDE while the second modification corresponds to multiplying the driver of the BSDE by I. The resulting BSDE is given in (3.18) and (3.13). The analysis of this BSDE is presented in Chapter 4. The main results of this chapter are: Proposition 4.2 and Proposition 4.3; the first of these proves the existence of the minimal super solution to the BSDE (3.18) and the second proves that this minimal super solution gives us the value function and optimal control of the modified stochastic optimal control problem (3.12). When the price process is assumed Markovian these BSDE also have PDE representations, which are also useful in the actual calculation of the value function and the optimal control. The PDE aspect is explored in Chapter 5. A popular choice for price dynamics in finance applications is the stochastic volatility model. To the best of our knowledge, it is rarely treated in the context of optimal liquidation; in Chapter 5 we assume the price dynamics to follow this model and explore its treatment in the PDE analysis of the problem.

The main novelty of the BSDE (3.18) is the following: relaxing the constraint $q_T = 0$ implies that the terminal condition (3.13) can be negative. But the driver of the BSDE (3.18) is not monotone in $y \in \mathbb{R}$. This means that prior results from [16, 17] establishing the existence of the minimal supersolution for singular terminal valued

BSDE (which assume monotone generators) are not directly applicable. Secondly, the same aspects of the problem (a terminal condition that can take negative values and a generator that is superlinear in y) imply that the minimal supersolution of the BSDE can explode to $-\infty$ backwards in time. The set of terminal conditions ξ that arise from modified IS orders have bounded negative parts, i.e., they satisfy $\xi^- < K < \infty$ for some constant $K > 0$. For such terminal conditions a simply way to deal with the nonmonotonicity of the driver is to replace it by a monotone generator over $(-K, \infty)$; this is what we do in Chapter 4. Once this is done results on BSDE with monotone generators can be applied to the problem. To deal with the possibility of explosion to $-\infty$ backward in time we derive an apriori lower bound process on any supersolution of the BSDE with a terminal condition whose negative part is bounded by K. We introduce an assumption on the permanent price impact parameter and market volume that guarantees the existence and boundedness of the lowerbound process over the interval $[0, T]$.

The main output of the standard IS order is the cash position X_T at time T generated by the trading algorithm, under the assumption that the price process is a Brownian motion, X_T turns out to be normally distributed whose mean and variance have simple formulas in terms of the model parameters. When we relax the IS order so that full liquidation is no longer required at terminal time, the output of the IS order consists of the pair of real random variables (q_T, X_T) where X_T is, as before, the total cash generated by the trading process and q_T is the remaining position at terminal time in the asset being traded. For the relaxed/modified IS orders, X_T is not normally distributed even when the price process is taken to be Brownian and the joint distribution of (X_T, q_T) doesn't have an explicit form. Define

$$A = \frac{X_T - (q_0 - q_T)S_0}{(q_0 - q_T)S_0} \tag{1.1}$$

A is the percentage deviation from the target price S_0 of the average price at which the position is (partially) closed in the time interval $[0, T]$. In Chapter 6 we study the joint distribution of $(q_T/q_0, A)$. We note that A can be divided into three pieces: one corresponding to permanent price impact (A_1), one corresponding to random fluctuations in the price (A_2) and one corresponding to transaction/bid-ask spread costs (A_3). A_1 turns out to be a linear function of $1 - q_T/q_0$; therefore, its distribution is fully determined by that of q_T/q_0. We provide a numerical study of the distribution of q_T/q_0

and the conditional distributions of A_2 and A_3 given q_T/q_0 under the assumption that $\bar{S}_t = \sigma W_t$ for the cases (3.14),(3.15),(3.16) and (3.17) The same Chapter also provides numerical examples of the sample path behavior of the optimal controls of these four modified IS orders. Chapter 7 comments further on the models presented in this work and on possible future research.

CHAPTER 2

LITERATURE REVİEW

After the liquidity crisis in 2007-2008 the optimal liquidation problem become really popular among researchers.The first paper on optimal execution of portfolio is Bertsimas and Lo 1998 [8]. They modeled a dynamic optimal trading strategies which minimize the expected cost of trading a large portfolio. The model proposed that the best way to trade will be simply breaking down the total portfolio into equal waves and trade them in equal time horizons. Optimality means the minimization of expected execution cost. The main weakness of the model was that it did not consider price movement risk while trading which effects the expected cost of trading immensely. That is the main reason why their model frequently suggests splitting the large order equally into small orders to be executed at constant speed.

In 1999 Almgren and Chriss proposed a model which takes into account both the expected cost of execution and the price risk during execution [2],[3]. The proposed model gives a closed form formula for the optimal execution strategy which is deterministic and can be computed before the execution starts.

In the literature there are two main impact functions which are used to model optimal execution problems. The first one is deterministic impact functions which were common in early research. This model is easily solved with calculus of variation methods. The second is stochastic impact models. The resulting stochastic control problem can be solved via dynamic programming or by stochastic maximum principle. Both cases induce to solution of a singular terminal condition of the Hamilton-Jacobi-Bellman (HJB) equation or the adjoint equation in the stochastic maximum principle [25].

Ankirchener et al [5] showed that the value function of stochastic optimal control problem of optimal liquidation can be represented as the solution of singular BSDE.

Ankirchener and Kruse [6] solve an stochastic price impact in liquidation problem and verified the optimality of their results with penalization method. Additionally they derive optimal trading strategies for closing financial asset positions in markets with stochastic price impact and non-zero returns.

Kruse and Popier [17] extend the model suggested by Ankirchener et al [5] allowing general filtration, driver and terminal cost, show the existence of the minimal supersolution of the associated BSDE with singular terminal values and prove that the value function and the optimal control can be expressed in terms of this minimal supersolution.

Grawe et all [13] extend the optimal liquidation problem to allow price-dependent impact function and passive orders sent to a "dark pool." They characterize the value function in terms of the unique smooth solution to a PDE with singular terminal value and provide a study of its asymptotic behavior at terminal time.

Horst et all [15] set up an optimal liquidation control problem with possibly degenerate coefficients under market impact. They describe the value function by a degenerate backward stochastic partial differential equation with singular terminal value.

Grawe et all [12] extend their work [13] to a non markovian case and establish existence, uniqueness, and regularity of solution results for a class of backward stochastic partial differential equations with singular terminal condition.

Popier and Zhou [23] consider the optimal liquidation problem in a setting where the underlying probability measure is not completely known and focus on the worst case expected transition costs. They provide a solution to this problem by a second order BSDE.

CHAPTER 3

DEFINITIONS

We adopt the price and trading dynamics of the Almgren Chriss framework as presented [14, Chapter 3]. Everything is assumed to be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a quasi left continuous and right continuous filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$ (see [24]). The market volume at time t is denoted by Vol_t, which is a nonnegative process adapted to the filtration $\mathbb{F}$. The position of the investor at time t is q_t. The process q is assumed to be absolutely continuous in the time variable, let v denote its derivative: $v_t = q'_t$; q and v are adapted to $\mathbb{F}$. The midprice process satisfies:

$$S_t = S_0 + \bar{S}_t + \int_0^t \kappa(v_s)ds, \tag{3.1}$$

where $\bar{S}$ is a martingale adapted to $\mathbb{F}$. and $v \mapsto \kappa(s)$ is the permanent price impact function. The standard choice for κ is a linear function, i.e., $\kappa(v) = kv$ for some constant $k > 0$, see [14, Chapter 3]. We will be primarily working with a linear price impact. The actual trading price at time t is $S_t + g_t(v_t/\text{Vol}_t)$ where g_t models transaction costs and the bid-ask spread (g depends on t, ω and v_t/Vol_t). The process g is often specified via the so-called execution cost function L_t: $L_t(\rho) = \rho g(\rho)$. The simplest and classical choice for L_t is a quadratic function of $\rho = v_t/\text{Vol}_t$: $L_t(\rho) = \eta_t\rho^2$, $\eta_t > 0$, where $\eta > 0$ is an adapted process; see [14, 4]. In this thesis we focus on this choice of L The actual trading price at time t, expressed in terms of L is

$$S_t + \frac{\text{Vol}_t}{v_t} L_t(v_t/\text{Vol}_t).$$

The cash position that q generates is

$$X'_T = -\int_0^T \left(S_t + \frac{\text{Vol}_t}{v_t} L_t\left(\frac{v_t}{\text{Vol}_t}\right)\right) v_t dt \; = -\int_0^T S_t v_t dt - \int_0^T \text{Vol}_t L_t\left(\frac{v_t}{\text{Vol}_t}\right) v_t dt \tag{3.2}$$

($v_t = q_t' < 0$ corresponds to selling, hence an increase in X, and $v_t > 0$ corresponds to buying, a decrease in X). Integrating the first term by parts gives:

$$\begin{aligned}\int_0^T S_t v_t dt &= S_T q_T - S_0 q_0 - \int_0^T q_t dS_t \\ &= S_T q_T - S_0 q_0 - \int_0^T q_t d\bar{S}_t - \int_0^T k v_t q_t dt \\ &= S_T q_T - S_0 q_0 - \int_0^T q_t d\bar{S}_t - \frac{k}{2}(q_T^2 - q_0^2)\end{aligned}$$

Then

$$X_T' = q_0 S_0 - q_T S_T + \frac{k}{2}\left(q_T^2 - q_0^2\right) + \int_0^T q_t d\bar{S}_t - \int_0^T \text{Vol}_t L_t \left(\frac{v_t}{\text{Vol}_t}\right) dt.$$

In the classical formulation of the problem the position is required to be closed fully at terminal time, i.e., $q_T = 0$. Then the terminal cash position is

$$X_T = q_0 S_0 - \frac{k}{2} q_0^2 + \int_0^T q_t d\bar{S}_t - \int_0^T \text{Vol}_t L \left(\frac{v_t}{\text{Vol}_t}\right) dt; \tag{3.3}$$

to identify the optimal liquidation strategy q one solves

$$\max_{q \in \mathcal{A}} \mathbb{E}[U(X_T)], \tag{3.4}$$

where

$$\mathcal{A}_0 = \{q : q \text{ is progressively measurable, absolutely continuous }, q_T = 0\}$$

and U is the utility function of the trader. A standard choice for the utility function is $U(x) = -e^{-\gamma x}$, where γ is the risk aversion and parameter of the investor. In the current work we will be focusing on the case $\gamma \to 0$, for which (3.5) reduces to

$$\max_{q \in \mathcal{A}_0} \mathbb{E}[X_T]. \tag{3.5}$$

Assuming that the local martingale $\int_0^t q_t d\bar{S}_t$ is a martingale and taking the expectation of X_T in (3.3) we see that the above control problem is equivalent to

$$\min_{q \in \mathcal{A}} \mathbb{E}\left[\int_0^T \text{Vol}_t L \left(\frac{v_t}{\text{Vol}_t}\right) dt\right] \tag{3.6}$$

This is the standard version of the stochastic optimal control formulation of the IS order in the Almgrenn Chriss framework for $\gamma = 0$. Our goal is to formulate modifications of this control problem; the modifications are proposed and discussed in section 3.1 below.

Let V denote the value function of the control problem (3.6)

$$V(q,t) = \min_{q \in \mathcal{A}_t} \mathbb{E}\left[\int_t^T \text{Vol}_t L\left(\frac{v_t}{\text{Vol}_t}\right) dt\right].$$

For $p > 1$, let $\hat{p}$ be the Holder conjugate of q, i.e., $1/p + 1/\hat{p} = 1$. For $p > 1$ let Y be the solution of the BSDE:

$$dY_t = (p-1)\frac{\text{Vol}_t^{\hat{p}-1}}{\eta_t} Y_t^{\hat{p}} dt + Z_t dW_t, \quad Y_T = \infty. \tag{3.7}$$

For $L_t(\rho) = \eta_t \rho^p$, [4] derive the following BSDE representation for the value function V:

$$V(0,q) = q^p Y_0.$$

Note that the terminal condition of the BSDE in (3.7) is ∞. This type of terminal condition is called "singular". This terminal condition corresponds to the constraint $q_T = 0$ of the stochastic optimal control problem. The work [17] extend these results to general driver, filtration and terminal conditions. We refer to further related work in Chapter 2.

3.1 Modifications

When full liquidation is no longer required, i.e., when we don't have the constraint $q_T = 0$, the output of the trading process at time T will be (X'_T, q_T) where X'_T is the cash generated by the trading process and q_T is the position remaining in the asset being traded. To formulate a utility maximization problem similar to (3.5) the q_T term must be represented in monetary terms, i.e., a money value must be assigned to the position q_T. In the present work we mainly focus on the following simple choice: the market value of the position at terminal time T ignoring trading costs, i.e., $q_T S_T$. With this choice, the monetary value of the position (X'_T, q_T) is

$$X_T = X'_T + q_T S_T = q_0 S_0 - \frac{k}{2} q_0^2 + \frac{k}{2} q_T^2 + \int_0^T q_t d\bar{S}_t - \int_0^T \text{Vol}_t L_t\left(\frac{v_t}{\text{Vol}_t}\right) dt. \tag{3.8}$$

Recall that our goal is to modify the IS order to not liquidate depending on price behavior in two ways 1) by relaxing the full liquidation constraint if the price is too low 2) stopping/pausing trade if the price is too low. The following formulation allows

both of these possibilities. Let I_t be an adapted process taking values in $\{0,1\}$. Let $S \in \mathcal{F}_T$ be a measurable set. Define

$$\mathcal{A}_{I,S} = \{q : q' \text{progressively measurable }, q_t' d_t \text{ absolutely continuous with respect to } I_t dt, q_T(\omega) = 0 \text{ if } \omega \in \boldsymbol{S}\}. \tag{3.9}$$

We modify (3.5) to

$$\max_{q \in \mathcal{A}_{I,S}} \mathbb{E}[X_T]. \tag{3.10}$$

The formula (3.8) implies that this control problem is equivalent to:

$$\min_{q \in \mathcal{A}_{I,S}} \mathbb{E}\left[\int_0^T \eta_t \frac{v_t^2}{\text{Vol}_t} dt - \frac{k}{2} q_T^2\right] \tag{3.11}$$

If we adopt the conventions

$$\infty \cdot 0 = 0, \int_0^T \eta_t \frac{v_t^2}{I_t \text{Vol}_t} dt = \infty \text{ if } v \text{ is not absolutely continuous with respect to } I,$$

(3.11) can be formulated as

$$\min_{q \in \mathcal{A}_0} \mathbb{E}\left[\int_0^T \eta_t \frac{v_t^2}{\text{vol}_t} dt + \xi q_T^2\right] \tag{3.12}$$

where

$$\text{vol}_t = \text{Vol}_t I_t,$$
$$\xi = \left(-\frac{k}{2} \mathbf{1}_{\boldsymbol{S}^c} + \infty \cdot \mathbf{1}_{\mathbf{S}}\right). \tag{3.13}$$

Note that the I process controls when trading takes place and the event $\boldsymbol{S}$ controls when full liquidation is required.

The midprice process S consists of two components: $\bar{S}_t$ and $k(q_t - q_0)$. The first component, $\bar{S}$, is the random component of the change in the midprice; any large and unpredictable drop in price that the investor may fear can arise from this component. Given this observation one possible formulation is that the investor puts a lowerbound L on this component and requires that the liquidation process take into account whether $\bar{S}$ goes below L. This is the point of view we take in this work. Define

$$\tau_L = \inf\{s > 0 : \bar{S}_s \leq L\}$$
$$\tau_{t,L} = \inf\{s > t : \bar{S}_s \leq L\}.$$

Given a lowerbound L on $\bar{S}$, here are some possible choices for I and $\boldsymbol{S}$:

$$I_t = I_t^{(1)} \doteq 1, \boldsymbol{S} = \boldsymbol{S}^{(1)} \doteq \{\bar{S}_T > L\} : \tag{3.14}$$

trading is allowed at all times, full liquidation is forced only when the terminal price $\bar{S}_T$ is above L

$$I_t = I_t^{(2)} \doteq \mathbf{1}_{\{t \leq \tau_L\}}, \boldsymbol{S} = \boldsymbol{S}^{(2)} \doteq \{\tau_L > T\} : \tag{3.15}$$

trading stops once $\bar{S}$ hits the lower bound L; full liquidation takes place if $\bar{S}$ remains above L throughout $[0, T]$.

$$I_t = I_t^{(3)} \doteq \mathbf{1}_{[L,\infty)}(\bar{S}_t), \boldsymbol{S} = \boldsymbol{S}^{(3)} = \{\tau_{L,T-\delta} \geq T\} : \tag{3.16}$$

trading pauses when the price $\bar{S}$ is below L, full liquidation takes place if the price process $\bar{S}$ remains above L in the time interval $[T - \delta, T]$ for a given time length $\delta > 0$.

Let us comment on the $\delta > 0$ parameter in this formulation: essentially we would like to continue with the liquidation when the price is not too below our target price S_0 and close the position fully if the terminal price is also near our target price. However, allowing trading (re)start arbitrarily close to T and forcing a full liquidation implies high transaction costs (in fact, ∞ transaction costs under the current model and not realistic in practice). This is the reason for the $\delta > 0$ parameter: full liquidation is forced only if the price remains above L in the time interval $[T - \delta, T]$.

In the last formulation trading pauses once $\bar{S}$ hits L; if $\bar{S}$ is a continuous diffusion process , once it hits L, it will hit L infinitely often and the trading process will switch on and off infinitely often as $\bar{S}$ crosses L. One can get a discrete sequence of on and off trading intervals by putting a buffer of size $b > 0$ above L between trading and no trading; once trading pauses, it is turned back on once $\bar{S}$ goes above $b + L$. The corresponding I and S are expressed through the following sequence of hitting

times:

$$\begin{aligned}
\tau_{L,0} &= \tau_L, \\
\tau_{b,0} &= \inf\{t : t > \tau_{L,0}, \bar{S}_t \geq b + L\}, \\
\tau_{L,k} &= \inf\{t : t > \tau_{b,k-1}, \bar{S}_t \leq L\}, \\
\tau_{b,k} &= \inf\{t : t > \tau_{L,k}, \bar{S}_t \geq b + L\}. \\
\bar{\tau}_{b,k} &= \begin{cases} \tau_{b,k} & \text{if } \tau_{b,k} + \delta < T \\ T & \text{otherwise.} \end{cases}
\end{aligned}$$

Adding a buffer of size $b > 0$ between no-trading and trading in (3.16) amounts to the following definitions:

$$I_t = I_t^{(4)} \doteq \sum_{k=-1}^{\infty} \mathbf{1}_{[\tau_{b,k} \leq t \leq \tau_{L,k+1}]}, \quad \boldsymbol{S} = \boldsymbol{S}^{(4)} \doteq \{I_T = 1\}. \tag{3.17}$$

The BSDE corresponding to the modified stochastic optimal control problem (3.12) is again of the form (3.7):

$$dY_t = \text{Vol}_t \frac{1}{\eta_t} Y_t^2 ds + dM_t, \tag{3.18}$$

with terminal condition

$$Y_T = \xi, \tag{3.19}$$

where

$$\xi = -\frac{k}{2} 1_{\boldsymbol{S}^c} + \infty \cdot 1_{\boldsymbol{S}}. \tag{3.20}$$

Parallel to (3.7), the singular term $\infty \cdot 1_{\boldsymbol{S}}$ in the terminal condition corresponds to the constraint $q_T = 0$ over the event $\boldsymbol{S}$ of the stochastic optimal control problem. A pair (Y, M) is said to be a supersolution of this BSDE if it satisfies (3.18) and

$$\liminf_{t \to T} Y_t \geq \xi \tag{3.21}$$

holds. It is said to be minimal if $Y'_t \geq Y_t$ for any other supersolution (Y', M'). Two of our main aims in the rest of the paper are the following: 1) prove that the BSDE (3.18, 3.19) have a minimal supersolution $(Y^{\min}, M^{\min})$ and 2) this minimal supersolution gives us the value function of (3.12). We tackle these problems in the next chapter. To ease notation we will assume $\eta_t = 1$; otherwise we can modify the Vol process to

$$\widetilde{\text{Vol}}_t = \text{Vol}_t / \eta_t.$$

and base all of our arguments on this modified volume process.

CHAPTER 4

BSDE ANALYSIS

The goal of this Chapter is the prove the existence of the minimal supersolution to the BSDE (3.18), (3.19) and to show that this minimal supersolution gives us both the value function and the optimal control for the modified stochastic optimal control problem (3.12). We will be working with a general terminal condition ξ that also cover the cases listed in (3.13) that arise from the stochastic optimal control formulation of the modified IS order. The key issue in the solution the BSDE (3.18,3.19) is the following: the terminal conditions that are of interest to us (given in (3.13)) can take negative values, this combined with the driver $t \mapsto \text{vol}_t Y_t^2$ means that the solution of (3.18,3.19) can explode to $-\infty$ backwards in time. The negative part of the terminal conditions (3.13) is bounded by $k/2$; for this reason, it suffices, for our purposes to focus on terminal conditions with bounded negative real part, i.e., we assume

$$\xi^- \leq K,$$

for some $K > 0$. We would like our BSDE to have solutions in the time interval $[0, T]$. To ensure that the solution of the BSDE doesn't explode to $-\infty$ before time 0, the K term and the driver of the BSDE have to balance each other; we express this in the following assumption:

Assumption 1. *Vol is non-negative and deterministic and satisfies*

$$K \int_0^T Vol_t dt < 1, \tag{4.1}$$

or it is bounded by a constant $\overline{vol} > 0$ *such that*

$$KT\overline{vol} < 1. \tag{4.2}$$

We will assume throughout that either (4.1) or (4.2) holds. Note that when Vol satisfies one of these, vol $= I$ Vol also does because $I \in \{0, 1\}$. Under Assumption 1 the lowerbound process z is defined as follows:

$$z_t = -\frac{1}{K^{-1} - \int_t^T \mathrm{vol}_s ds}, \tag{4.3}$$

if (4.1) holds; and

$$z_t = -\frac{1}{K^{-1} - \overline{\mathrm{vol}}(T-t)} \tag{4.4}$$

if (4.2) holds.

Lemma 1. *z of* (4.3) *satisfies*

$$\frac{dz}{dt} - \mathrm{vol}_t z^2 = 0, \tag{4.5}$$

and z of (4.4) *satisfies*

$$\frac{dz}{dt} - \overline{\mathrm{vol}}\, z^2 = 0. \tag{4.6}$$

Both z satisfy $z_T = -K$. Under Assumption 1 z is increasing on $[0, T]$ and satisfies for any t, $-\infty < z_0 \leq z_t \leq -K$.

Proof. Assumption 1 implies

$$K^{-1} - \int_t^T \mathrm{vol}_s ds > 0 \quad \text{or} \quad K^{-1} - \overline{\mathrm{vol}}(T-t) > 0, \tag{4.7}$$

for $t \in [0, T]$. Therefore, $z_t < 0$ on $[0, T]$. Nonnegativity of vol and $\overline{\mathrm{vol}}$ imply that z is increasing. One can check by differentiation that z of (4.3) satisfies (4.5) and z of (4.4) satisfies (4.6). □

The terminal condition ξ of (3.20) is singular, i.e., it can take the value ∞. The standard way to obtain the minimal supersolution of a BSDE with a singular terminal condition (a terminal condition that can take the value ∞) is approximation below, i.e., we truncate the terminal condition ξ to $\xi \wedge L$, solve the resulting BSDE and let $L \nearrow \infty$. Therefore, the treatment of singular terminal values requires the solution of the same BSDE with bounded/integrable terminal values. The next Proposition adressses integrable terminal values:

Proposition 4.1. *Suppose Assumption 1 holds and assume:*

- $\xi^+ \in L^\varrho(\Omega)$ *for some* $\varrho > 1$,
- ξ^- *is bounded by* K *for some* $K > 0$.

Then BSDE (3.18) *has a unique solution* (Y, Z) *such that* Y^- *is bounded and*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^+|^\varrho + \langle M\rangle_T^{\varrho/2}\right] < +\infty.$$

Moreover if ξ^+ *is bounded,* Y *is also bounded.*

Proof. Recall the lack of monotonicity for the generator $y \mapsto -\text{vol}_t y^2$. However if the negative part Y^- of the solution is bounded by some constant $\kappa > 0$, that is Y is bounded from below by $-\kappa$, then we can replace the generator $(t, y) \mapsto -\text{vol}_t y^2$ by a monotone continuous generator

$$\widetilde{f}_{-\kappa}(s, y) = -\text{vol}_t y^2 \mathbf{1}_{y\geq -\kappa} + \text{vol}_t \kappa(2y + \kappa)\mathbf{1}_{y<-\kappa}. \tag{4.8}$$

Indeed for any t, y, y'

$$(y - y')(\widetilde{f}_{-\kappa}(t, y) - \widetilde{f}_{-\kappa}(t, y')) \leq 2\text{vol}_t \kappa(y - y')^2.$$

And since $Y \geq -\kappa$,

$$Y_t = \xi + \int_t^T f(s, Y_s)ds - \int_t^T dM_s = \xi + \int_t^T \widetilde{f}_{-\kappa}(s, Y_s)ds - \int_t^T dM_s.$$

We can use uniqueness result for BSDEs driven by monotone generator. See [19, Proposition 5.24], our setting implying that a.s. vol belongs to $L^1(0, T)$.Therefore, if the solution exists and if the negative part is bounded, the solution of (3.18) is unique.

By Lemma 1, process z is bounded from below by $z_0 < 0$. Now we define the generator $\widetilde{f}_{z_0}$ and the BSDE

$$Y_t = \xi + \int_t^T \widetilde{f}_{z_0}(s, Y_s)ds - \int_t^T dM_s. \tag{4.9}$$

Since $\widetilde{f}_{z_0}$ is monotone w.r.t. y and since vol $\in L^1(0, T)$, BSDE (4.9) has a unique solution (Y, Z) (see again [19, Proposition 5.24]). Remark that for $\xi = -K$, the solution is $(z, 0)$. Since $\xi \geq -K$, the comparison principle ([19, Proposition 5.33]) states that a.s. for any t, $Y_t \geq z_t \geq z_0$. In other words (Y, M) is a solution of the BSDE (3.18), and this achieves the proof of the proposition. □

Remark 1 (On the negative part of Y)**.** *Applying Itô-Tanaka formula implies that the negative part of Y is a subsolution of the BSDE*

$$U_t = \xi^- + \int_t^T \widetilde{f}(s, U_s)ds - \int_t^T dV_s, \tag{4.10}$$

with $\widetilde{f}(t,y) = vol_t y^2 \mathbf{1}_{y\geq 0}$. Generator $\widetilde{f}$ is not monotone. However it is increasing and positive. From Lemma 1, $(U^*, V^*) = (-z, 0)$ is a bounded supersolution of this BSDE.

Following [11], we deduce the existence of a minimal bounded supersolution (U, V) which is also bounded and non-negative (see [11, Theorems 3.3 and 4.1]). Using again $\widetilde{f}_{-\kappa}$, we deduce easily that (U, V) in fact is the unique solution of the BSDE (4.10)*.*

As a by-product, we obtain a better bound: a.s. for any t, $0 \leq Y_t^- \leq U_t \leq -z_t$.

Remark 2 (On the positive part of Y)**.** *Consider the BSDE*

$$\begin{aligned}\Upsilon_t &= \xi^+ + \int_t^T vol_s \left[-(\Upsilon_s - U_s)^2 + (U_s)^2\right] \mathbf{1}_{\Upsilon_s \geq 0} ds - \int_t^T d\mathcal{M}s \\ &= \xi^+ + \int_t^T \hat{f}(s, \Upsilon_s)ds - \int_t^T d\mathcal{M}_s\end{aligned}$$

where U is the solution of (4.10)*. Since U is bounded by κ,*

$$\partial_y \hat{f}(s,y) \leq 2\mathrm{vol}_s \kappa,$$

thus driver $\hat{f}$ is monotone:

$$(y - y')(\hat{f}(t,y) - \hat{f}(t,y')) \leq 2\mathrm{vol}_t \kappa (y - y')^2.$$

Hence existence and uniqueness of the solution holds, if ξ^+ belongs to some space $L^\varrho(\Omega)$. Define $Y = \Upsilon - U$ and $M = \mathcal{M} - V$:

$$\begin{aligned}Y_t = \Upsilon_t - U_t &= \xi^+ - \xi^- + \int_t^T (\hat{f}(s, \Upsilon_s) - \widetilde{f}(s, U_s))ds - \int_t^T dM_s \\ &= \xi - \int_t^T \mathrm{vol}_s (\Upsilon_s - U_s)^p ds - \int_t^T dM_s.\end{aligned}$$

Hence (Y, M) solves BSDE (3.18)*. And a.s. for any t, $0 \leq Y_t^+ \leq \Upsilon_t$.*

Finally, we treat the case when ξ is singular: either $\mathbb{P}(\xi = +\infty) > 0$ or ξ^+ does not belong to some $L^\varrho(\Omega)$. To guarantee the existence of a minimal supersolution in this case we need the following assumption on vol and $\boldsymbol{S}$:

Assumption 2. *There exists some $\ell > 1$ and some $\epsilon > 0$ such that*

$$\mathbb{E}\left[1_{\boldsymbol{S}} \int_{T-\epsilon}^{T} \frac{1}{(\mathrm{vol}_s)^\ell} ds\right] < +\infty. \tag{4.11}$$

Recall that $\{\xi = \infty\} = \boldsymbol{S}$ corresponds to the constraint $q_T = 0$, i.e., the position is to be fully closed at time T. Assumption 2 can be interpreted as the availability of liquidity (both through Vol and I) at terminal time if full closure constraint is imposed. Our main result on singular terminal conditions is the following:

Proposition 4.2. *Suppose $\xi^- \leq K$ and Assumptions 1 and 2 hold. Then there exists a minimal supersolution $(Y^{\min}, M^{\min})$ to the BSDE* (3.18) *with terminal condition $Y_T = \xi$ such that $Y^{\min}$ has a left-limit at time T and the negative part of this minimal supersolution is bounded.*

Proof. The proof follows the same arguments as [17, Proposition 3]. Let us consider for any $L \geq 0$

$$\xi^L = \xi \wedge L.$$

Note that the related solution (Y^L, M^L) of

$$Y_t^L = \xi^L - \int_t^T \mathrm{vol}_s (Y_s^L)^2 ds - \int_t^T dM_s^L,$$

has the same upper bound $U = -z$ for the negative part $(Y^L)^-$ for any L. And by comparison principle for monotone BSDEs, arguing as in [17] leads to:

$$Y^L \nearrow Y.$$

From Remark 2, $Y^L \leq \Upsilon^L$ with

$$\Upsilon_t^L = (\xi^+ \wedge L) - \int_t^T \mathrm{vol}_s (\Upsilon_s^L)^2 \mathbf{1}_{\Upsilon_s^L \geq 0} ds - \int_t^T dM_s^L$$

From [17, Lemma 1], we have a.s. for any $t \in [T-\epsilon, T]$

$$\Upsilon_t^L \leq \frac{1}{(T-t)^2} \mathbb{E}\left[\int_t^T \frac{1}{\mathrm{vol}_s} ds \middle| \mathcal{F}_t\right].$$

Therefore from (4.11), Y_t^L, $T - \epsilon \leq t \leq T$, is finite and bounded in $L^\ell(\Omega)$ uniformly w.r.t. L. In particular for any $\eta < \epsilon$, there exists a constant C such that for any L,

$$\mathbb{E}(Y_{T-\eta}^L)^\ell \leq C.$$

Standard estimates for BSDEs show that (Y^L, M^L) converges to (Y, M) in the space $\mathbb{S}^\ell(0, T-\eta) \times \mathbb{H}^\ell(0, T-\eta)$, for any $\eta > 0$, that is (Y, M) solves (3.18) for any $0 \leq t \leq r < T$

$$Y_t = Y_r - \int_t^r \mathrm{vol}_s(Y_s)^2 ds - \int_t^r dM_s.$$

Moreover $Y^- \leq U$. And since we assume the filtration to be quasi left-continuous, we obtain that a.s.

$$\liminf_{t \to T} Y_t \geq \xi.$$

Finally minimality can be obtained as in the proof of [17, Proposition 4]. If $(\widetilde{Y}, \widetilde{Z})$ is another supersolution, we add to both solutions $\widetilde{Y}$ and Y^L the quantity $-z$ and the same arguments on $\widetilde{Y} - z$ and $Y^L - z$ lead to a.s. $\widetilde{Y} - z \geq Y^L - z$.

The only remaining problem concerns the existence of a limit at time T. Compared to [21, Theorem 2.1], we only have to deal with the negative part of $Y^{\min}$ or of Y^L, which approximates $Y^{\min}$. However we can apply the arguments of the proof of [21, Theorem 2.1] with the function $\Theta(y) = \frac{\pi}{2} - \arctan(y)$ for example:

$$\Theta(Y_t) = \mathbb{E}\left[\Theta(\xi)|\mathcal{F}_t\right] + \psi_t^+ - \psi_t^-$$

where ψ^+ and ψ^- are two non-negative supermartingales such that ψ^+ converges a.s. to zero. To obtain this result, we use that the negative part of Y^L is bounded uniformly with respect to L and also that $Z^L \mathbf{1}_{Y^L \leq 0}$ is bounded in $\mathbb{H}^2(0, T)$ (consequence of the Itô-Tanaka formula for $(Y^L)^-$). This achieves the proof.

□

Our next task is to relate the solution/minimal supersolution of the BSDE to the solution of the stochastic optimal control problem (3.12).

4.1 Solution of the stochastic optimal control problem

For ease of reference let us restate our stochastic optimal control problem:

$$\min_{q \in \mathcal{A}_0} \mathbb{E}\left[\int_0^T \frac{v_t^2}{\mathrm{vol}_t} dt + \xi q_T^2\right]; \tag{4.12}$$

as in the analysis of the BSDE, we will work with an arbitrary $\xi \geq -K$.

Our goal is to prove the following result:

Proposition 4.3. *Suppose Assumptions 1 and 2 hold, suppose* $\xi^- \leq K$ *and let* $(Y^{\min}, M^{\min})$ *be the minimal supersolution of* (3.18), (3.19). *Then*

$$q_t^* = q_0 \exp\left(-\int_0^t Y_s^{\min} vol_s ds\right), t \in [0,T), \tag{4.13}$$

(equivalently, $v_t^* = -Y_t^{\min} vol_t q_t^*$*) is the optimal state process for the stochastic optimal control problem* (4.12). *Moreover the value function of* (4.12) *at time* t *is given by*

$$V(t,q) = q^2 Y_t^{\min}.$$

The proof directly follows from the next three lemmas and is given at the end of this section. Let's call J the expression inside the $\min$ in (4.12):

$$J(v) = \int_0^T \frac{v_t^2}{\text{vol}_t} dt + \xi q_T^2 = \int_0^T \frac{v_t^2}{\text{vol}_t} dt + \xi \left(q_0 + \int_0^T v_t dt\right)^2, \quad v \in \mathcal{A}_0.$$

We start with the following observation:

Lemma 2. *If*

$$\xi^- \int_0^T \text{vol}_t dt < 1 \tag{4.14}$$

almost surely, then the functional $v \mapsto J(v)$ *is strictly convex. The Gâteaux derivative of* J *at point* v *in direction* w*, is given by*

$$\langle DJ(v), w\rangle = 2\int_0^T \frac{v_t w_t}{\text{vol}_t} dt + 2\xi \left(q_0 + \int_0^T v_t dt\right)\left(\int_0^T w_t dt\right).$$

Proof. Taking v and $\widetilde{v}$ in $\mathcal{A}_0$, and $\theta \in [0,1]$, we have

$$\begin{aligned}
&J(\theta v + (1-\theta)\widetilde{v}) - \theta J(v) - (1-\theta) J(\widetilde{v}) \\
&= -\theta(1-\theta)\left[\int_0^T \frac{(v_t - \widetilde{v}_t)^2}{\text{vol}_t} dt + \xi\left(\int_0^T (v_t - \widetilde{v}_t) dt\right)^2\right] \\
&\leq -\theta(1-\theta)\left[\int_0^T \frac{(v_t - \widetilde{v}_t)^2}{\text{vol}_t} dt - \xi^-\left(\int_0^T (v_t - \widetilde{v}_t) dt\right)^2\right] \\
&\leq \theta(1-\theta)\left[-1 + \xi^- \int_0^T \text{vol}_t dt\right] \int_0^T \frac{(v_t - \widetilde{v}_t)^2}{\text{vol}_t} dt \leq 0.
\end{aligned}$$

We use the Cauchy-Schwarz inequality for the inequality. Now for any $\epsilon > 0$ and v and w in $\mathcal{A}_0$,

$$\begin{aligned}
&\frac{1}{\epsilon}(J(v+\epsilon w) - J(v)) \\
&= 2\int_0^T \frac{v_t w_t}{\text{vol}_t} dt + \epsilon \int_0^T \frac{w_t^2}{\text{vol}_t} dt \\
&\quad + 2\xi \left(q_0 + \int_0^T v_t dt\right)\left(\int_0^T w_t dt\right) - \epsilon\xi \left(\int_0^T w_t dt\right)^2.
\end{aligned}$$

Letting ϵ to zero gives the desired formula. □

We will use the following process to relate $Y^{\min}$ and q^* to the stochastic optimal control problem (4.12):

$$N_t \doteq 2Y_t^{\min} q_t^*.$$

By its definition q^* satisfies

$$\frac{dq^*}{dt} = -Y_t^{\min} \text{vol}_t q_t^*.$$

This and integration by parts give:

$$N_t = 2\int_0^t q_s \text{vol}_s (Y_s)^2 ds - \int_0^t 2Y_s Y_s \text{vol}_s q_s ds + 2\int_0^t q_s dM_s = 2\int_0^t q_s dM_s.$$

Hence N is a local martingale. To prove that q^* is the optimal control and $Y^{\min}$ determines the value function of the stochastic optimal control problem, it suffices to prove that N is a true martingale:

Lemma 3. *Suppose N is a true martingale. Then q^* is the optimal control of* (4.12) *and*

$$Y_t^{\min} q_t^2 = \mathbb{E}\left[\int_t^T \frac{(v_s^*)^2}{vol_s} ds + \xi q_T^2 \middle| \mathcal{F}_t\right].$$

Proof. For $w \in \mathcal{A}_0$ define $\widehat{q}_t = \int_0^t w_s ds$. Integration by parts implies:

$$\int_0^T w_t \frac{2v_t^*}{\text{vol}_t} dt = \int_0^T \widehat{q}_t dN_t - N_T \widehat{q}_T = \int_0^T \widehat{q}_t dN_t - 2\xi q_T \widehat{q}_T.$$

This and Lemma 2 give

$$\langle DJ(v^*), w\rangle = \int_0^T \widehat{q}_t dN_t.$$

This, the convexity of J and the assumption that N is martingale imply

$$\mathbb{E}(J(v^*) - J(v)) \le \mathbb{E}\langle DJ(v^*), v^* - v\rangle = 0.$$

Therefore, v^* is the optimal control (unique from the strict convexity of J). Itô's formula applied to $Y_t(q_t)^2$ gives

$$\begin{aligned} d(Y_t(q_t)^2) &= (q_t)^2 \text{vol}_t(Y_t)^2 dt + (q_t)^2 dM_t + 2Y_t q_t(-Y_t \text{vol}_t q_t)dt \\ &= -\text{vol}_t(q_t Y_t)^2 dt + (q_t)^2 dM_t = -\frac{(v_t^*)^2}{\text{vol}_t} dt + (q_t)^2 dM_t. \end{aligned}$$

Therefore, again if the local martingale is a martingale,

$$Y_t q_t^2 = \mathbb{E}\left[\int_t^T \frac{(v_s^*)^2}{\text{vol}_s} ds + \xi q_T^2 \middle| \mathcal{F}_t \right].$$

In other words $Y_t q_t^2 = V(t, q_t)$ is the value function of the control problem. □

The last intermediate result we need is a version of Proposition 4.3 where ξ is integrable:

Lemma 4. *Suppose ξ is integrable. If (Y, M) is the solution of* (3.18) *with terminal condition $Y_T = \xi$, then the optimal state process q^* (resp. optimal control v^*) is given by*

$$q_t^* = q_0 - \int_0^t (Y_s \text{vol}_s) q_s^* ds \quad (\textit{resp.} \ -Y_s \text{vol}_s q_s^*).$$

Moreover $Y_0(q_0)^2$ is the value function of the control problem.

Proof. By Lemma 3 we only have to prove that the local martingales are martingales. Indeed since ξ is bounded, from Proposition 4.1, Y is bounded and for any $\ell > 1$

$$\mathbb{E}\left([M]_T^{\ell/2} \right) < +\infty.$$

Since

$$q_t^* = q_0 \exp\left(-\int_0^t Y_s \text{vol}_s ds \right)$$

is also bounded, $\int (q^*)^2 dM$ is a martingale and for $w \in \mathcal{A}_0$,

$$\int_0^T (\hat{q}_s q_s^*)^2 d[M]_s \leq C \sup_{t\in[0,T]} (\hat{q}_s)^2 [M]_T,$$

and thus

$$\mathbb{E}\int_0^T (\hat{q}_s q_s^*) d[M]_s \leq C \left[\mathbb{E} \sup_{t\in[0,T]} (\hat{q}_s)^{2\ell} \right]^{\frac{1}{\ell}} \left[\mathbb{E}\, [M]_T^{\frac{\ell}{\ell-1}} \right]^{\frac{\ell-1}{\ell}} < +\infty.$$

In other words $\int \hat{q} q^* dM$ is also a martingale. □

We now give

Proof of Proposition 4.3. That $N = 2Y^{\min}q^*$, is a martingale on $[0,T)$ follows from the product rule and the definitions of $Y^{\min}$ and q^*. Since $(Y^{\min})^-$ is bounded (by κ), note that

$$\exp\left(-\int_0^t Y_s^{\min}\text{vol}_s ds\right) \leq \exp\left(\int_0^t (Y^{\min})_s^- \text{vol}_s ds\right) \leq \exp\left(\kappa \int_0^t \text{vol}_s ds\right).$$

This and Assumption 1 imply that q^* is also bounded. Since $(Y^{\min})^-$ is also bounded, the martingale N is bounded from above. Therefore, the limit at time T of N exists in $\mathbb{R}$ and

$$q_t^* = \frac{N_t}{2Y_t^{\min}}$$

tends to zero a.s. on the set $\boldsymbol{S} = \{\xi = +\infty\}$, since $\lim_{t\to T} Y_t^{\min}\mathbf{1}_{\boldsymbol{S}}(T) = +\infty$.

Now we apply Itô's formula on $Y^{\min}(q^*)^2$: for any $0 \leq t \leq r < T$

$$\begin{aligned} Y_t^{\min}(q_t^*)^2 &= Y_r^{\min}(q_r^*)^2 - \int_t^r (q_s^*)^2 \text{vol}_s (Y_s^{\min})^2 ds \\ &\quad - \int_t^r Y_s^{\min} 2(q_s^*)(-Y_s^{\min}\text{vol}_s q_s^*) ds - \int_t^r (q_s^*)^2 dM_s^{\min} \\ &= Y_r^{\min}(q_r^*)^2 + \int_t^r \frac{(v_s^*)^2}{\text{vol}_s} ds - \int_t^r (q_s^*)^2 dM_s^{\min} \end{aligned}$$

with $v_s^* = -Y_s^{\min} q_s^* \text{vol}_s$. Taking the conditional expectation we get

$$Y_t^{\min}(q_t^*)^2 = \mathbb{E}\left[Y_r^{\min}(q_r^*)^2 + \int_t^r \frac{(v_s^*)^2}{\text{vol}_s} ds \middle| \mathcal{F}_t\right].$$

By monotone convergence theorem

$$\liminf_{r\to T} \mathbb{E}\left[\int_t^r \frac{(v_s^*)^2}{\text{vol}_s} ds \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T \frac{(v_s^*)^2}{\text{vol}_s} ds \middle| \mathcal{F}_t\right].$$

And by Fatou's lemma ($(Y^{\min})^-$ is bounded)

$$\liminf_{r\to T} \mathbb{E}\left[Y_r^{\min}(q_r^*)^2 \middle| \mathcal{F}_t\right] \geq \mathbb{E}\left[\liminf_{r\to T}(Y_r^{\min}(q_r^*)^2) \middle| \mathcal{F}_t\right].$$

Recall the definition of $N = 2Y^{\min}q^*$ and that the limit of N at time T exists in $\mathbb{R}$. Moreover $\lim_{r\to T} q_r^* = 0 = q_T^*$, when $\lim_{r\to T} Y_r^{\min} = +\infty$. Therefore, if $\lim_{r\to T} Y_r^{\min} = +\infty$, then

$$\liminf_{r\to T}(Y_r^{\min}(q_r^*)^2) = \liminf_{r\to T}(N_r) \lim_{r\to T} q_r^* = 0 = \xi(q_T^*)^2$$

(with the convention $\infty \cdot 0 = 0$). If $\liminf_{r\to T} Y_r^{\min} < +\infty$, then

$$\liminf_{r\to T}(Y_r^{\min}(q_r^*)^2) \geq \xi(q_T^*)^2.$$

In any case, we obtain

$$Y_t^{\min}(q_t^*)^2 \geq \mathbb{E}\left[\int_t^T \frac{(v_s^*)^2}{\text{vol}_s} ds + \xi(q_T^*)^2 \middle| \mathcal{F}_t\right].$$

Thus $Y^{\min}q^2$ dominates the value function $V(\cdot, q)$ of the constrained control problem.

Now if v is in $\mathcal{A}_1$, it is in $\mathcal{A}_0$. Therefore, the value function V dominates the value function of the unconstrained control problem with terminal penalty $\xi \wedge K$, for any K. Denote by Y^K the solution of the BSDE (3.18) with bounded terminal value $Y_T^K = \xi \wedge K$. From Lemma (4), $Y^K q^2$ is the value function of the unconstrained control problem. We deduce that for any K

$$Y_t^K q^2 \leq V(t, q) \leq Y_t^{\min} q^2.$$

Since Y^K converges to $Y^{\min}$, we obtain that $Y^{\min}q^2$ is the value function of the constrained control problem and that q^* is the optimal state process. □

Note that the proof implies that the value function is finite at time 0, that is

$$\mathbb{E}\left[\int_0^T \frac{(v_s^*)^2}{\text{vol}_s} ds + \xi(q_T^*)^2\right] < +\infty.$$

Using (4.11), we can also deduce that

$$\mathbb{E}\left[\left(\int_0^T \frac{(v_s^*)^2}{\text{vol}_s} ds + \xi(q_T^*)^2\right)^{\ell}\right] < +\infty.$$

4.2 Reduction to time interval $[0, \tau_L \wedge L]$ for $I_t = 1_{\{t<\tau_L\}}$

For $I = I^{(2)} = 1_{\{t\leq\tau_L\}}$, by the definition (3.9) of $\mathcal{A}_{I,\boldsymbol{s}}$ we have $q_t = q_{\tau_L}$, $v_t = 0$ for $t > \tau_L$. Therefore, the stochastic optimal control problem (3.12) can also be expressed as

$$\min_{q\in\mathcal{A}_0} \mathbb{E}\left[\int_0^{\tau_L\wedge T} \eta_t \frac{v_t^2}{\text{Vol}_t} dt + \left(-\frac{k}{2}\mathbf{1}_{\{\tau_L<T\}} + \infty \cdot \mathbf{1}_{\{\tau_l>T\}}\right) q_T^2\right]. \tag{4.15}$$

for $I = 1_{\{t\leq\tau_L\}}$. The corresponding BSDE is again (3.18) but with terminal condition

$$Y_{\tau_L\wedge T} = \xi^{(2)} = -\frac{k}{2}\mathbf{1}_{\{\tau_L<T\}} + \infty \cdot \mathbf{1}_{\{\tau_l>T\}}. \tag{4.16}$$

The next proposition states that the BSDE (3.18) has a minimal supersolution for terminal conditions of the form

$$Y_{\tau\wedge T} = \xi = \zeta \mathbf{1}_{\{\tau \geq T\}} + \psi_\tau \mathbf{1}_{\{\tau < T\}}, \tag{4.17}$$

where τ is a stopping time and ζ, ψ are bounded from below by $-K$ and ψ is also bounded from above; (4.16) is a special case of (4.17). The reduction of the problem to the time interval $[0, T \wedge \tau_L]$ is also useful for the PDE analysis of the problem (section 5.2).

Proposition 4.4. *Let ξ be as in* (4.17)*. If*

$$\mathbb{E} \int_0^T \mathrm{vol}_s(\mathbb{E}[\zeta|\mathcal{F}_s])^2 ds < +\infty,$$

then the BSDE (3.18) *with terminal condition* (4.17) *has a unique solution (Y, Z). Without any integrability condition on ζ, but with* (4.11)*, there exists a minimal supersolution $(Y^{\min}, Z^{\min})$ for the BSDE.*

Proof. For L sufficiently large:

$$\xi \wedge L = (\zeta \wedge L)\mathbf{1}_{\tau \geq T} + \psi_\tau \mathbf{1}_{\tau < T}.$$

Again we construct a sequence of solutions (Y^L, Z^L), such that (Y^L) is non-decreasing. It only remains to control Y^L, uniformly in L.

We define $(\widehat{U}, \widehat{V})$ as the solution with terminal condition $\psi_\tau \mathbf{1}_{\tau<T} - \zeta^- \mathbf{1}_{\tau \geq T}$. Remark that $\widehat{U}$ is bounded. Now for any $t \leq s \leq T$ we have

$$\begin{aligned} Y^L_{t\wedge\tau} - \widehat{U}_{t\wedge\tau} = Y^L_{s\wedge\tau} - \widehat{U}_{s\wedge\tau} &- \int_{t\wedge\tau}^{s\wedge\tau} \mathrm{vol}_\rho \left[(Y^L_\rho - \widehat{U}_\rho + \widehat{U}_\rho)^2 - \widehat{U}^2_\rho \right] d\rho \\ &- \int_{t\wedge\tau}^{s\wedge\tau} (Z^L_\rho - \widehat{V}_\rho) d\rho. \end{aligned}$$

Hence $\Upsilon^L = Y^L - \widehat{U}$ solves the BSDE with terminal condition $(\zeta^+ \wedge L)\mathbf{1}_{\tau \geq T}$ and generator

$$g(s, y) = -\mathrm{vol}_s \left[(y + \widehat{U}_s)^2 - \widehat{U}^2_s \right].$$

The map $y \mapsto g(s, y)$ is not monotone on $\mathbb{R}$. But since the negative part of the data is bounded, we can modify g on $(-\infty, -\kappa)$, such that g becomes monotone (see Equation (4.8)). And again by the comparison principle, Υ^L is non-negative.

To control Υ^L, we again follow the arguments of [17, Lemma 1]. Young's inequality implies that for any $y \geq 0$ and $c \geq 0$

$$(y + \widehat{U}_s)^2 \geq 2(y + \widehat{U}_s) - c^2.$$

Thus with $c = 1/(\text{vol}_s(T-s))$

$$g(s,y) \leq -\frac{2}{T-s}y + \frac{1}{\text{vol}_s}\left(\frac{1}{T-s}\right)^2 - \frac{2}{T-s}\widehat{U}_s + \text{vol}_s(\widehat{U}_s)^2.$$

Since $\Upsilon^L_{T\wedge\tau} = 0$ if $\tau < T$, explicit solution for linear BSDE and comparison principle imply that for $t < T$

$$\begin{aligned}\Upsilon^L_{\tau\wedge t} \leq{}& \frac{1}{(T-\tau\wedge t)^2}\mathbb{E}\left[\int_{t\wedge\tau}^{T\wedge\tau}\frac{1}{\text{vol}_s}ds\middle|\mathcal{F}_{\tau\wedge t}\right] \\ &+ \frac{1}{(T-\tau\wedge t)^2}\mathbb{E}\left[\int_{t\wedge\tau}^{T\wedge\tau}(T-s)[\text{vol}_s(T-s)(\widehat{U}_s)^2 - 2\widehat{U}_s]ds\middle|\mathcal{F}_{\tau\wedge t}\right].\end{aligned}$$

This uniform bound on Υ^L, thus on Y^L, allows us to define the solution of the BSDE with terminal time $\tau \wedge T$ and a singular terminal condition. □

The next proposition connects the value function of (4.15) to the minimal supersolution whose existence was derived above. The proof is parallel to that of Proposition 4.3 and is omitted.

Proposition 4.5. *Let* $(Y^{\min}, M^{\min})$ *be the minimal supersolution of* (3.18) *and* (4.16). *For any* $t \in [0,T)$,

$$q^*_t = q_0 \exp\left(-\int_0^t Y^{\min}_s vol_s ds\right),$$

(equivalently, $v^*_t = -Y^{\min}_t vol_t q^*_t$*) is the optimal control for the stochastic optimal control problem the control problem* (4.15). *Moreover the value function of the same control problem at time* t *equals* $q_t^2 Y^{\min}_t$.

4.3 Reduction to time interval $[0, T-\delta]$ for $(I,S) = (I^{(3)}, S^{(3)})$ and $(I,S) = (I^{(4)}, S^{(4)})$

Both $(I, \boldsymbol{S}) = (I^{(3)}, \boldsymbol{S}^{(3)})$ and $(I, \boldsymbol{S}) = (I^{(4)}, \boldsymbol{S}^{(4)})$ consist of two phases: before and after time $T-\delta$, the reason for this was explained in the paragraph following (3.16). Both of these choices of I imply that the trading process for $I^{(3)}$ and $I^{(4)}$ proceeds

exactly as in $I^{(2)}$ after time $T-\delta$: if the algorithm is in trading mode at time $T-\delta$, the position is fully closed only when the price remains above L throughout the interval $[T-\delta, T]$; trading stops (and doesn't restart) if the price hits L. This implies that the stochastic optimal control problem (4.12) can be reduced to the following

$$\min_{q\in\mathcal{A}_0} \mathbb{E}\left[\int_0^{T-\delta} \frac{v_t^2}{\text{vol}_t} dt + \xi q_T^2\right]; \tag{4.18}$$

with terminal cost

$$\xi = I_{T-\delta}^{(j)} Y_{T-\delta}^{\min,1} - (1-I^{(j)})k/2. \tag{4.19}$$

where $Y^{\min,1}$ is the minimal supersolution of the BSDE (3.18) and the terminal condition (3.13) with $(I, \boldsymbol{S}) = (1, \boldsymbol{S}^{(1)})$. Corresponding to the reduction (4.18) the BSDE (3.18) and the terminal condition (3.13) for the cases $(I, \boldsymbol{S}) = (I^{(j)}, \boldsymbol{S}^{(j)})$ $j \in \{3, 4\}$ reduce to the same BSDE but solved over time interval $[0, T-\delta]$ with terminal condition

$$Y_{T-\delta} = \xi \tag{4.20}$$

where ξ is as in (4.19).

CHAPTER 5

PDE ANALYSIS

In this Chapter, we will assume the price process to be Markovian and the cost structure to be a function of the price process;under these assumptions, our goal is to relate the value function of the stochastic optimal control problem (3.12) to a PDE version of the BSDE for the four choices of I and $\boldsymbol{S}$ given in (3.14)-(3.17). For a direct PDE representation (i.e., identifying the value function as a solution of a related PDE), the process I and the measurable set $\boldsymbol{S}$ must also be functions of the price process. Of the four possible choices for I and $\boldsymbol{S}$ given in (3.14)-(3.17), only $(I^{(1)}, \boldsymbol{S}^{(1)})$ is given as a function of the price process. In section 5.1 we present the PDE representation of this case. In section 5.2 we present a PDE representation of the value function corresponding to $(I^{(2)}, \boldsymbol{S}^{(2)})$ by reducing the stochastic optimal control problem (3.12) to the time interval $[0, \tau_L \wedge T]$. A PDE treatment of the case $(I^{(3)}, \boldsymbol{S}^{(3)})$ is possible by treating the time intervals $[0, T-\delta]$ and $[T-\delta, T]$ separately. We do this in section 5.3. The PDE representation of $(I^{(4)}, \boldsymbol{S}^{4})$ begins also with a reduction to the time interval $[0, T-\delta]$; we then write the value function as a limit of a sequence of value functions V_n where n is the number of switches allowed between trading and no trading, each V_n is written as the solution of the PDE; this is treated in section 5.4.

As noted in the introduction, we assume the price process $\bar{S}$ to be driven by a stochastic volatility model:

$$\bar{S}_t = \int_0^t \sqrt{\nu_t} dW_t^{(1)},$$

where ν_t is the stochastic volatility process:

$$d\nu_t = \kappa(\theta - \nu_t)dt + c\sqrt{\nu_t} dW_t^{(2)}; \tag{5.1}$$

$(W^{(1)}, W^{(2)})$ is a standard Brownian motion in $\mathbb{R}^2$ (independent components, $W_1^{(i)}$,

$i = 1, 2$, have unit variance). Let $\mathcal{L}$ denote the second-order differential operator corresponding to these dynamics:

$$\mathcal{L}u = \frac{1}{2}\nu\partial^2_{ss}u + \frac{1}{2}\nu c^2\partial^2_{\nu\nu}u + \kappa(\theta - \nu)\partial_\nu u + c\nu\rho\partial^2_{s\nu}u. \tag{5.2}$$

To get a PDE representation we take η and Vol processes to be functions of $(\bar{S}, \nu)$. With a slight abuse of notation, we assume the market volume process to be $t \mapsto \text{Vol}(t, \bar{S}_t, \nu_t)$ where $\text{Vol} : [0, T] \times D \to \mathbb{R}$ is a nonnegative valued function; similarly the transaction cost process is $t \mapsto \eta(t, \bar{S}_t, \nu_t)$ where $\eta : [0, T] \times D \to \mathbb{R}_+$ is a strictly positive valued function. In this section we will use the assumption (4.2) on Vol:

$$0 \leq \text{Vol}(t, s, \nu) \leq \overline{\text{vol}}, \qquad k\overline{\text{vol}} < 2. \tag{5.3}$$

5.1 PDE representation for $I = 1, S = \{\bar{S}_T > L\}$

In this section we assume $I = 1$, i.e., $\text{vol}_t = \text{Vol}_t$. Let us now consider terminal values of the form

$$\xi = \Phi(\bar{S}_T, \nu_T); \tag{5.4}$$

where

$$\Phi : D = (0, +\infty) \times \mathbb{R} \to (-K, \infty] \tag{5.5}$$

is a measurable function. By (3.13), the choice $\boldsymbol{S} = \{\bar{S}_T > L\}$ corresponds to the Φ function

$$\Phi = \Phi^{(1)}(\bar{s}) = -\frac{k}{2}1_{(-\infty, L)}(\bar{s}) + \infty \cdot 1_{[L,\infty)}(\bar{s}).$$

Under the Markovian assumptions of the present section, and for $I = 1$ the BSDE (3.18) and the terminal condition (5.4) corresponds to the following PDE: for any $(\nu, s) \in D$ and $t \in [0, T)$

$$\partial_t u + \mathcal{L}u - \text{Vol}_t u^2 = 0. \tag{5.6}$$

with the terminal constraint

$$u(T, \cdot, \cdot) = \Phi. \tag{5.7}$$

Our goal, under the Markovian assumptions of the present section, is to prove that the value function of the stochastic optimal control problem (3.12) with $I = 1$ and $\boldsymbol{S} = \boldsymbol{S}^{(1)}$ can be expressed as a multiple of the unique solution of this PDE with

terminal condition $\Phi = \Phi^{(1)}$. The terminal condition $\Phi^{(1)}$ is singular, i.e., takes the values ∞ with positive probability. As in Chapter 4, we will start with a general terminal condition Φ that is more regular; the result for $\Phi^{(1)}$ will be obtained as a limiting case of regular terminal values. To express the value function as a solution to the above PDE, we first extend the stochastic optimal control problem (3.12) to allow it to start from any time point t. Accordingly, we define

$$\begin{aligned}
\bar{S}_r^{t,\nu,s} &= s + \int_t^r \sqrt{\nu_\rho^{t,\nu,s}} dW_\rho^{(1)}, \\
\nu_r^{t,\nu,s} &= \nu + \int_t^r \kappa(\theta - \nu_\rho^{t,\nu,s}) d\rho + \int_t^r c\sqrt{\nu_\rho^{t,\nu,s}} dW_\rho^{(2)}, \\
\xi = \Phi(\bar{S}_T^{t,\nu,s}, \nu_T^{t,\nu,s}), &\qquad \mathrm{vol}_\rho^{t,\nu,s} = \mathrm{vol}(t, \bar{S}_\rho^{t,\nu,s}, \nu_\rho^{t,\nu,s}).
\end{aligned}$$

Under our assumptions on Vol and Φ, Proposition 4.2 implies that the following BSDE has a unique minimal supersolution:

$$Y_r^{t,\nu,s} = \xi - \int_r^T \mathrm{vol}_\rho^{t,\nu,s} (Y_\rho^{t,\nu,s})^p ds - \int_r^T Z_\rho^{t,\nu,s} dW_\rho;$$

set

$$u^\Phi(t,\nu,s) := Y_t^{t,\nu,s}; \tag{5.8}$$

by Proposition 4.3, $u^\Phi(t,\nu,s)q^p$ is the value function of the extended version of the stochastic optimal control problem. With a slight abuse of language, we will refer to u^Φ simply as the value function of the extended stochastic optimal control problem with terminal cost Φq_T^p. Our goal in this section is to prove that u^Φ is the minimal supersolution (or the unique solution if Φ is finite) of the PDE (5.6) for any Φ of the form (5.5) (and in particular for $\Phi = \Phi^{(1)}$). Our first step in this direction is the following:

Lemma 5. *If Φ is continuous and with polynomial growth on D and if Vol is also continuous on $[0,T] \times D$, then u^Φ is a continuous function of $(t,\nu,s) \in D$ and is the unique viscosity solution of the PDE* (5.6) *with polynomial growth on D.*

Proof. See [19, Theorem 5.37] or [7, Theorems 3.4 and 3.5] (see also [22]). If Φ is bounded, the solution $Y^{t,\nu,s}$ and thus u are also bounded. Hence our generator is Lipschitz continuous.

If Φ satisfies $-k/2 \le \Phi(s,\nu) \le K(1 + |\nu|^\varrho + |s|^\varrho)$, then ξ^+ satisfies the condition imposed in Proposition 4.1. Thus $Y^{t,\nu,s}$ and thus u are bounded from below, and we

can modify our generator such that it becomes monotone (see the proof of Proposition 4.1). Then existence and uniqueness follows from [19, Theorem 5.37] (this result is stated for $(\nu, s) \in \mathbb{R}^2$ in [19] but all arguments continue to work when $(\nu, s) \in D$). □

Now suppose that Φ is a continuous function from D to $[-k/2, +\infty]$. We use the proof of Proposition 4.2. For any $L \geq 0$, we consider the bounded function $\Phi^L = \Phi \wedge L$. By the previous lemma there exists a unique bounded viscosity solution $u^{\Phi \wedge L}$ and by comparison principle,

$$u^{\Phi}(t, \nu, s) = \lim_{L \to +\infty} u^{\Phi \wedge L}(t, \nu, s)$$

is well-defined with a bounded negative part. Suppose now that for some $\varrho \geq 1$ and some $\epsilon > 0$:

$$\forall (t, \nu, s) \in [T - \epsilon, T] \times D, \quad \frac{1}{\mathrm{vol}(t, \nu, s)} \leq C(1 + |\nu|^{\varrho} + |s|^{\varrho}).$$

Then Condition (4.11) holds and we have on $[T - \epsilon, T] \times D$,

$$\begin{aligned} u^L(t, \nu, s) &\leq \frac{1}{(T-t)^2} \mathbb{E}\left[\int_t^T \frac{1}{\mathrm{vol}_u^{t,\nu,s}} du \right] \\ &\leq \frac{C}{(T-t)} (1 + |\nu|^{\varrho} + |s|^{\varrho}) \end{aligned} \tag{5.9}$$

On the rest of the interval $[0, T]$, the bound of the solution u^L is controlled by the previous estimate with $t = T - \epsilon$. In other words we have a bound on u^L which does not depend on L. Hence u is lower semi-continuous on $[0, T] \times D$ and finite (even locally bounded) on $[0, T) \times D$. These considerations give us the following result:

Lemma 6. *Suppose Φ is a continuous function from D to $[-k/2, +\infty]$; then u^{Φ} is the minimal viscosity solution of the PDE* (5.6) *on $[0, T) \times D$ (among all viscosity solutions with bounded negative part).*

Proof. See [22, Theorem 1]. □

The concept of a viscosity solution allows even a discontinuous solution and doesn't address the issue of smoothness/regularity of the solution; therefore the previous result doesn't say anything about the regularity of u^{Φ}. The properties of the operator $\mathcal{L}$

(and smoothness assumptions on Vol and η) allows us to establish the smoothness of u^Φ, with a regularization bootstrap argument for parabolic PDE:

Lemma 7. *Suppose that Vol and η are continuously differentiable with respect to all of their arguments. Assume that Φ_n is a sequence of continuous functions, converging to Φ, such that the related (viscosity) solutions u_n of the PDE* (5.6) *converge pointwise to u. Then u belongs to $C^{1,2}([0,T)\times D)$ and is a classical solution of the PDE* (5.6)*.*

Proof. Fix some $\epsilon > 0$ and K a compact subset of D. First note that from (5.9), the bound of u_n on $[0, T-\epsilon]\times K$ does not depend on the terminal value, that is on n, but only on ϵ and K.

Moreover the operator $\mathcal{L}$ can be written as follows:

$$\begin{aligned}\mathcal{L}u &= \frac{1}{2}\nu\partial^2_{ss}u + \frac{1}{2}\nu c^2\partial^2_{\nu\nu}u + \kappa(\theta-\nu)\partial_\nu u + c\nu\rho\partial^2_{s\nu}u \\ &= \frac{1}{2}\text{div}\left(a(\nu,s)\nabla u\right) + b(\nu,s)\nabla u,\end{aligned}$$

with

$$\nabla u(\nu,s) = \begin{pmatrix}\partial_s u \\ \partial_\nu u\end{pmatrix}$$

and

$$a(\nu,s) = \nu\begin{pmatrix}1 & c\rho \\ c\rho & c^2\end{pmatrix}, \quad b(\nu,s) = \begin{pmatrix}\frac{1}{2} + \frac{c\rho}{2} \\ \kappa(\theta-\nu) + \frac{c^2}{2} + \frac{c\rho}{2}\end{pmatrix}.$$

Our coefficients a and b are bounded on K, and a is uniformly elliptic on K. Since vol is also continuously differentiable with respect to all of their arguments, then we can easily check that all conditions called a)-c) of [18, Theorem VI.4.4] hold (with $m = 2$). And from this theorem, if there exists a function ψ continuous on $[0, T-\epsilon]\times K$ and of class $H^{1+\beta/2,2+\beta}(]0,T-\epsilon[\times\overset{\circ}{K})$ for some $\beta > 0$[1] then the PDE

$$\partial_t v + \mathcal{L}v - \text{vol}_t v^2 = 0$$

with the boundary condition $u = \psi$, has a unique solution v with the same regularity as ψ.

Now our viscosity solutions u_n are continuous and bounded on $[0, T-\epsilon]\times K$. Let us consider a sequence of smooth mollifiers ζ_m and define $\psi_m = u_n \star \zeta_m$ (where $\star$

[1] For the spaces used in this argument see [22, 1]

denotes the convolution operation). There exists a classical smooth solution $u_{n,m}$ of the PDE (5.6) with boundary condition ψ_m and pointwise $u_{n,m}$ converges to u_n as m goes to $+\infty$.

Note that Conditions (1.2) and (7.1) of [18, Chapter 3] are satisfied. Suppose that v is a smooth solution of the PDE (5.6) on $[0,T) \times D$: for any $(t,\nu,s) \in [0,T) \times D$

$$\partial_t v + \mathcal{L} v - \mathrm{vol}_t v^2 = 0,$$

such that for any $\epsilon > 0$ and any compact subset K of D , v is bounded on $[0, T-\epsilon] \times K$. Thus v solves on $[0, T-\epsilon] \times K$ the PDE

$$\partial_t v + \mathcal{L} v = \mathrm{vol}_t v^2 = f,$$

where f is a bounded function. We can apply [18, Theorem III.10.1]. Hence v is in $H^{\alpha,\alpha/2}([0, T-\epsilon] \times K)$ (space of functions which α-Hölder continuous in the space variable x and $\alpha/2$-Hölder continuous in the times variable t). The value of $\alpha > 0$ and the Hölder norm of v depend on ϵ, K and the bound on v. In other words α does not depend on the terminal value. Therefore, $u_{n,m}$ belongs to $H^{\alpha,\alpha/2}([0, T-\epsilon] \times K)$ and solves the PDE

$$\partial_t v + \mathcal{L} v = \mathrm{vol}_t(u_{n,m})^2 = f_{n,m}.$$

Assume that vol is also in $H^\alpha(D)$. Then from [18, Theorem IV.10.1], $u_{n,m}$ is in $H^{2+\alpha,1+\alpha/2}([0, T-\epsilon'] \times K')$ for any $\epsilon' > \epsilon$ and $K' \subset K$, and the norm depends only on the H^α-norm of $f_{n,m}$. Therefore, u_n, and thus u, belong to the same space[2], that is on any subset $[0, T-\epsilon] \times K$, u_n and u are in $C^{1,2}$. □

The proof also shows that the regularity of any solution does not depend on the terminal value. In other words, far from $t = T$ and $\nu = 0$, the solutions are smooth and classical solutions.

Let us summarize the foregoing results:

Proposition 5.1. *Suppose that* vol *is continuously differentiable on* $[0,T] \times D$ *and that* Φ *is bounded from below by* $-k/2$*. If one of the next conditions holds:*

- Φ *is continuous and with polynomial growth on* D*,*

[2] The Arzela-Ascoli theorem implies that $u_{n,m}$ (up to a subsequence) converges to some function $\widetilde{u}_n \in H^{\alpha,\alpha/2}([\eta, L-\eta] \times [0, T-\epsilon])$. Here $\widetilde{u}_n = u_n$ since pointwise convergence has been proved before.

- Φ *is continuous from* D *to* $[-k/2, +\infty]$ *and for some* $\varrho \geq 1$ *and* $\epsilon > 0$

$$\forall (t, \nu, s) \in [T - \epsilon, T] \times D, \quad \frac{1}{\mathrm{vol}(t, \nu, s)} \leq C(1 + |\nu|^{\varrho} + |s|^{\varrho}),$$

then there exists a viscosity solution u *of the PDE* (5.6) *with terminal value* $\Phi(\cdot, \cdot)$. *Moreover* u *is of class* $C^{1,2}([0, T) \times D)$ *and is the minimal viscosity solution (among all viscosity solutions with bounded negative part).*

Let us next comment on the smoothness of u on the boundary $\nu = 0$, called the hyperbolic part of the boundary. Recall that our operator $\mathcal{L}$ is defined by (5.2) and that the dynamics of ν is given by (5.1). The Feller condition ensuring a positive process ν is $2\kappa\theta > c^2$. Under this condition, the Fichera function

$$b(\nu) = \kappa(\theta - \nu) - \frac{1}{2}c^2$$

is positive when ν goes to zero. Hence no boundary condition has to be supplied on $\nu = 0$ (see for example [9]).

In the Markovian setting, that is if vol is also a function of (t, ν, s), then the value function V_L is of the form

$$V_L(t, q, \nu, s) = q^2 u_L(t, \nu, s). \tag{5.10}$$

The function u_L is a solution of the PDE (5.6) with $p = 2$ and with the terminal condition

$$u_L(T, \nu, \cdot) = \infty \cdot \mathbf{1}_{(L,\infty)}(\cdot) - \frac{k}{2}\mathbf{1}_{(-\infty,L]}(\cdot) = \Phi(\cdot).$$

Note that we cannot directly apply Proposition 5.1, since Φ is not continuous. Nonetheless

Lemma 8. *There exists a minimal viscosity solution* u_L, *which is of class* $C^{1,2}$ *on* $[0, T) \times D$.

Proof. Indeed let us define

$$\Phi_K(\cdot) = K \cdot \mathbf{1}_{(L,\infty)}(\cdot) - \frac{k}{2}\mathbf{1}_{(-\infty,L]}(\cdot)$$

and

$$\Phi_{K,n}(\cdot) = K \cdot \mathbf{1}_{(L+1/n,\infty)}(\cdot) - \frac{k}{2}\mathbf{1}_{(-\infty,L]}(\cdot) + \left[(K+k/2)\frac{(\cdot+k/2)}{K+k/2} - k/2\right]\mathbf{1}_{(L,L+1/n]}(\cdot).$$

$\Phi_{K,n}$ is continuous and non-decreasing w.r.t. n and converges to Φ_K. Therefore, the related continuous viscosity solutions $u_{K,n}$ converge to u_K. Arguing as in the proof of Proposition 5.1, we obtain a uniform norm of u_K in the space $H^{1+\alpha,2+\alpha}([0,T-\epsilon]\times\mathcal{K})$, for any compact subset $\mathcal{K}$ of D. Then we pass on the limit on K to obtain the desired result. Minimality can be obtained as for Proposition 5.1. □

These give us the main result of this section:

Proposition 5.2. *Suppose* (5.3) *holds. Then the value function* $u^{\Phi^{(1)}}$ *is the minimal supersolution of the PDE* (5.6) *with boundary condition* (5.7) *with* $\Phi = \Phi^{(1)}$ *and the optimal control is given by*

$$q_t^* = q_0 \exp\left(-\int_0^t u^{\Phi^{(1)}}(\bar{S}_t, \nu_t, t) vol_s ds\right). \tag{5.11}$$

5.2 PDE representation for $I = 1_{\{t<\tau_L\}}$ and $S = \{\tau_L > T\}$

To get a PDE representation of the BSDE (3.18), (3.19) for $I_t = I_t^{(2)} = 1_{\{t\le\tau_L\}}$; we consider the problem in the interval $[0, \tau_L \wedge T]$. As discussed in section 4.2, the corresponding BSDE is again (3.18) but with terminal condition (4.16). This reduced formulation of the problem is indeed Markovian. The PDE is the same as before (5.6) but solved over the domain $D = (0,\infty)\times[L,\infty)$ and with boundary conditions

$$u|_{[0,T]\times(0,\infty)\times\{L\}} = -k/2 \tag{5.12}$$

$$u|_{\{T\}\times(0,\infty)\times[L,\infty)} = \infty. \tag{5.13}$$

The value function $u^{(2)}$ of the extended version of the stochastic control problem is again defined through (5.8) and we have:

Proposition 5.3. $u^{(2)}$ *is the minimal viscosity solution of* (5.6) *and boundary conditions* (5.12).

The proof proceeds parallel to the argument given in the previous section. We therefore provide an outline. We begin by considering the case where the boundary condition is given by

$$\begin{aligned} u|_{[0,T]\times(0,\infty)\times\{L\}} &= \psi \\ u|_{\{T\}\times(0,\infty)\times[L,\infty)} &= \Phi. \end{aligned} \tag{5.14}$$

where ψ is a continuous and bounded function. If Φ is bounded and if the compatibility constraint $\Phi(\nu, L) = \psi(T, \nu)$ is verified, we can directly apply [19, Theorem 5.41] to obtain the existence of a unique viscosity bounded and continuous solution u of the PDE (5.6) with the boundary condition (5.14) The regularity inside the domain can be obtained by the same arguments of Lemma 7. If the compatibility condition does not hold, the solution still exists but is not continuous up to the boundary. Finally the ∞ terminal condition can be handled via approximation from below (as was done in the previous section as well as in Chapter 4 in the treatment of the BSDE (3.18) and the singular terminal condition (3.19).

5.3 PDE representation for $I_t = 1_{\{\bar{S}_t > L\}}$ and $S = \{\tau_{T-\delta,L} > T\}$

The case $(I, \boldsymbol{S}) = (I^{(3)}, \boldsymbol{S}^{(3)})$ is Markovian in its driver term: $I^{(3)} = 1_{\{\bar{S}_t > L\}}$ is a function of the price process only but it is not Markovian in $\boldsymbol{S}^{(3)} = \{\tau_{T-\delta,L} > T\}$ because $\tau_{T-\delta,L}$, depends on the entire trajectory of the price process. But we saw in Section 4.3 that this case can be decomposed into two components, before and after time $T - \delta$. We saw in the same section that after time $T - \delta$ this case either reduces to $(I, \boldsymbol{S}) = (I^{(2)}, \boldsymbol{S}^{(2)}) = (1_{(0,\tau_L]}(t), \{\tau_L > T\})$ for $I^{(3)}_{T-\delta} = 1$ and to $(I, \boldsymbol{S}) = (0, \emptyset)$ for $I^{(3)}_{T-\delta} = 0$. We already derived the PDE representation of the value function for the case $(I^{(2)}, \boldsymbol{S}^{(2)})$ in the previous section. $(I, \boldsymbol{S}) = (0, \emptyset)$ is trivially Markovian (the BSDE simply reduces to $dY_t = 0$ with terminal condition $Y_T = -k/2$. These PDE representations imply that the terminal condition (4.19) equals

$$\xi = 1_{\{\bar{S}_{T-\delta} > L\}} u^{\Phi^{(1)}}(\bar{S}_{T-\delta}, \nu_{T-\delta}, T - \delta) - 1_{\{\bar{S}_{T-\delta} \leq L\}} \frac{k}{2}. \tag{5.15}$$

With this, the problem (4.18) is completely Markovian, since both the terminal condition and the cost functions are functions of the price process; the BSDE (3.18) and

the boundary condition $Y_{T-\delta} = \xi$ reduces to the following PDE:

$$\partial_t u + \mathcal{L}u - r(t, s, u) = 0, \tag{5.16}$$

where

$$r(t, s, v) = v^2 vol_t \mathbf{1}_{s>L}.$$

The PDE is solved over over the domain $[0, T - \delta] \times (0, +\infty) \times \mathbb{R}$ with terminal boundary condition

$$g(\nu, s) = u^{\Phi^{(1)}}(T - \delta, \nu, s)\mathbf{1}_{s>L} - k/2\mathbf{1}_{s\leq L}.$$

Note that the terminal boundary condition is bounded and continuous, i.e., no singularities. In this respect the analysis of this PDE is simpler to the case covered in Section 5.1. The difficulty as compared to the analysis of Section 5.1 is that the r function has a discontinuity at $s = L$. In what follows we focus on this aspect of the PDE (5.16) and how it can be solved when r is discontinuous.

Lemma 9. *There exists a function $u^{(3)}$ such that $u^{(3)}$ is bounded and continuous on $[0, T - \delta] \times (0, +\infty) \times \mathbb{R}$ and is a solution of class $C^{1,2}$ of the PDE* (5.16) *on $[0, T - \delta) \times (0, +\infty) \times (\mathbb{R} \setminus \{L\})$.*

Proof. To circumvent the discontinuity of r, let us introduce

$$\phi^\epsilon(s) = \left(\mathbf{1}_{s>L+\epsilon} + \frac{(s - L)}{\epsilon}\mathbf{1}_{L<s\leq L+\epsilon}\right), \quad r^\epsilon(t, s, v) = v^2 vol_t \phi^\epsilon(s).$$

This function is Lipschitz continuous w.r.t. s, satisfies $r^\epsilon \leq r$ and converges increasingly and pointwise to r when ϵ tends to zero.

From standard arguments (see [19, Theorem 5.37]), there exists a unique bounded and continuous viscosity solution $u^{3,\epsilon}$ of the PDE

$$\partial_t u + \mathcal{L}u - r^\epsilon(t, s, u) = 0$$

with the same terminal condition g. Note that the bounds on $u^{3,\epsilon}$ do not depend on ϵ. Thus arguing as in Lemma 7, we can prove that $u^{3,\epsilon}$ is of class $C^{1,2}$ on $[0, T - \delta) \times (0, +\infty) \times (\mathbb{R} \setminus \{L\})$ with a norm independent of ϵ.

The comparison principle shows that $u^{3,\epsilon}$ is a decreasing sequence and thus we can define $u^{(3)}$ as the decreasing limit of $u^{3,\epsilon}$ as ϵ tends to zero. We obtain immediately

that $u^{(3)}$ is bounded and upper semi-continuous and is a viscosity subsolution of PDE (5.16) (well-known result on stability for viscosity solutions [10]). Now an Ascoli's type argument shows that $u^{(3)}$ is in fact continuous on $[0, T-\delta] \times (0, +\infty) \times (\mathbb{R} \setminus \{L\})$ and thus by some regularization argument, it is a classical solution of the PDE (5.16) on $[0, T-\delta) \times (0, +\infty) \times (\mathbb{R} \setminus \{L\})$.

The only remaining point concerns the continuity of $u^{(3)}$ on the set $\{s = L\}$. Let us define another approximating sequence w^ϵ defined as the solution of PDE (5.16) where r is replaced by $\widetilde{r}^\epsilon$:

$$\psi^\epsilon(s) = \left(\mathbf{1}_{s>L} + \frac{(s-L)}{\epsilon} \mathbf{1}_{L-\epsilon<s\leq L} \right), \quad \widetilde{r}^\epsilon(t,s,v) = v^2 vol_t \psi^\epsilon(s).$$

w^ϵ converges to $w_\star$, which is lower semi-continuous and is a viscosity supersolution of PDE (5.16). Moreover by comparison principle, $w^\epsilon \leq w_\star \leq u^{(3)} \leq u^{3,\epsilon}$. Comparing sub and supersolutions imply that $w_\star = u^{(3)}$ (standard result for viscosity solutions). Let us prove this statement in our case. For any (ν, s) we have

$$\begin{aligned}
&u^{3,\epsilon}(t,\nu,s) - w^\epsilon(t,\nu,s) = Y_t^{\epsilon,t,\nu,s} - \widetilde{Y}_t^{\epsilon,t,\nu,s} \\
&= Y_{T-\delta}^{\epsilon,t,\nu,s} - \widetilde{Y}_{T-\delta}^{\epsilon,t,\nu,s} - \int_t^{T-\delta} r^\epsilon(u, \bar{S}_u^{t,\nu,s}, Y_u^{\epsilon,t,\nu,s}) - \widetilde{r}^\epsilon(u, \bar{S}_u^{t,\nu,s}, \widetilde{Y}_u^{\epsilon,t,\nu,s}) du \\
&\quad - \int_t^{T-\delta} (Z_u^{\epsilon,t,\nu,s} - \widetilde{Z}_u^{\epsilon,t,\nu,s}) dW_u \\
&= - \int_t^{T-\delta} vol(u, \nu_u^{t,\nu,s}, \bar{S}_u^{t,\nu,s}) \phi^\epsilon(\bar{S}_u^{t,\nu,s}) (Y_u^{\epsilon,t,\nu,s} - \widetilde{Y}_u^{\epsilon,t,\nu,s})(Y_u^{\epsilon,t,\nu,s} + \widetilde{Y}_u^{\epsilon,t,\nu,s}) du \\
&\quad - \int_t^{T-\delta} vol(u, \nu_u^{t,\nu,s}, \bar{S}_u^{t,\nu,s}) (\widetilde{Y}_u^{\epsilon,t,\nu,s})^2 \left(\phi^\epsilon(\bar{S}_u^{t,\nu,s}) - \widetilde{\phi}^\epsilon(\bar{S}_u^{t,\nu,s}) \right) du \\
&\quad - \int_t^{T-\delta} (Z_u^{\epsilon,t,\nu,s} - \widetilde{Z}_u^{\epsilon,t,\nu,s}) dW_u.
\end{aligned}$$

Using the boundedness of $Y^{\epsilon,\cdot,\cdot,\cdot}$ and $\widetilde{Y}^{\epsilon,\cdot,\cdot,\cdot}$ (uniformly w.r.t. ϵ) and standard stability result for BSDE, we obtain the existence of a constant C independent of ϵ such that

$$\begin{aligned}
|u^{3,\epsilon}(t,\nu,s) - w^\epsilon(t,\nu,s)|^2 &\leq C\mathbb{E}\left[\int_t^{T-\delta} \left(\phi^\epsilon(\bar{S}_u^{t,\nu,s}) - \widetilde{\phi}^\epsilon(\bar{S}_u^{t,\nu,s}) \right)^2 du \right] \\
&\leq 2C \int_t^{T-\delta} \mathbb{P}\left(L - \epsilon \leq \bar{S}_u^{t,\nu,s} \leq L + \epsilon \right) du
\end{aligned}$$

Fix some $\eta > 0$. The uniform ellipticity of $\mathcal{L}$ implies that there exists ϵ_0 such that for any $\epsilon < \epsilon_0$,

$$\mathbb{P}\left(L - \epsilon \leq \bar{S}_u^{t,\nu,s} \leq L + \epsilon \right) \leq \eta^2/(2C).$$

Hence letting ϵ go to zero, we get for any $\eta > 0$

$$|u^{(3)}(t,\nu,s) - w_\star(t,\nu,s)| \leq \eta.$$

Thus $u^{(3)} = w_\star$ and thus $u^{(3)}$ is continuous. □

We can now construct the minimal supersolution of the BSDE (3.18) and the boundary condition (4.19) over the time interval $[0, T-\delta]$ using the function $u^{(3)}$ constructed in the previous result by setting

$$Y_t = u^{(3)}(t, \nu_t, \bar{S}_t), t \leq T - \delta \tag{5.17}$$

Using Itô's formula (allowed since $\bar{S}_t \neq L$ a.s.) we get for any $0 \leq t \leq T - \delta$

$$\begin{aligned} Y_t &= Y_{T-\delta} - \int_t^T (Y_r)^2 vol_r \mathbf{1}_{\bar{S}_r > L} dr - \int_t^T \nabla u(r, \nu_r, \bar{S}_r) dW_r \\ &= u^{\Phi^{(1)}}(T-\delta, \nu_{T-\delta}, \bar{S}_{T-\delta}) \mathbf{1}_{\bar{S}_{T-\delta} > L} - \frac{k}{2} \mathbf{1}_{\bar{S}_{T-\delta} \leq L} \\ &\quad - \int_t^T (Y_r)^2 \widetilde{vol}_r dr - \int_t^T \nabla u(r, \nu_r, \bar{S}_r) dW_r. \end{aligned}$$

This and (5.15) imply that Y as defined in (5.17) is indeed the unique solution of the BSDE (3.18) over the time interval $[0, T-\delta]$ with terminal condition (4.19). By (4.18) the optimal control for this case is given by

$$q_t = q_0 \exp\left(- \int_0^t u^{(3)}(s, \nu_s, \bar{S}_s) \text{vol}(s, \nu_s, \bar{S}_s) ds \right).$$

5.4 PDE representation for $I^{(4)}$ and $S^{(4)}$

Let $V^{(4)}$ denote the value function of (4.12) for $(I, \boldsymbol{S}) = (I^{(4)}, \boldsymbol{S}^{(4)})$. A natural method to compute $V^{(4)}$ is to first put an upperbound n on the number of trading intervals and then let $n \nearrow \infty$. The value function of the control problem with a limit on the number of active intervals is:

$$V^{4,n}(q, \nu, s) = \inf_{q \in \mathcal{A}_0} \mathbb{E}\left[\int_0^T \frac{(q'_s)^2}{\text{vol}_s^n} ds - \frac{k}{2} q_T^2 \right], \tag{5.18}$$

where $\text{vol}_t^n = \text{Vol}_t I_t^{4,n}$ and

$$I_t^{4,n} = \sum_{k=-1}^{n} \mathbf{1}_{[\bar{\tau}_{b,k} \leq t \leq \tau_{L,k+1}]}$$

Proposition 5.4.

$$V^{4,n}(q,\nu,s) \searrow V^4(q,\nu,s)$$

as n tends to ∞.

Proof. Note that $I^{4,n} \nearrow I^{(4)}$; this implies 1) $V^{4,n}$ is decreasing and 2) $V^{4,n} \geq V^{(4)}$. Therefore we have

$$\lim_n V^{4,n} \geq V^{(4)}.$$

Next we prove that we can replace the inequality $\geq$ above with $=$. For this choose any $q \in \mathcal{A}_0$ define $\widetilde{q}$ equal to q on the random interval $[\![0, \tau_{L,n+1}]\!]$ and q' equal to zero after $\tau_{L,n+1}$, we get:

$$\begin{aligned}\int_0^T \frac{(q'_s)^2}{\mathrm{vol}_s} ds - \frac{k}{2} q_T^2 &= \int_0^{\tau_{L,n+1}} \frac{(q'_s)^2}{\mathrm{vol}_s} ds + \int_{\tau_{L,n+1}}^T \frac{(q'_s)^2}{\mathrm{vol}_s} ds - \frac{k}{2} q_T^2 \\ &= \int_0^{\tau_{L,n+1}} \frac{(\widetilde{q}'_s)^2}{\mathrm{vol}_s^n} ds - \frac{k}{2} \widetilde{q}_T^2 + \int_{\tau_{L,n+1}}^T \frac{(q'_s)^2}{\mathrm{vol}_s} ds - \frac{k}{2}(q_T^2 - \widetilde{q}_T^2).\end{aligned}$$

Taking the expectation, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^T \frac{(q'_s)^2}{\mathrm{vol}_s} ds - \frac{k}{2} q_T^2\right] &= \mathbb{E}\left[\int_0^{\tau_{L,n+1}} \frac{(\widetilde{q}'_s)^2}{\mathrm{vol}_s} ds - \frac{k}{2} \widetilde{q}_T^2\right] \\ &\quad + \mathbb{E}\left[\int_{\tau_{L,n+1}}^T \frac{(q'_s)^2}{\mathrm{vol}_s} ds - \frac{k}{2}(q_T^2 - \widetilde{q}_T^2)\right] \\ &\geq V_n(q,\nu,s) + \mathbb{E}\left[\int_{\tau_{L,n+1}}^T \frac{(q'_s)^2}{\mathrm{vol}_s} ds - \frac{k}{2}(q_T^2 - \widetilde{q}_T^2)\right].\end{aligned}$$

That $\widetilde{q}'_t = 0$ for $t > \tau_{L,n+1}$ implies $\widetilde{q}_T = q_{\tau_{L,n+1}}$, therefore:

$$\begin{aligned}\mathbb{E}\left[q_T^2 - \widetilde{q}_T^2\right] &= \mathbb{E}\left[(q_T + \widetilde{q}_T)\left(\int_{\varpi_n}^T q'_s ds\right)\right] \\ &\leq \left[\mathbb{E}(q_T + \widetilde{q}_T)^2\right]^{\frac{1}{2}} \left[\mathbb{E}\left(\int_{\varpi_n}^T q'_s ds\right)^2\right]^{\frac{1}{2}} \\ &\leq C T^{\frac{1}{2}} \left[\mathbb{E}(q_T + \widetilde{q}_T)^2\right]^{\frac{1}{2}} \left[\mathbb{E}\left(\int_{\tau_{L,n+1}}^T \frac{(q'_s)^2}{\mathrm{vol}_s} ds\right)\right]^{\frac{1}{2}}\end{aligned}$$

under the condition of Theorem 4.1 on vol. Therefore, we obtain for any strategy

$q \in \mathcal{A}_0$ and any n:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \frac{(q_s')^2}{\mathrm{vol}_s} ds - \frac{k}{2} q_T^2\right] &\geq V_n(q,\nu,s) + \mathbb{E}\left[\int_{\tau_{L,n+1}}^T \frac{(q_s')^2}{\mathrm{vol}_s} ds\right] \\
&\quad - C\frac{k}{2}\sqrt{T}\left[\mathbb{E}\left(q_T + \widetilde{q}_T\right)^2\right]^{\frac{1}{2}} \left[\mathbb{E}\left(\int_{\tau_{L,n+1}}^T \frac{(q_s')^2}{\mathrm{vol}_s} ds\right)\right]^{\frac{1}{2}} \\
&\geq V(q,\nu,s) + \mathbb{E}\left[\int_{\tau_{L,n+1}}^T \frac{(q_s')^2}{\mathrm{vol}_s} ds\right] \\
&\quad - C\frac{k}{2}\sqrt{T}\left[\mathbb{E}\left(q_T + \widetilde{q}_T\right)^2\right]^{\frac{1}{2}} \left[\mathbb{E}\left(\int_{\tau_{L,n+1}}^T \frac{(q_s')^2}{\mathrm{vol}_s} ds\right)\right]^{\frac{1}{2}}
\end{aligned}$$

$Y_t \geq z_t \geq z_0$ and (4.13) imply that we can assume q_T to be bounded. Letting $n \to \infty$ in the last display, bounded and dominated convergence theorems imply

$$\mathbb{E}\left[\int_0^T \frac{(q_s')^2}{\mathrm{vol}_s} ds - \frac{k}{2} q_T^2\right] \geq \lim_n V_n(q,\nu,s) \geq V(q,\nu,s).$$

Since this holds for any $q \in \mathcal{A}_0$, we obtain the desired result. □

Instead of deriving a PDE representation for $V^{(4)}$, which turns out to be difficult to do directly, we will work with the sequence $V^{n,4}$. To get such a representation we extend the problem as in the previous sections: Now we consider the dynamical version of this control problem. For any $t \in [0,T]$,

$$V^{4,n}(t,q,\nu,s,\chi) = \inf_{q\in\mathcal{A}_0} \mathbb{E}\left[\int_t^T \frac{(q_s')^2}{\mathrm{vol}_s^n} ds - \frac{k}{2} q_T^2\right], \tag{5.19}$$

the value function of the control problem starting at time t (instead of zero). Note the additional parameter χ which can take the values $\{0,1\}$; these are the values taken by the process I, $I = 1$ represents trading is on and $I = 0$ represents trading is off.

As in Section 4.2, the problem (5.19) can be reduced to the time interval $[0, \tau_L]$ for $\xi = 1$ and to the time interval $[0, \tau_{b+L}]$ for $\xi = 0$ by conditioning on the first transition from one state to the other (from active to waiting or from waiting to active); doing this gives the following two equations:

$$\begin{aligned}
V^{4,n}(t,q,\nu,s,0) &= \mathbb{E}\left[V^{4,n-1}(\bar{\tau}_{b,0}, q, \nu_{\bar{\tau}_{b,0}}, L+b, 1)\mathbf{1}_{\{\bar{\tau}_{b,0}<T\}} - \frac{k}{2} q_T^2 \mathbf{1}_{\{\bar{\tau}_{b,0}\geq T\}}\right], \\
V^{4,n}(t,q,\nu,s,1) &= \inf_{q\in\mathcal{A}_0} \mathbb{E}\left[\int_t^{\tau_{L,0}\wedge T} \frac{(q_s')^2}{\mathrm{Vol}_s} ds + V^{4,n-1}(\tau_{L,0}, q_{\tau_{L,0}}, \nu_{\tau_{L,0}}, L, 0)\mathbf{1}_{\{\tau_{L,0}\leq T\}}\right].
\end{aligned}$$

As in all of the previous cases the quadratic cost structure implies

$$V^{4,n}(t,q,\nu,s,\chi) = q^2 u^{4,n}(t,\nu,s,\chi), \tag{5.20}$$

Note that

$$u^{4,0}(\cdot,0) = -k/2^3$$

(i.e., if we start with no trading $I_t = 0$ and there can be no further transitions to trading the trader only pays the terminal cost $-(k/2)q^2$. The above recursions and the argument given in Section 5.3 imply the following sequence of PDE to compute $u^{4,n}$ for $n \geq 1$: for $u^{4,n}(\cdot,0)$:

$$\partial_t u^{4,n}(\cdot,0) + \mathcal{L} u^{4,n}(\cdot,0) = 0, \tag{5.21}$$

where $\mathcal{L}$ is defined by (5.2) and the above PDE is solved in the region $[0, T-\delta] \times (0,\infty) \times (-\infty, L+b)$ with boundary conditions

$$u^{4,n}(T-\delta,\nu,s,0) = -\frac{k}{2}, u^{4,n}(t,\nu,L+b,0) = u^{4,n}(t,\nu,L+b,1);$$

for $u^{4,n}(\cdot,1)$:

$$\partial_t u^{4,n}(\cdot,1) + \mathcal{L} u^{4,n}(\cdot,1) - \text{vol}_t u^2_{a,n}(\cdot,1) = 0, \tag{5.22}$$

solved in the region $[0,T] \times (0,\infty) \times (L,\infty)$ with boundary conditions

$$\begin{aligned}
&u^{4,n}(T,\nu,s,1) = \infty, \\
&u^{4,n}(t,\nu,L,1) = u^{4,n-1}(t,\nu,L,0), t < T-\delta, \\
&u_{4,n}(t,\nu,L,1) = -\frac{k}{2}, t \geq T-\delta.
\end{aligned}$$

The base case is $n = 1$, for which we have: $u_{4,1}(\cdot,\cdot,\cdot,1) = u^{(2)}$ where $u^{(2)}$ is the value function for the case $I = 1_{\{t \leq \tau_L\}}$ and $\boldsymbol{S} = \{\tau_L > T\}$ derived in Section 5.2 (see Proposition 5.3). The arguments of Lemma 7 show that $u^{4,1}(\cdot,\cdot,\cdot,1)$ is of class $C^{1,2}$ on $[0,T-\delta] \times (0,\infty) \times (L,\infty)$ and continuous on $[0,T-\delta] \times (0,\infty) \times [L,\infty)$. Once $u^{a,1}$ is available, the computation follows the sequence:

$$u^{4,1}(\cdot,1) \to u^{4,1}(\cdot,0) \to u^{4,2}(\cdot,1) \to u_{4,2}(\cdot,0) \cdots \tag{5.23}$$

Applying the same arguments recursively we see that $u^{4,n}$ satisfies the same regularity properties for all n.

[3] $\cdot$ stands for the parameters (t, nu, s).

The representation (5.20) and Proposition 4.3 imply

$$Y_t^{\min} = u^{4,n}(t, \nu_t, \bar{S}_t, 1), t \leq \tau_{L,0},$$

for $I = I^{(4),n}$. This and (4.13) imply that the optimal control for this case is

$$q_t^* = q_0 \exp\left(-\int_0^t u_n(s, \nu_s, \bar{S}_s, 1)\text{Vol}(s, \nu_s, \bar{S}_s)ds\right), \quad t \leq \tau_{L,0}.$$

For $t > \tau_{L,0}$ we replace $u^{4,n}$ with $u^{4,k}$ where k is the remaining number of trading intervals.

CHAPTER 6

NUMERICAL EXAMPLES

In this Chapter we present some numerical examples that give an idea how the modified IS order behaves and performs for four possible choices of I and $\mathbf{S}$ given in (3.14)-(3.17). In the previous chapter we saw how the computation of the value function and the optimal control for each of these choices could be reduced to the solution of PDE. In this chapter we will be using these PDE based calculations. Recall A of (1.1), which is the percentage deviation from the target price S_0 of the average price at which the position is (partially) closed in the time interval $[0, T]$. In our numerical examples we will focus on two aspects of the modified IS order: the optimal q^* given in (4.13), which is the actual trading algorithm and the distribution of the pair (q_T, A) which is the actual output of the trading algorithm. Compared to the original IS order, the modified IS order considered in the present work has two additional parameters: the process I that determines when trading takes place and the event $\boldsymbol{S}$ that determines when full liquidation takes place. We will show the behavior of q^* and (q_T^*, A) and check the similarity and differences between all of the possible choices for I and $\boldsymbol{S}$ given in (3.14)-(3.17).

To simplify the presentation and the calculations we take $\bar{S} = \sigma W$, where W is a standard Brownian motion and $\sigma > 0$ a constant and $\text{vol}_t = V > 0$; these are also the choices made for these parameters in the standard Almgren-Chriss framework [14, Chapter 3]. Under this assumption and the PDE analysis of the previous chapter the optimal control q^* given in (4.13) becomes

$$q_t^* = q_0 \exp\left(- \int_0^t u(s, W_s) \text{vol}_s ds \right),$$

where u is the relevant value function. We will compute u and q^* by discretizing and

numerically solving the corresponding PDE, which becomes:

$$u_t + \frac{1}{2}\sigma^2 u_{xx} - \frac{V}{\eta} I_t u^2 = 0, \tag{6.1}$$

where the domain of the equation and its boundary conditions depend $\mathbf{S}$ and I may depend on x.

One of our goals is to explore how q^* and the distribution of $(q_T^*/q_0, A)$ change with the model parameters. To better understand this, we factor out as many parameters as possible from the calculations. We begin with factoring out q_0 from q^*:

$$\mathfrak{q}_t = \frac{q_t^*}{q_0} = \exp\left(-\int_0^t u(s, W_s)\text{vol}_s ds\right), \tag{6.2}$$

this is the percentage of the initial position q_0 remaining at time t. If we let

$$v(t, x) = u(t, \sigma x)\left(\frac{V}{\eta}\right) \tag{6.3}$$

the equation (6.1) reduces to

$$v_t + \frac{1}{2}u_{xx} - I_t v^2 = 0. \tag{6.4}$$

To see how A depends on model parameters let us reduce the expression (1.1) as much as possible under the above assumptions:

$$A = \frac{X_T - (q_0 - q_T)S_0}{(q_0 - q_T)S_0}$$

By (3.2) (the expression for X_T') and the assumption $L(v) = \eta v^2$:

$$= \frac{-\int_0^T S_t v_t dt - \frac{\eta}{V}\int_0^T I_t v_t^2 dt - S_0(q_0 - q_T)}{S_0(q_0 - q_T)}$$

By the definition (3.1) of S_t and the assumptions $\kappa(v) = kv$, $\bar{S}_t = \sigma W_t$:

$$= \frac{k/2(q_T - q_0)^2 + \sigma\int_0^T W_t v_t dt - \frac{\eta}{V}\int_0^T v_t^2 dt}{(q_0 - q_T)S_0}.$$

Simplifying the last expression we get

$$A = -\frac{kq_0}{2S_0}\left(1 - \frac{q_T}{q_0}\right) + \frac{\sigma}{S_0}\frac{1}{(q_0 - q_T)}\int_0^T W_t v_t dt - \frac{\eta}{VS_0}\frac{1}{(q_0 - q_T)}\int_0^T v_t^2 dt. \tag{6.5}$$

From this expression we see that A consists of three components: 1) one due to the permanent price impact 2) one due to random fluctuations in price and 3) one due to

transaction costs. All components consist of a coefficient term and a term depending on q or its derivative v:

$$\text{Permanent market impact term: } A_1 = 1 - \mathfrak{q}_T \qquad \text{coefficient: } -\frac{kq_0}{2S_0}$$

$$\text{Random fluctuations term: } A_2 = \frac{1}{1-\mathfrak{q}_T}\int_0^T W_t(v_t/q_0)dt, \text{coefficient: } \frac{\sigma}{S_0}$$

$$\text{Transaction costs term: } A_3 = \frac{1}{1-\mathfrak{q}_T}\int_0^T (v_t/q_0)^2 dt. \qquad \text{coefficient: } \frac{\eta q_0}{VS_0}$$

The permanent impact term $1 - \mathfrak{q}_T$ is the portion of the initial position that is closed; A depends linearly on this portion with coefficient $\frac{kq_0}{2S_0}$. Secondly note that if S_0, k and η are parameterized as multiples of σ then none of the coefficients appearing in A depend on σ. We will comment on the behavior of the other two terms in the following sections.

Before we move on let us note the following for comparison. The case $I = 1$ and $\boldsymbol{S} = \Omega$ corresponds to the standard Almgren Chriss liquidation algorithm with $\gamma = 0$ for which the optimal control is known to be

$$q_t^{*,S} = q_0 \frac{T-t}{T}, \tag{6.6}$$

i.e., closing the position with uniform speed over the time interval $[0, T]$. Then $v_t^{*,S}/q_0 = 1$ and $\mathfrak{q}_T = 1$. These reduce A_3 to

$$A_3^S = 1 \tag{6.7}$$

for the standard IS algorithm. Similarly, for A_2 we have

$$A_2^S = \int_0^T W_t dt, \tag{6.8}$$

which is a standard normal random variable by the iid increments of W.

6.1 $I = 1$ and $S = \{W_T > L\}$

We continue or analysis with the choices $I = I^{(1)} = 1$ and $\boldsymbol{S} = \boldsymbol{S}^{(1)} = \{\bar{S}_T > L\}$ for I and $\boldsymbol{S}$ given in (3.14); these choices correspond to: no restriction on trading and closing the position fully is required only when the terminal price $\bar{S}_T$ is above a given threshold L. Parallel to the change of variable in (6.3) we assume L is given as

a multiple of $\sigma > 0$; with this convention and the assumption $\bar{S}_t = \sigma W_t$, $\boldsymbol{S}$ becomes $\boldsymbol{S} = \{W_T > L\}$. For $I_t = 1$, the PDE (6.4) is

$$v_t + \frac{1}{2}u_{xx} - v^2 = 0; \tag{6.9}$$

for $\boldsymbol{S} = \{W_T > L\}$ the domain and the boundary conditions for this PDE are: $(t, x) \in [0, T] \times \mathbb{R}$ and

$$v(T, x) = \infty \cdot \mathbf{1}_{[L,\infty)}(x) - \frac{kV}{2\eta} \cdot \mathbf{1}_{(-\infty,L)}(x), \tag{6.10}$$

$x \in \mathbb{R}$, where we again use the scaling (6.3). Recall our convention that k and η are specified as multiples of σ; it follows that PDE (6.9) and its boundary condition (6.10) are independent of σ. The optimal control q^* is computed from v via the formula (6.2):

$$q_t^* = q_0 \mathfrak{q}_t = q_0 \exp\left(-\int_0^T v(W_t, t)dt\right) \tag{6.11}$$

we note that q^* is independent of σ. We had already noted that the coefficients in (6.3) are independent of σ. We have observed above that the same is true also for q^*, therefore all of A_1, A_2 and A_3 are independent of σ as well. The same analysis in fact holds for all of $I = I^{(i)}$, $\boldsymbol{S} = \boldsymbol{S}^{(i)}$, $i = 1, 2, 3, 4$. treated in the following sections. This gives us the following result:

Proposition 6.1. *Suppose all of S_0, k, η and L are parameterized as multiples of σ. Then q^* and S' do not depend on σ for $I = I^{(i)}$, $\boldsymbol{S} = \boldsymbol{S}^{(i)}$, $i = 1, 2, 3, 4$.*

As already noted q^* is computed via the solution of the PDE (6.9) which obviously doesn't have an explicit solution. To see how q^* behaves we will solve (6.9) numerically; for the parameter values we begin by considering those used in [14, Chapter 3]: $T = 1$, $\eta = 0.1$, $V = 4 \times 10^6$, $S_0 = 45$, $\sigma = 0.6$. Recall that k parameter doesn't appear in the control problem corresponding to the original IS order, so no value for k is specified in [14, Chapter 3] A k value of $k = 2 \times 10^{-7}$ accompanying these parameter values is given in [14, Chapter 8] in the context of block trade pricing. The assumption (1) in the present case reduces to

$$kV/2\eta < 1;$$

for the above parameter values we have $kV/2\eta = 4$, therefore the above parameter values do not satisfy Assumption 1 To continue with our numerical example, we take

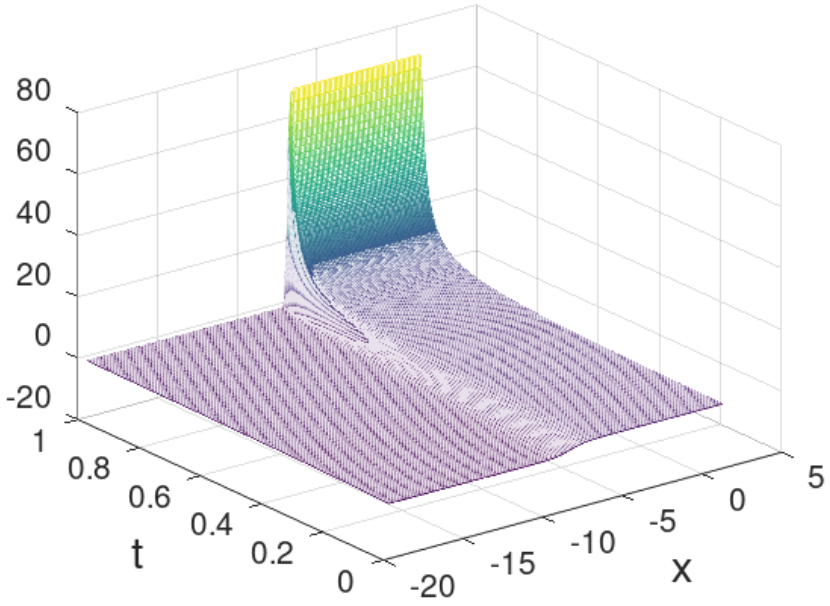


Figure 6.1: Graph of u for $I = 1$ and $\boldsymbol{S} = \{W_T < L\}$

$\eta = 0.3$, $V = 4 \times 10^6$ and $k = 0.5 \times 10^{-7}$ for these values we have$kV/\eta = 2/3$ which satisfies (1). In addition to these we need to provide a value for the L parameter, which we choose as $L = 1.4 * \sigma$. Several q^* for these parameter values are shown in Figure 6.3.

Unless otherwise noted, these are the main parameter values used through all of the numerical examples.

The graph of v for the parameter values above are shown in Figures 6.1 and 6.2.

We note that for $x > L$ and x away from L, $u(x, \cdot)$ behaves like $t \mapsto y_t$ and for $x < L$ and x away from L, $u(x, \cdot)$ behaves like $t \mapsto z_t$. The negative boundary condition for u means that $u(x, t)$ takes negative values for $x < L$; (3.2) implies that whenever u is negative, the corresponding q^* is actually buying the underlying stock.

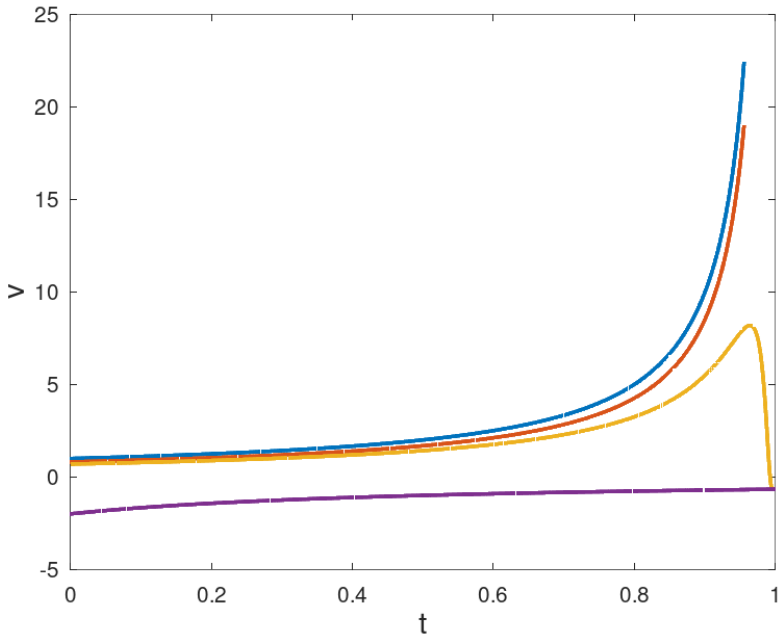


Figure 6.2: Graph of $u(x, \cdot)$ for $x \in \{0, L, 1.2L, 5L\}$

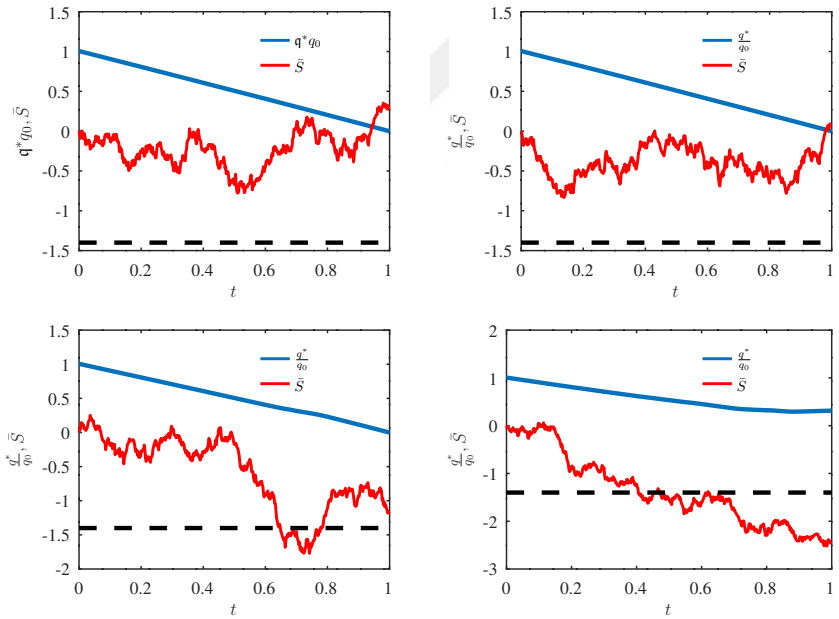


Figure 6.3: Sample paths of $\bar{S}$ and q^* for $I = 1$ and $\boldsymbol{S} = \{\bar{S}_T > L\}$; the dashed line shows $L = -1.4\sigma$

Figure 6.3 shows four sample paths of $\bar{S}$ and q^*. In the first two examples $\bar{S}$ stays above L at all times and the corresponding q^* goes parallel to $q^{*,S}$ of (6.6), the optimal liquidation path for the standard IS order. In the third example $\bar{S}$ is below L

approximately in the time interval $[0.6, 0.8]$ when trading slows down, it goes above L around 0.8 and closes above L; correspondingly q^* speeds up trading after 0.8 and closes the position at terminal time. In the fourth example, $\bar{S}$ hits L around the middle of the trading interval and remains below L till the end; correspondingly q^* slows down and stops trading and the position is only partially closed at terminal time. In the last example q^* is in fact slightly increasing near $t = T = 1$ (i.e., q^* buying the underlying asset) ; this is due to the negative value that the terminal value takes for $x < L$. These examples suggest that q^* behaves approximately as follows: when $\bar{S}$ is above L, it behaves like the standard IS algorithm $q^{*,S}$, linearly closing the remaning position; when $\bar{S}$ goes below L, q^* slows down/ stops trading. The negative boundary condition implies that the algorithm can in fact execute buy trades especially when the price is below L near terminal time T.

6.1.1 Distribution of $(\mathfrak{q}_T, A)$

For $\boldsymbol{S} = \{\bar{S}_T > L\}$, the position fully closes when the closing price is above the lowerbound L, therefore, the probability that the algorithm closes the position at terminal time is:

$$\mathbb{P}(q_T = 0) = \mathbb{P}(W_T > L) = 1 - N_{0,1}(L/\sqrt{T})$$

where $N_{0,1}$ denotes the standard normal distribution.

A random variable E is said to be exponentially distributed with rate λ if $\mathbb{P}(E > x)) = e^{-\lambda x}$, i.e.,

$$-\log(\mathbb{P}(E > x)) = \lambda x; \tag{6.12}$$

a well known fact is

$$\mathbb{E}[E] = \frac{1}{\lambda}. \tag{6.13}$$

The distribution of $\mathfrak{q}_T$ over $(0, \infty)$ depends on u via (6.11) and it obviously doesn't have an explicit formula. Figure 6.4 shows graphs of $x \mapsto \mathbb{P}(\mathfrak{q}_T > x | \mathfrak{q}_T > 0)$, $x \mapsto -\log(\mathbb{P}(\mathfrak{q}_T > x | \mathfrak{q}_T > 0))$ and $x \mapsto \dfrac{x}{\mathbb{E}[\mathfrak{q}_T | q^*_T > 0]}$ (all estimated via simulating 10^4 sample paths).

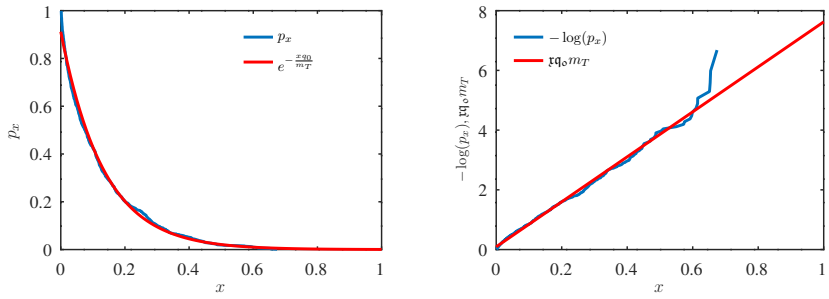


Figure 6.4: On the left: graph of $p_x = \mathbb{P}(\mathfrak{q}_T > x | q_T^* > 0)$, on the right: graphs of $-\log(p_x)$ and $\dfrac{xq_0}{m_T}$, $m_T = \mathbb{E}[\mathfrak{q}_T | q_T^* > 0]$

These graphs, (6.12) and (6.13) suggest that the exponential distribution provides a rough approximation for the conditional distribution of $\mathfrak{q}_T$ given the event $\{q_T^* > 0\}$. An exponentially distributed random variable satisfies $\mathbb{E}[E] = \dfrac{1}{\lambda}$ and $var(E) = \dfrac{1}{\lambda^2}$. In the case of $\mathfrak{q}_T$ conditioned over $\{q_T^* > 0\}$ we have the Monte Carlo estimates $\mathbb{E}[\mathfrak{q}_T | \mathfrak{q}_T > 0] = 0.1218$ and $var(\mathfrak{q}_T | \mathfrak{q}_T > 0)^{1/2} = 0.1387$ for the parameter values specified above.

q^* depends on L via the domain of the PDE (6.9) and on $kV/2\eta$ via the terminal condition (6.10). Figure 6.5 shows how $\mathbb{E}[\mathfrak{q}_T | \mathfrak{q}_T > 0]$ and $var(\mathfrak{q}_T | \mathfrak{q}_T > 0)^{1/2}$ vary with these parameters.

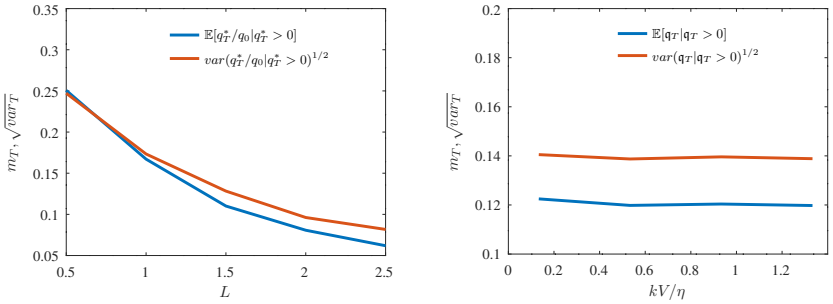


Figure 6.5: Graph of $m_T = \mathbb{E}[\mathfrak{q}_T | \mathfrak{q}_T > 0]$ and $\sqrt{var_T} = var(\mathfrak{q}_T | \mathfrak{q}_T > 0)^{1/2}$ as a function of L

We have already noted that the permanent impact factor term A_1 of (6.5) is fully determined by $\mathfrak{q}_T$. We now consider the joint distribution of $(A_2, \mathfrak{q}_T)$. This distribution consists of two parts: the distribution of A_2 conditioned on $\mathfrak{q}_T = 0$ (i.e., the cases where the algorithm closes the initial position q_0 fully) and the conditional distribution of A_2 given $\mathfrak{q}_T$ for $\mathfrak{q}_T > 0$ (the cases where the algorithm closes the initial position q_0

partially). If q^* were a deterministic function (as in the case of the standard IS order), A_2 would be normally distributed by the normal and independent increments of W. The q-q plot of the conditional distribution of A_2 given $q_T^* = 0$ and $q_T^* = x$ for several values of x is shown Figure 6.6; (for $x > 0$ we approximate $P(A_2 \in A | q_T^*/q_0 = x)$ with $P(A_2 \in A | \mathfrak{q}_T \in (x, x+\delta)$ where $\delta > 0$ is small and we estimate the latter by simulating 2×10^5 sample paths of W and q^*). These plots suggest that the conditional distribution A_2 given $\mathfrak{q}_T$ is approximately normal even though q^* is random and a function of W.

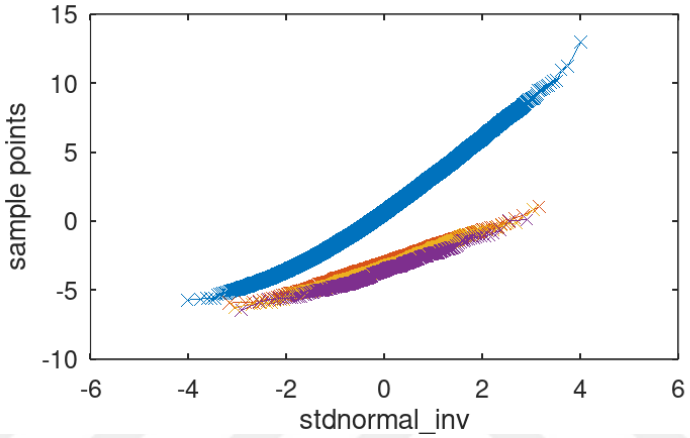


Figure 6.6: q-q plots of the conditional distribution of A_2 given $\mathfrak{q}_T = x$ for $x = 0$, $x = 0.06 + j0.07$, $j \in \{1, 2, 3, 4\}$

Figure 6.7 shows the graphs of $\mathbb{E}[A_2 | \mathfrak{q}_T = x]$ and $\sqrt{var(A_2 | \mathfrak{q}_T = x)}$ (using the same approximation as above and then simulation).

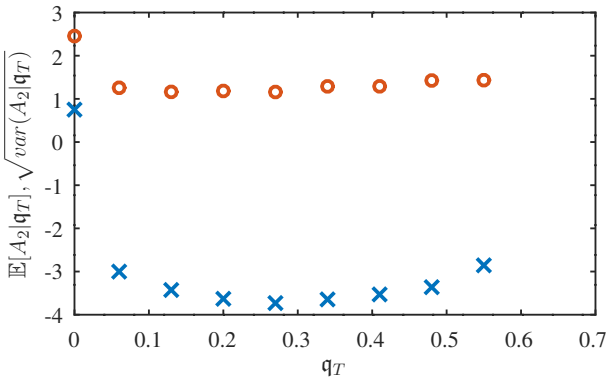


Figure 6.7: Graphs of $\mathbb{E}[A_2 | \mathfrak{q}_T = x]$ and $\sqrt{var(A_2 | \mathfrak{q}_T = x)}$ for $x = 0$ and $x = 0.06 + j0.07$, $j \in \{1, \cdots, 8\}$

For different L values $(-\sigma, -1.4\sigma)$ there is almost no difference for the fully liqui-

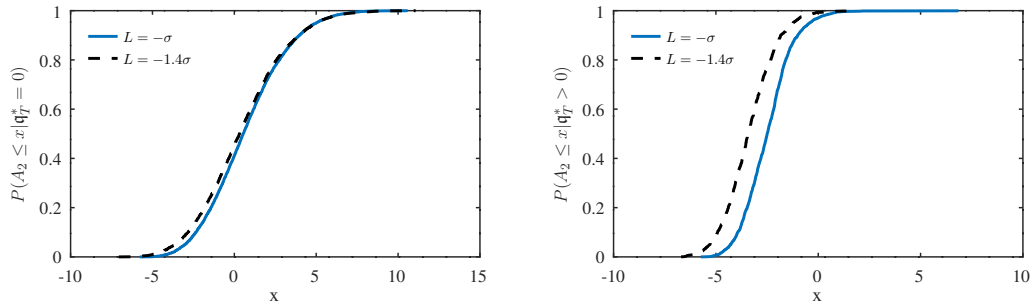


Figure 6.8: Graphs of $P(A_2 \leq x)|\mathfrak{q}_T = 0)$ and $P(A_2 \leq x)|\mathfrak{q}_T > 0)$ for $L = -\sigma$ and $L = -1.4\sigma$

dated cases $P(A_2 \leq x)|\mathfrak{q}_T = 0)$. But when we checked the cases that portfolio is not liquidated fully $P(A_2 \leq x)|\mathfrak{q}_T > 0)$ there is a slight difference.

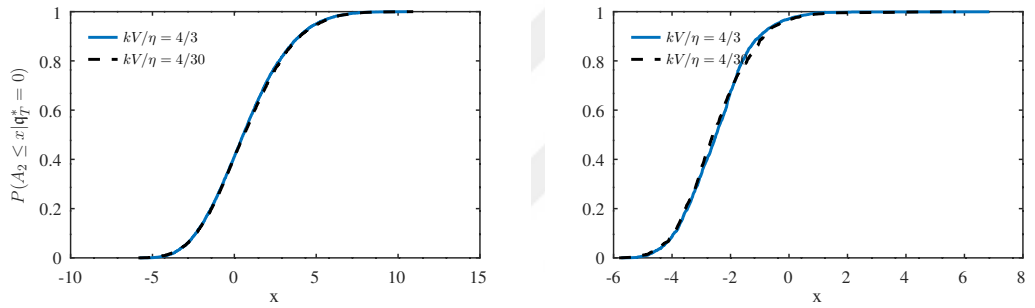


Figure 6.9: Graphs of $P(A_2 \leq x)|\mathfrak{q}_T = 0)$ and $P(A_2 \leq x)|\mathfrak{q}_T > 0)$ for $kV/\eta = 4/3$ and $kV/\eta = 4/30$

For different kV/η values $(\frac{4}{3}, \frac{4}{30})$ there is no difference for the fully liquidated cases $P(A_2 \leq x)|\mathfrak{q}_T = 0)$. As well as for the cases that portfolio is not liquidated fully $P(A_2 \leq x)|\mathfrak{q}_T > 0)$.

We now consider the joint distribution of $(A_3, \mathfrak{q}_T)$. The left graph in Figure 6.10 shows the conditional distribution of A_3 given $\mathfrak{q}_T = 0$; we see that most of the mass of this distribution is concentrated around a point near 1. This is similar to the behavior $A_3 = 1$ for the standard IS order (See (6.7)). The right graph of Figure 6.10 shows how the conditional mean and variance of A_3 changes with $\mathfrak{q}_T$; as opposed to A_2 this graph suggests that these conditional mean and variance are relatively flat as a function of $\mathfrak{q}_T$.

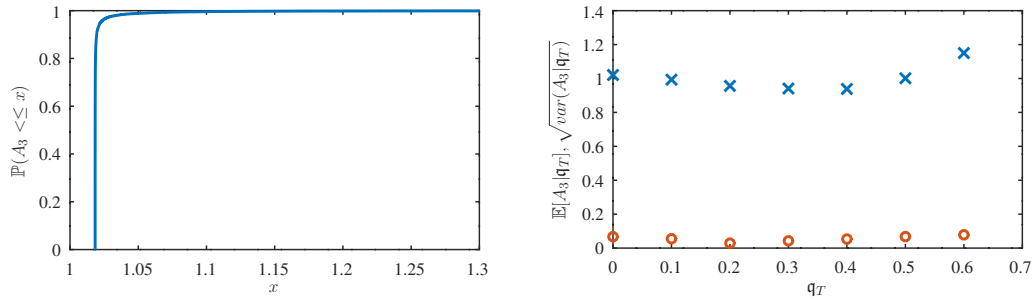


Figure 6.10: On the left: distribution of A_3 conditioned on $\mathfrak{q}_T = 0$, on the right: graphs of $\mathbb{E}[A_3|\mathfrak{q}_T = x]$ $(\times)$ and $var(A_3|\mathfrak{q}_T = x)^{1/2}$ (o)

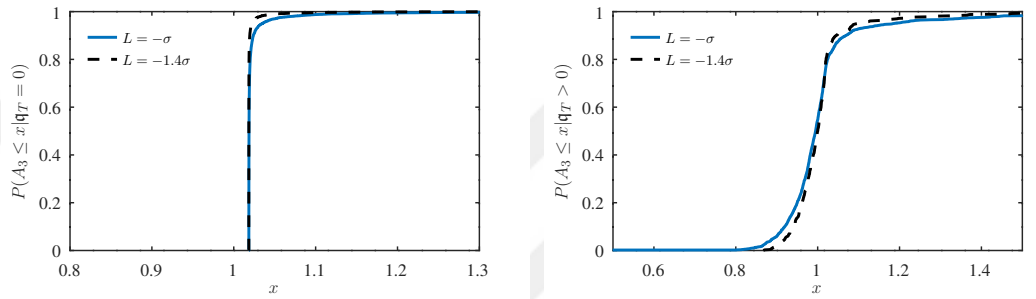


Figure 6.11: Graphs of $P(A_3 \leq x)|\mathfrak{q}_T = 0)$ and $P(A_3 \leq x)|\mathfrak{q}_T > 0)$ for $L = -\sigma$ and $L = -1.4\sigma$

For different L values $(-\sigma, -1.4\sigma)$ there is almost no difference for the fully liquidated cases $P(A_3 \leq x)|\mathfrak{q}_T = 0)$. But when we checked the cases that portfolio is not liquidated fully $P(A_3 \leq x)|\mathfrak{q}_T > 0)$ there is a slight difference. The behavior of probability is changed for fully liquidated cases $\mathfrak{q}_T = 0$ around 90% take same value. For the not fully liquidated cases $\mathfrak{q}_T > 0$ the values of A_3 distributed between $0.6 - 1.4$.

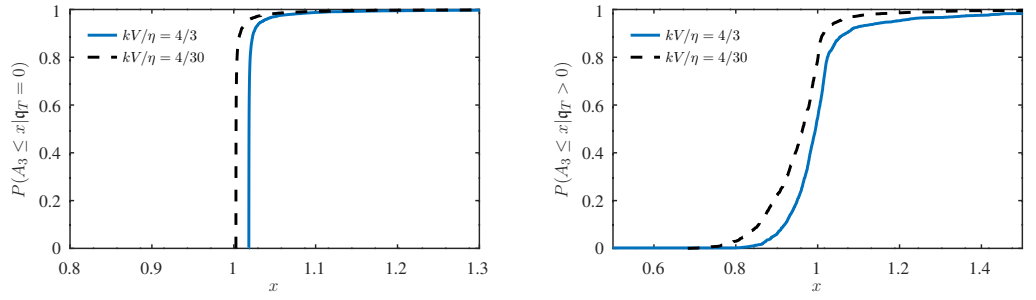


Figure 6.12: Graphs of $P(A_3 \leq x)|\mathfrak{q}_T = 0)$ and $P(A_3 \leq x)|\mathfrak{q}_T > 0)$ for $kV/\eta = \dfrac{4}{3}$ and $kV/\eta = \dfrac{4}{30}$

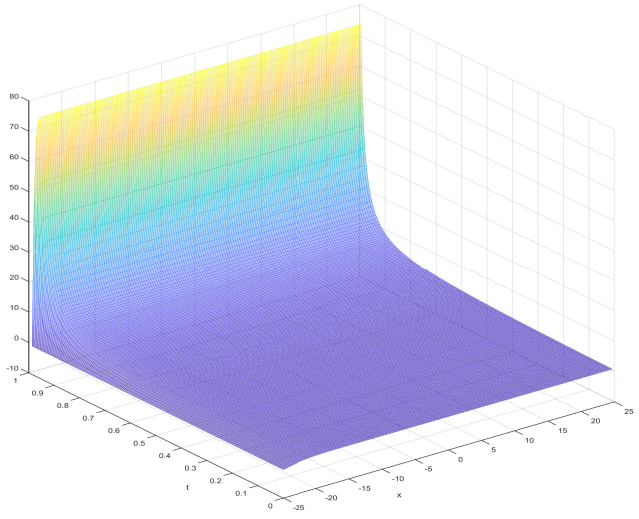


Figure 6.13: graph of $u^{(2)}$ for $I = 1_{\{t<\tau_L\}}$ and $S = \{\tau_L > T\}$

We note that decreasing kV/η shifts both distributions to the left.

6.2 $I = 1_{\{t<\tau_L\}}$ and $S = \{\tau_L > T\}$

In this model the full liquidation take place only when $S = \{\tau_L > T\}$ and trading takes place until $\tau_L \wedge T$. The value function and the optimal control is computed as in the previous section using the PDE represenation given in Section 5.2.

As before we begin by a graph of the solution of the PDE in Figures 6.13 and 6.14.

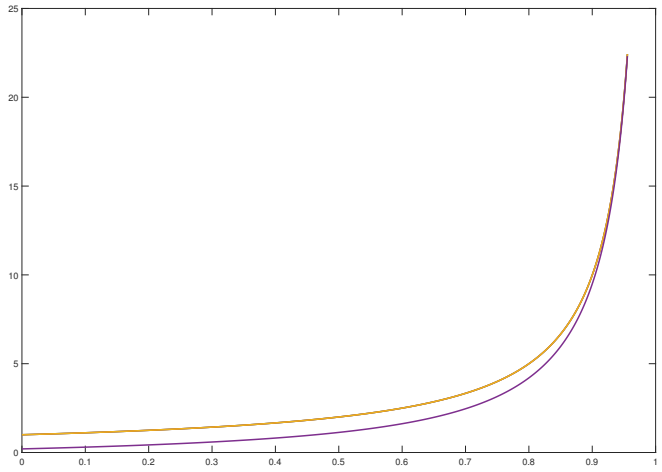


Figure 6.14: Graph of $u^{(2)}(x, \cdot)$ for $x \in \{0, L, 1.2L, 5L\}$

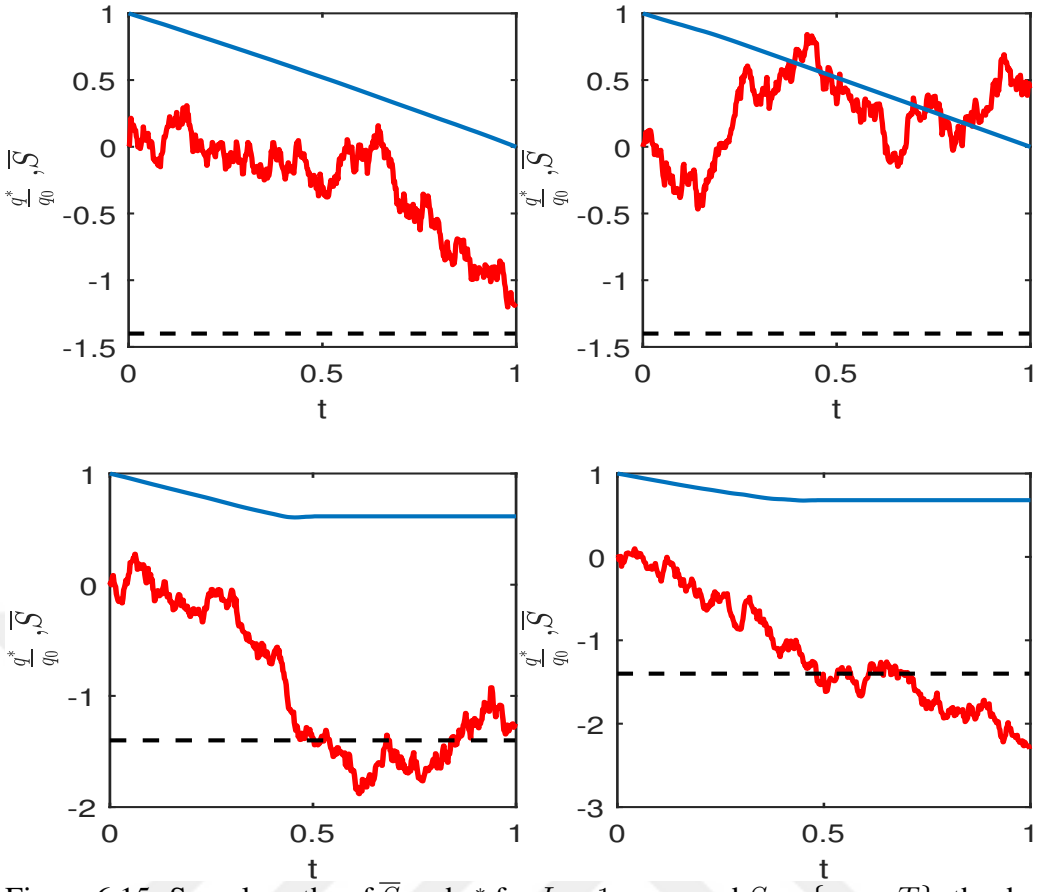


Figure 6.15: Sample paths of $\overline{S}$ and q^* for $I = 1_{\{t<\tau_L\}}$ and $S = \{\tau_L > T\}$; the dashed line shows $L = -1.4\sigma$ where $\sigma = 1$

Qualitatively these figures look similar to the PDE solutions given in the previous section.

Figure 6.15 shows four sample paths of $\bar{S}$ and q^*. Qualitatively these sample paths are similar to those given in the previous section. The most important difference is when price hits L in these sample paths trading stops completely (rather than slowing down) and never restarts. This behavior is explicitly encoded in the choice $I_t = \mathbf{1}_{\{t \leq \tau_L\}}$.

6.2.1 Distribution of $(\mathfrak{q}_T, A)$

To estimate the density of $\mathfrak{q}_T$ given $q_T^* > 0$ we use the following kernel smoothing:

$$\hat{f}(x) = \frac{1}{\delta} \sum_{i=1}^{n} \mathfrak{n}(0, 1, \frac{x - \mathfrak{q}_T^i}{\delta}) \frac{1}{n}, \tag{6.14}$$

where $\mathfrak{q}_T^i$ are the simulated independent samples of $\mathfrak{q}_T$ and $\mathfrak{n}(0, 1, \cdot)$ is the standard normal density.

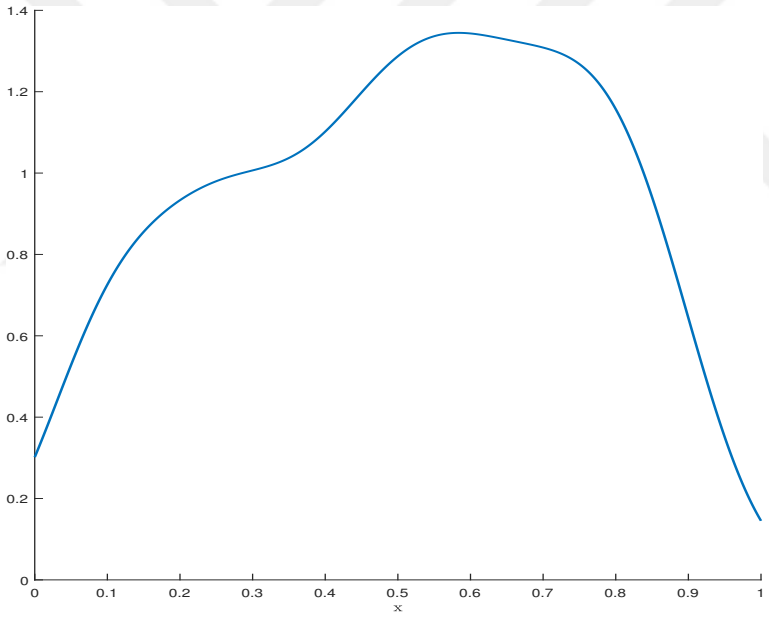


Figure 6.16: Density of $(\mathfrak{q} \mid q_T^* > 0)$

The graph of $\hat{f}$ is presented in Figure 6.16; recall that in the previous section this conditional distribution looked approximately exponential. This graph suggests that this is not the case for $I = 1_{\{t \leq \tau_L\}}$.

The remaining figure in the present section reproduce the figures in the previous section given for the case $I = 1$ and $\boldsymbol{S} = \{W_T > L\}$ which show the conditional distribution of A_2 and A_3 given $\mathfrak{q}_T$. Qualitatively the behavior of these distributions

are similar to their counterparts in the previous section therefore, we only comment on those graphs where the behavior is different compared to the results in the previous section.

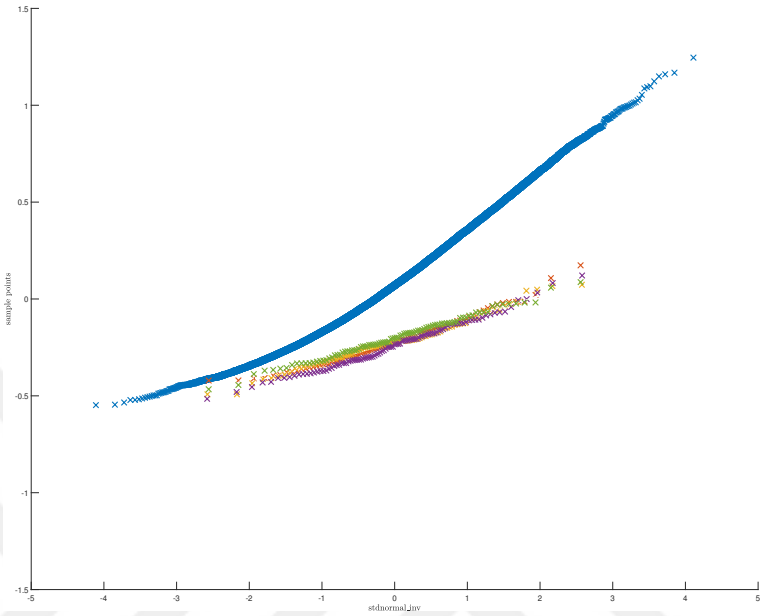


Figure 6.17: q-q plots of the conditional distribution of A_2 given $\mathfrak{q}_T = 0$ for $x = 0$,$x = 0.06 + j0.07$,$j \in \{1, 2, 3, 4\}$

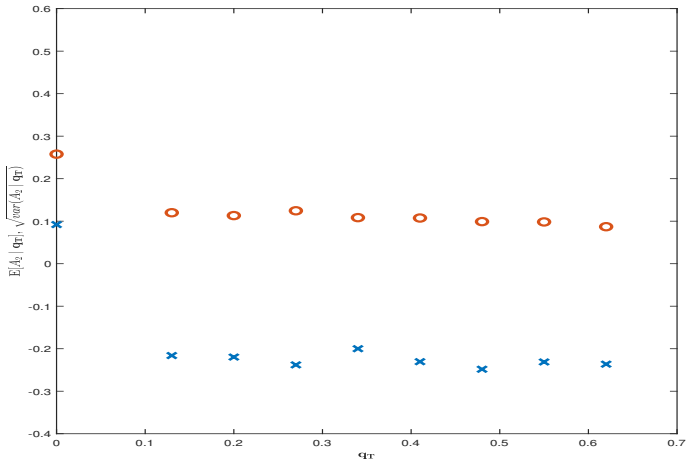


Figure 6.18: Graphs of $\mathbb{E}[A_2 \mid \mathfrak{q}_T = x]$, $\sqrt{var(A_2 \mid \mathfrak{q}_T = x)}$ for $x = 0$,$x = 0.06 + j0.07$,$j \in \{1, \ldots, 8\}$

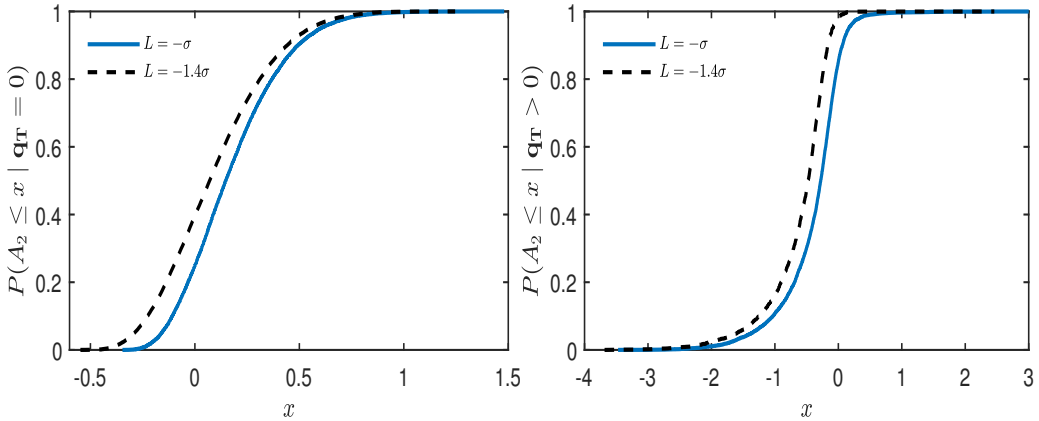


Figure 6.19: Graphs of $P(A_2 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_2 \leq x \mid \mathfrak{q}_T > 0)$ for $L = -\sigma$ and $L = -1.4\sigma$

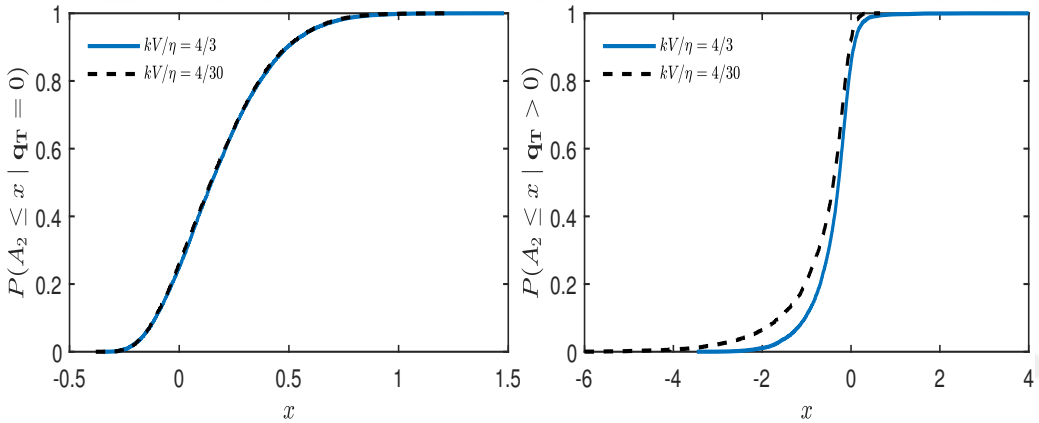


Figure 6.20: Graphs of $P(A_2 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_2 \leq x \mid \mathfrak{q}_T > 0)$ for $\kappa V/\eta = 4/3$ and $\kappa V/\eta = 4/30$

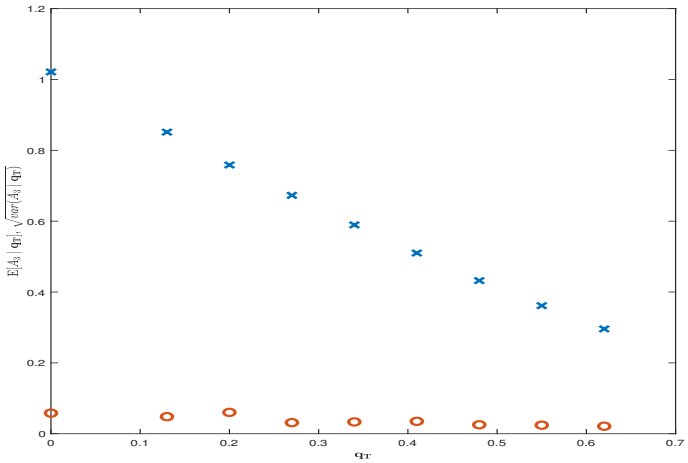


Figure 6.21: Graphs of $\mathbb{E}[A_3 \mid \mathfrak{q}_T]$, $\sqrt{var(A_3 \mid \mathbf{q}_T = x)}$ for $x = 0$,$x = 0.06 + j0.07$,$j \in \{1, \ldots, 8\}$

A notable difference as compared to Figure 6.10 is that the expectation $\mathbb{E}[A_3|\mathfrak{q}_T = x]$ here is significantly more decreasing in x than in Figure 6.10.

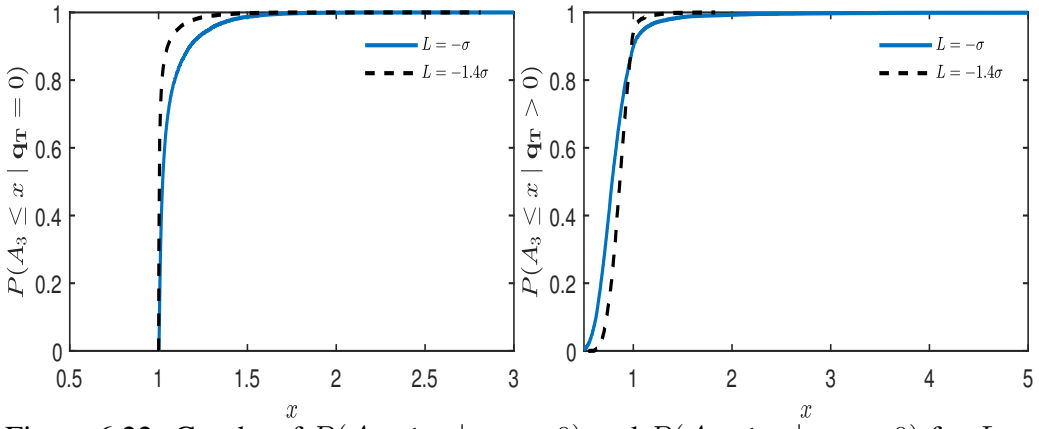


Figure 6.22: Graphs of $P(A_3 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_3 \leq x \mid \mathfrak{q}_T > 0)$ for $L = -\sigma$ and $L = -1.4\sigma$

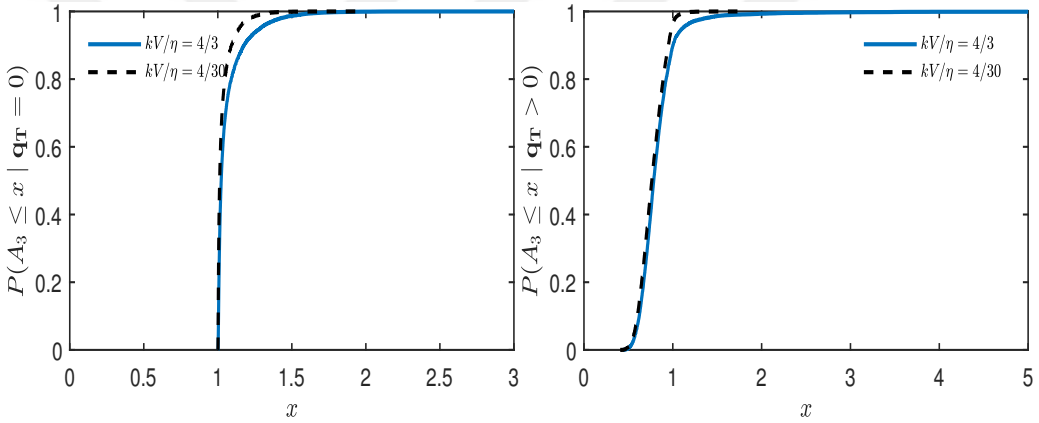


Figure 6.23: Graphs of $P(A_3 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_3 \leq x \mid \mathfrak{q}_T > 0)$ for $\kappa V/\eta = 4/3$ and $\kappa V/\eta = 4/30$

6.3 $I_t = 1_{\{W_t > L\}}$ and $S = \{\tau_{T-\delta,L} > T\}$

In this model trading pauses when the price $\bar{S}$ is below the lower bound L and Full liquidation take place if the price process $\bar{S}$ remains above L in the time interval $[T - \delta, T]$. The value function and the optimal control is computed as in the previous sections using the PDE represenation given in Section 5.3. The results in this section are qualitatively similar to those given in the previous section; therefore we will only comment on the differences from the observations made in the previous section.

The graph of v for the present case is shown Figures 6.24 and 6.25.

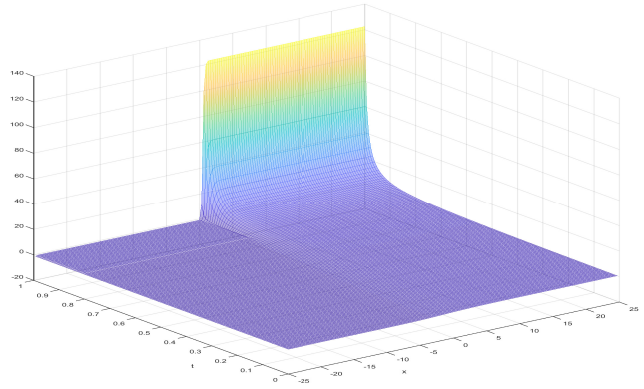


Figure 6.24: graph of u for $I_t = 1_{\{W_t > L\}}$ and $S = \{\tau_{T-\delta,L} > T\}$

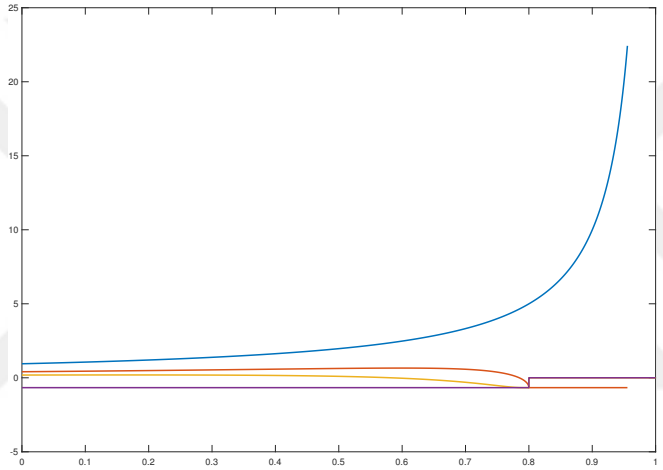


Figure 6.25: Graph of $u(x, \cdot)$ for $x \in \{0, L, 1.2L, 5L\}$

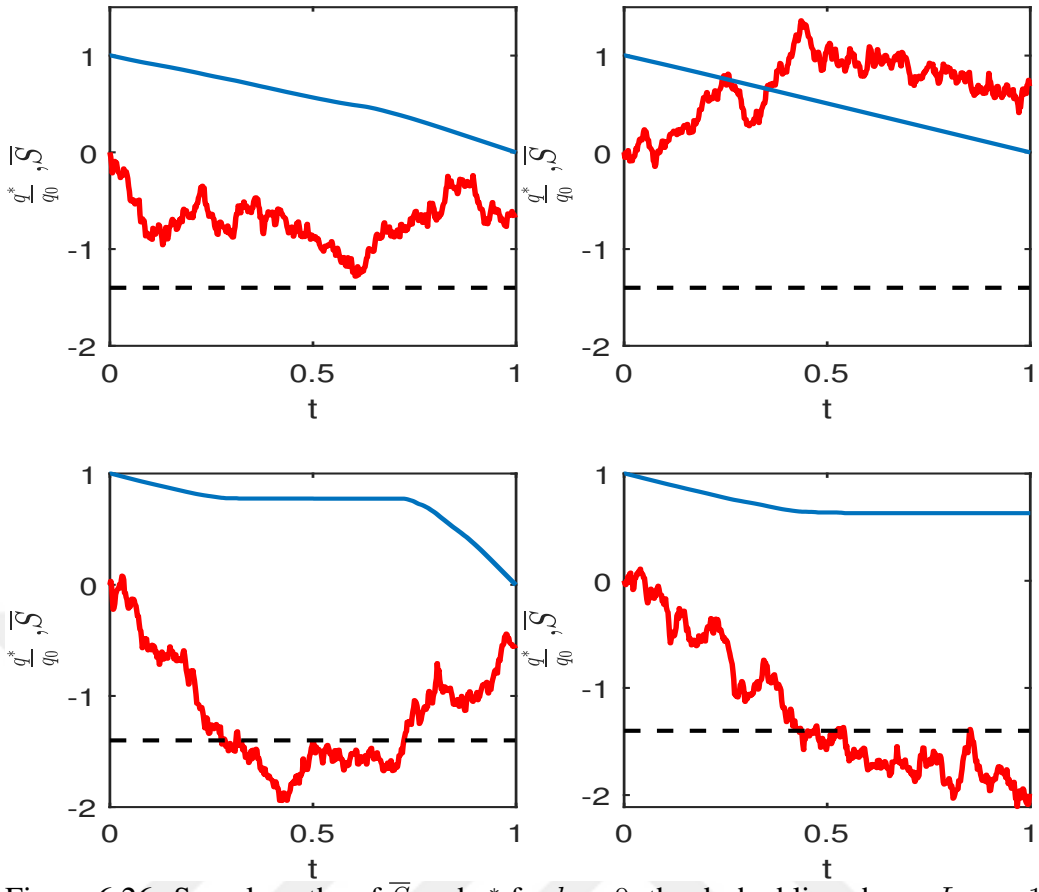


Figure 6.26: Sample paths of $\overline{S}$ and q^* for $b = 0$; the dashed line shows $L = -1.4\sigma$ where $\sigma = 1$

The only qualitative difference in these figures as compared to Figure 6.14 is that $v(\cdot, x)$ is flat on the interval $[T - \delta, T]$ for $x < L$; this arises from the fact that trading stops if the price is below L after time $T - \delta$.

Figure 6.26 shows four sample paths of $\bar{S}$ and q^*. As already noted these sample paths looks similar to those given in Figure 6.15 except for the following important difference: in the current setup the trading continues when $\bar{S}$ is above L.

6.3.1 Distribution of $(\mathfrak{q}_T, A)$

Figure 6.27 shows the approximate density of $\mathfrak{q} \mid q_T^* > 0$ computed with the kernel density estimation of the previous section. All of the quantities presented in Figures 6.27-6.36 show similar behavior to their counterparts whose graphs were presented in the previous section; and the comments made in the previous section mostly apply

in this section as well:

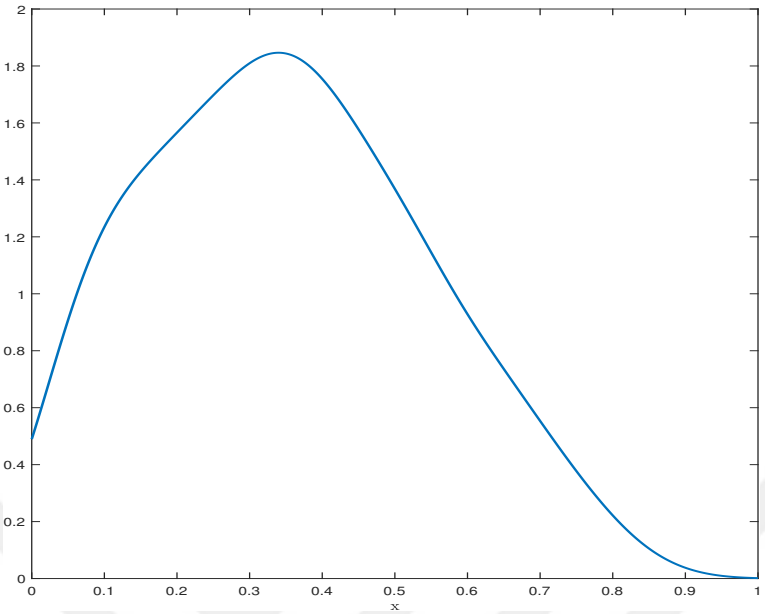


Figure 6.27: Density of $(\mathfrak{q} \mid q_T^* > 0)$

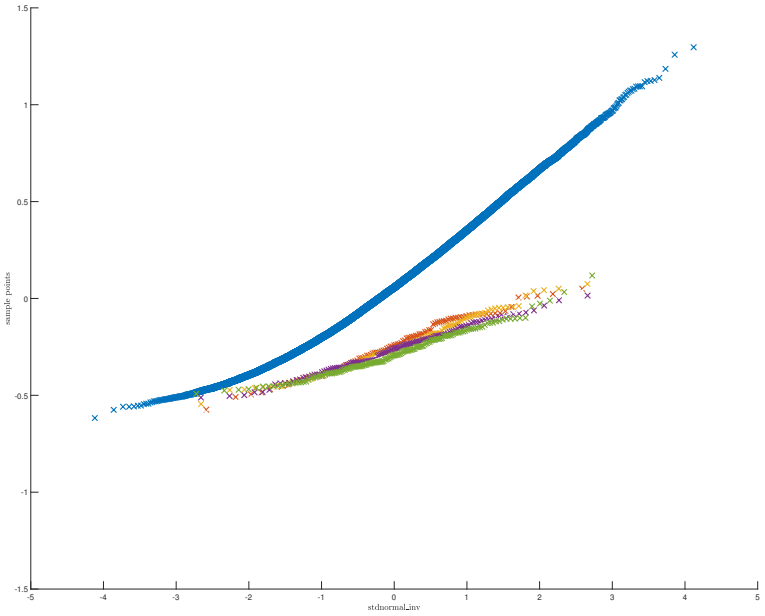


Figure 6.28: q-q plots of the conditional distribution of A_2 given $\mathfrak{q}_T = 0$ for $x = 0$,$x = 0.06 + j0.07$,$j \in \{1, 2, 3, 4\}$

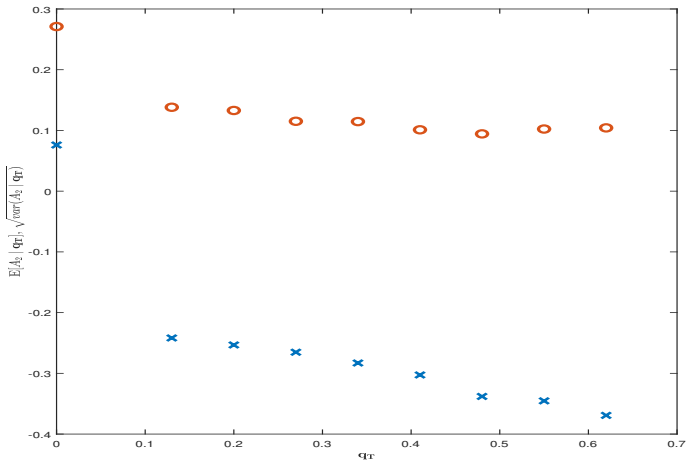


Figure 6.29: Graphs of $\mathbb{E}[A_2 \mid \mathfrak{q}_T]$, $\sqrt{var(A_2 \mid \mathbf{q}_T = x)}$ for $x = 0$,$x = 0.06 + j0.07$,$j \in \{1, \ldots, 8\}$

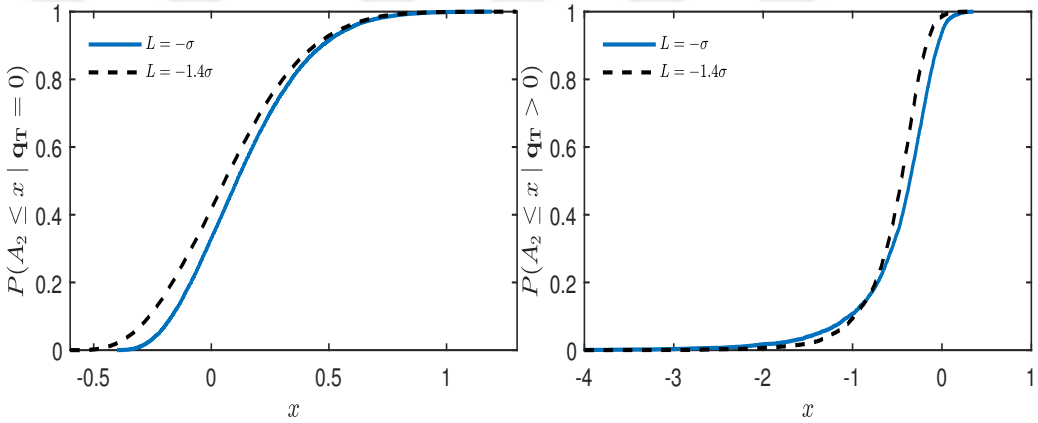


Figure 6.30: Graphs of $P(A_2 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_2 \leq x \mid \mathfrak{q}_T > 0)$ for $L = -\sigma$ and $L = -1.4\sigma$

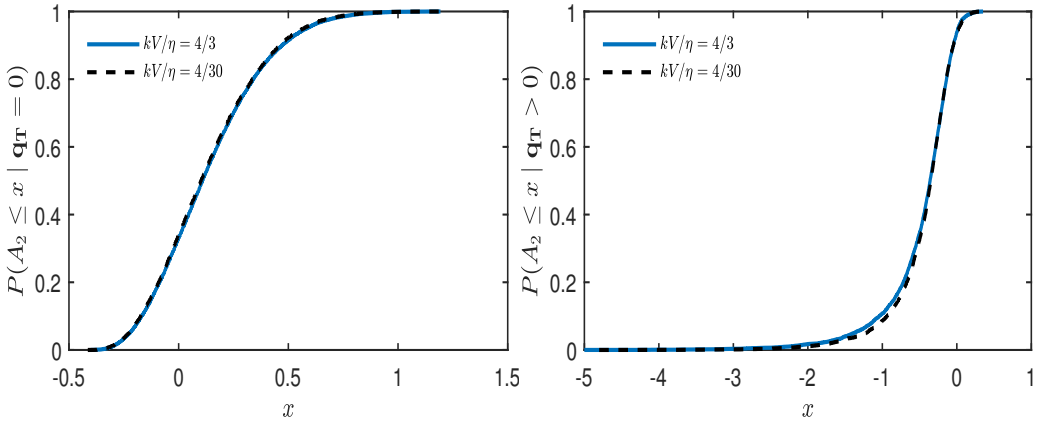


Figure 6.31: Graphs of $P(A_2 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_2 \leq x \mid \mathfrak{q}_T > 0)$ for $\kappa V/\eta = 4/3$ and $\kappa V/\eta = 4/30$

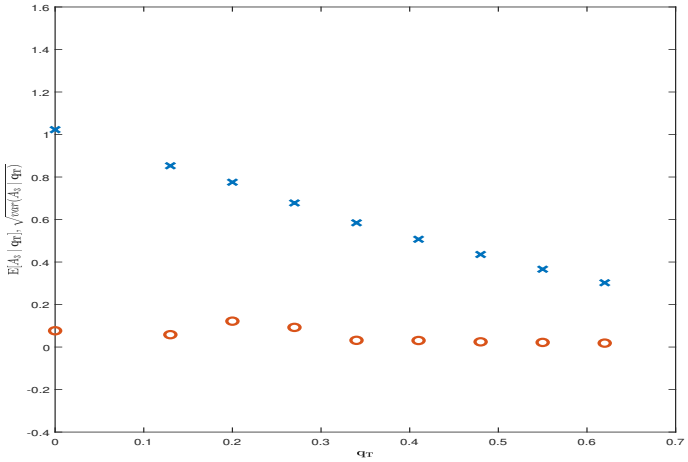


Figure 6.32: Graphs of $\mathbb{E}[A_3 \mid \mathfrak{q}_T]$, $\sqrt{var(A_3 \mid \mathbf{q}_T = x)}$ for $x = 0$,$x = 0.06 + j0.07$,$j \in \{1, \ldots, 8\}$

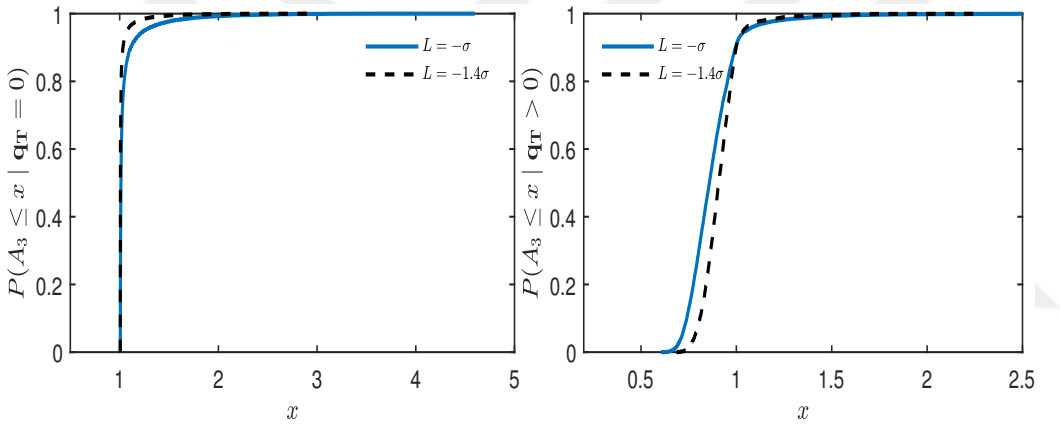


Figure 6.33: Graphs of $P(A_3 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_3 \leq x \mid \mathfrak{q}_T > 0)$ for $L = -\sigma$ and $L = -1.4\sigma$

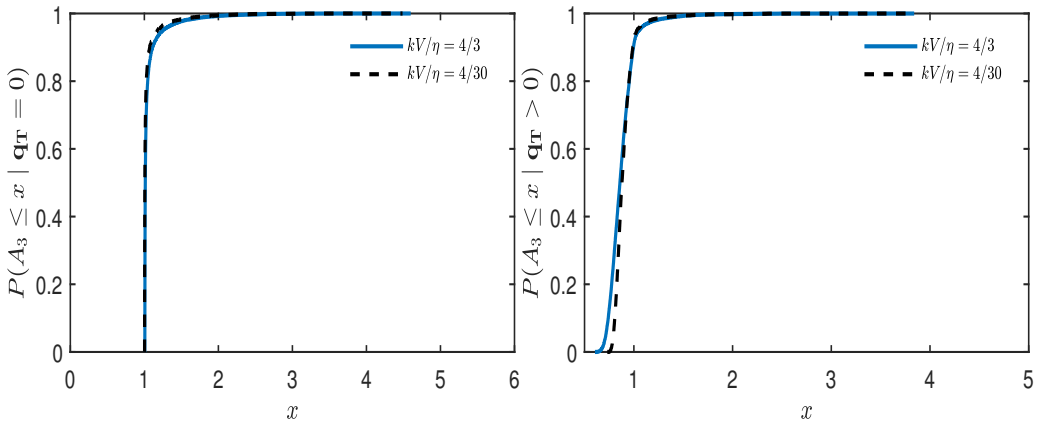


Figure 6.34: Graphs of $P(A_3 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_3 \leq x \mid \mathfrak{q}_T > 0)$ for $\kappa V/\eta = 4/3$ and $\kappa V/\eta = 4/30$

A novel feature of the algorithm in the present section is the δ parameter; recall that in the present case we have $\boldsymbol{S} = \{\tau_{T-\delta,L} > T\}$, i.e., the position is required to be fully closed if the price process remains above L in the time interval $[T-\delta, T]$. We next look at how δ influences the distributions of A_2 and A_3 in the following figures:

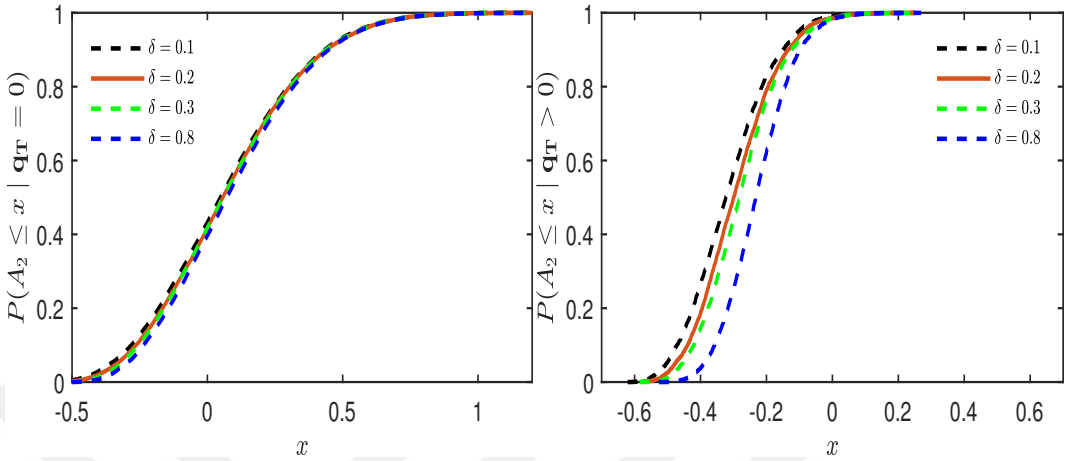


Figure 6.35: Graphs of $P(A_2 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_2 \leq x \mid \mathfrak{q}_T > 0)$ for $\delta = 0.1$, $\delta = 0.2$, $\delta = 0.3$, $\delta = 0.8$

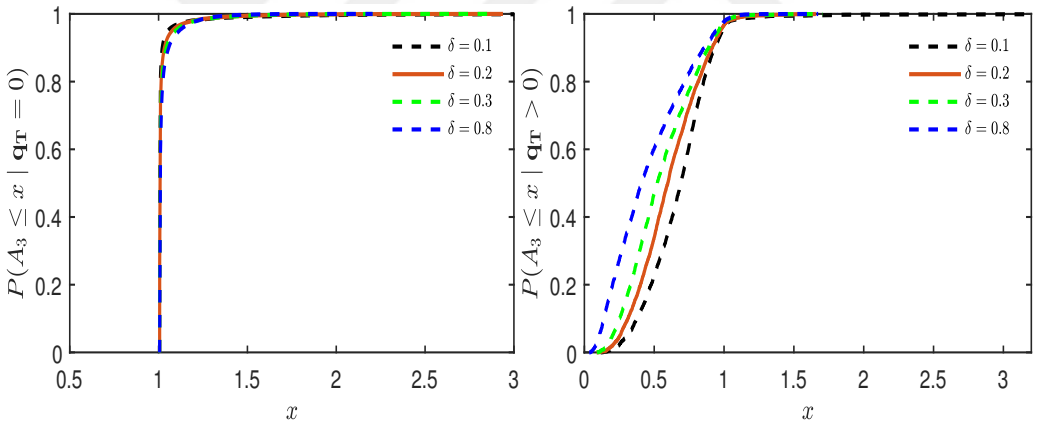


Figure 6.36: Graphs of $P(A_3 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_3 \leq x \mid \mathfrak{q}_T > 0)$ for $\delta = 0.1$, $\delta = 0.2$, $\delta = 0.3$, $\delta = 0.8$

For both A_2 and A_3 the impact of δ seems to be the following: for $\mathfrak{q}_T = 0$ δ has a limited impact; for $\mathfrak{q}_T > 0$ the distribution shifts to the left with increasing δ.

6.4 $I^{(4)}$ and $S^{(4)}$

Finally in this section we consider the case $I = I^{(4)}$ and $\boldsymbol{S} = \boldsymbol{S}^{(4)}$, which corresponds to the following trading scheme: trading stops when $\bar{S}$ hits L and restarts when it

goes back up to $L+b$. The full liquidation condition is almost the same as in previous section: the position is fully closed if $I^{(4)}_{T-\delta} = 1$ (the algorithm must be in trading mode at time $T-\delta$) and $\bar{S}$ remains above L in the time interval $[T, T-\delta]$. The value function and the optimal control is computed as in the previous sections using the PDE represenation given in Section 5.4. The results in this section are qualitatively similar to those given in the previous section; therefore we will only comment on the differences from the observations made in the previous section.

We again begin with a graph of the value function; which is qualitatively similar to the ones in the previous sections:

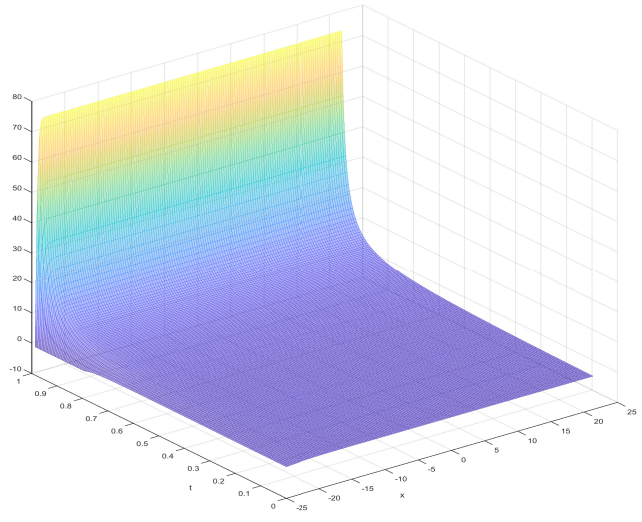


Figure 6.37: graph of v for $I^{(4)}$ and $S^{(4)}$

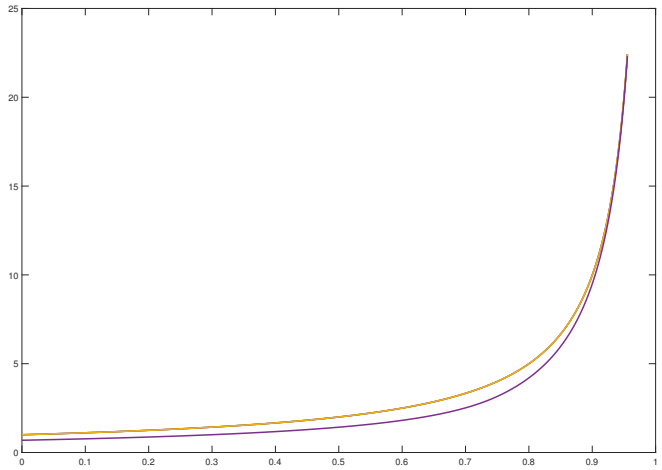


Figure 6.38: Graph of $u(x, \cdot)$ for $x \in \{0, L, 1.2L, 5L\}$

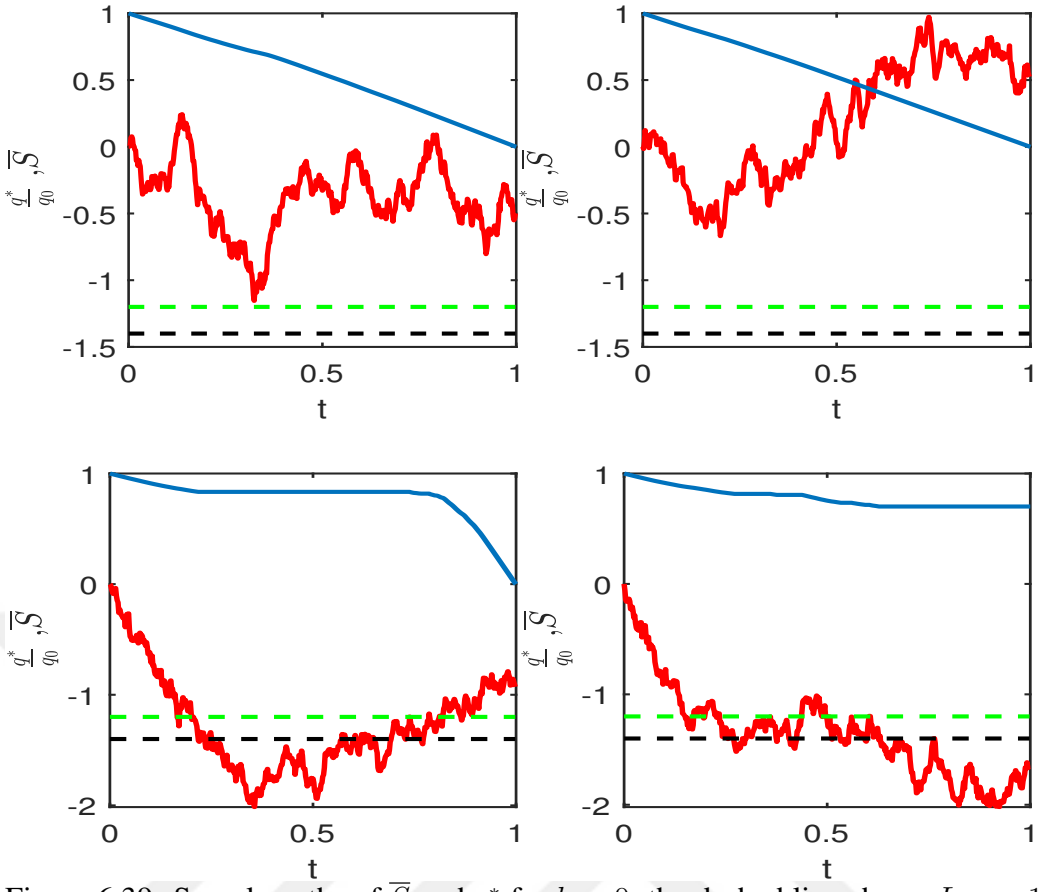


Figure 6.39: Sample paths of $\overline{S}$ and q^* for $b = 0$; the dashed line shows $L = -1.4\sigma$ where $\sigma = 1$

Figure 6.39 shows four sample paths of $\bar{S}$ and q^*. These are qualitatively very similar to those given in the previous section; the main difference is that once trading stops when $\bar{S}$ hits L it doesn't restart until $\bar{S}$ hits $L + b$; this is again an explicit feature of the choice $I = I^{(4)}$.

6.4.1 Distribution of $(\mathfrak{q}_T, A)$

Figure 6.40 shows the approximate density of $\mathfrak{q} \mid q_T^* > 0$ computed with the kernel density estimation of the previus section. All of the quantities presented in Figures 6.40-6.49 show similar behavior to their counterparts whose graphs were presented in the previous section; and the comments made in the previous section mostly apply in this section as well:

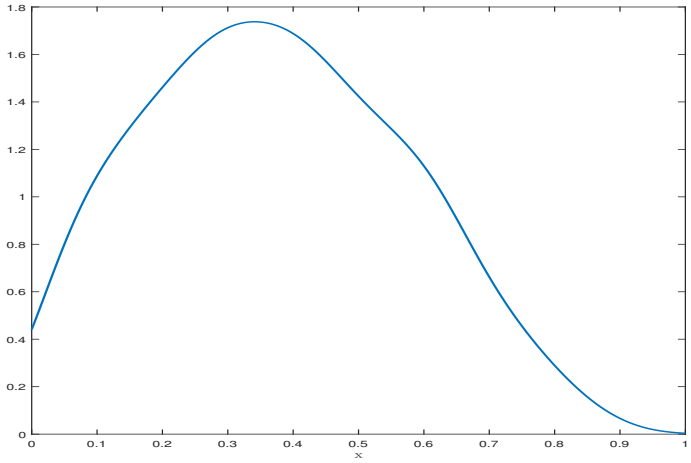


Figure 6.40: Density of $(\mathfrak{q} \mid q_T^* > 0)$

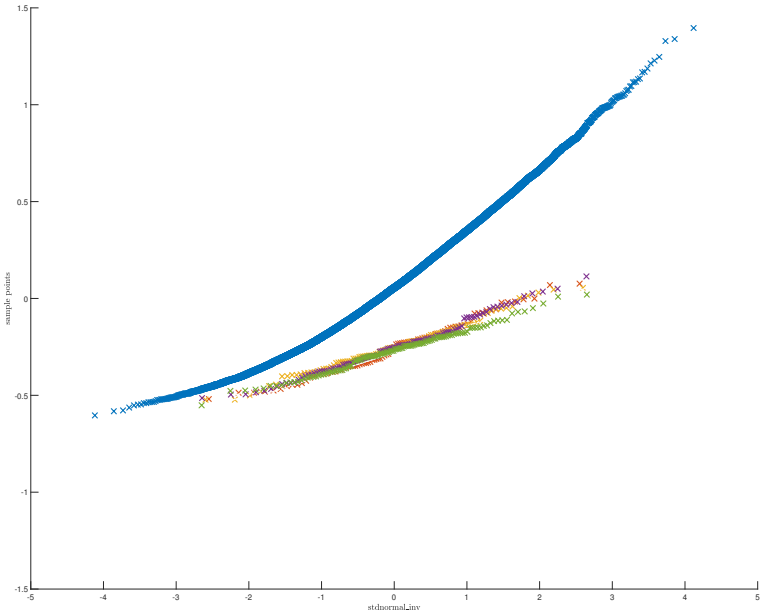


Figure 6.41: q-q plots of the conditional distribution of A_2 given $\mathfrak{q}_T = 0$ for $x = 0$,$x = 0.06 + j0.07$,$j \in \{1, 2, 3, 4\}$

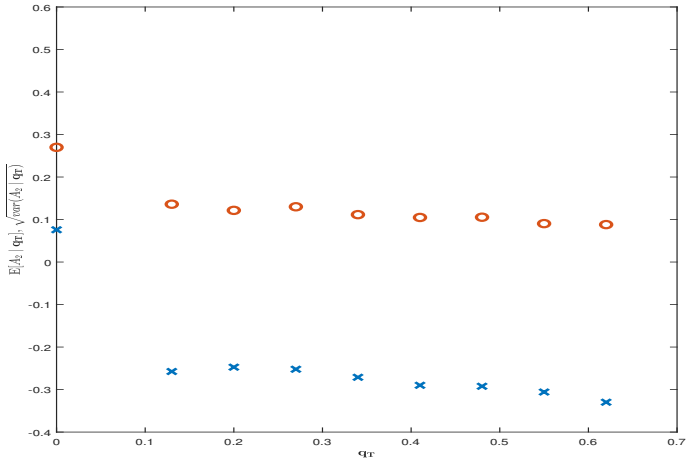


Figure 6.42: Graphs of $\mathbb{E}[A_2 \mid \mathfrak{q}_T]$, $\sqrt{var(A_2 \mid \mathbf{q_T} = x)}$ for $x = 0$,$x = 0.06 + j0.07$,$j \in \{1, \ldots, 8\}$

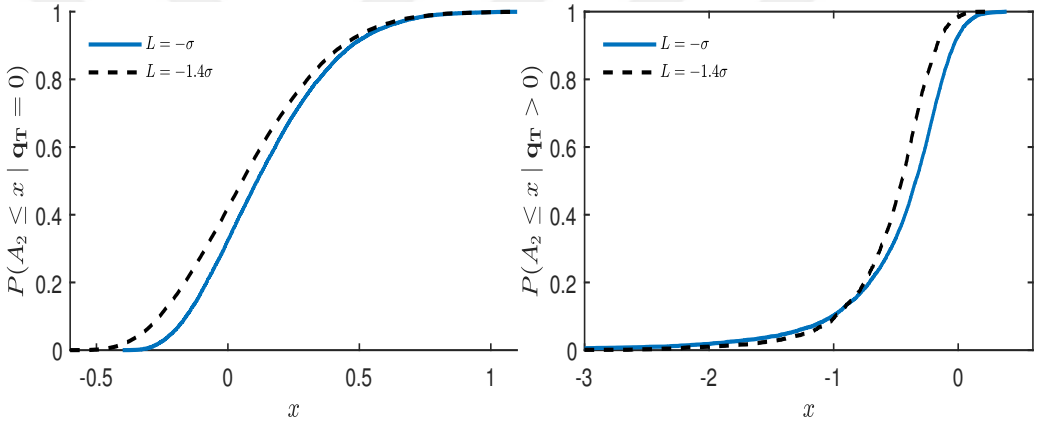


Figure 6.43: Graphs of $P(A_2 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_2 \leq x \mid \mathfrak{q}_T > 0)$ for $L = -\sigma$ and $L = -1.4\sigma$

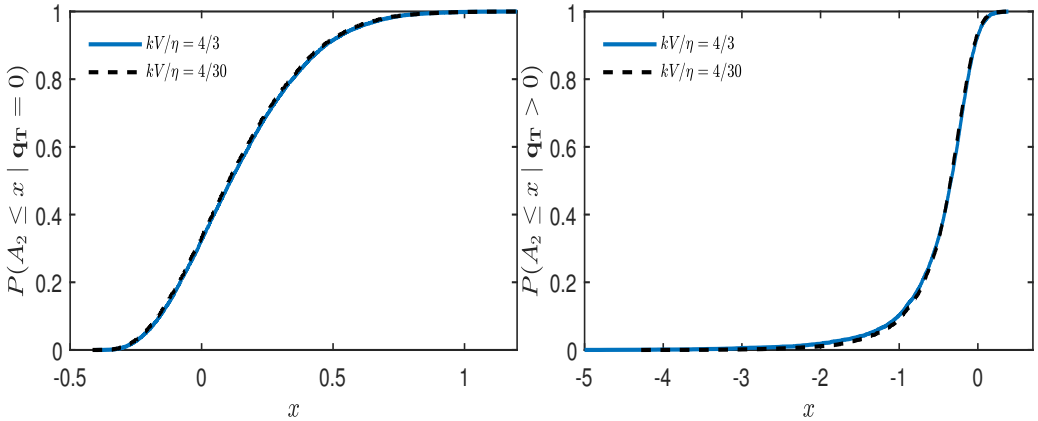


Figure 6.44: Graphs of $P(A_2 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_2 \leq x \mid \mathfrak{q}_T > 0)$ for $\kappa V/\eta = 4/3$ and $\kappa V/\eta = 4/30$

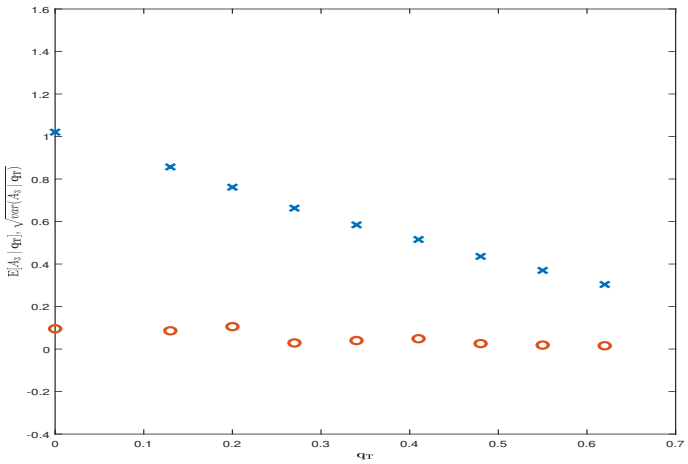


Figure 6.45: Graphs of $\mathbb{E}[A_3 \mid \mathfrak{q}_T]$, $\sqrt{var(A_3 \mid \mathbf{q}_T = x)}$ for $x = 0$,$x = 0.06 + j0.07$,$j \in \{1, \ldots, 8\}$

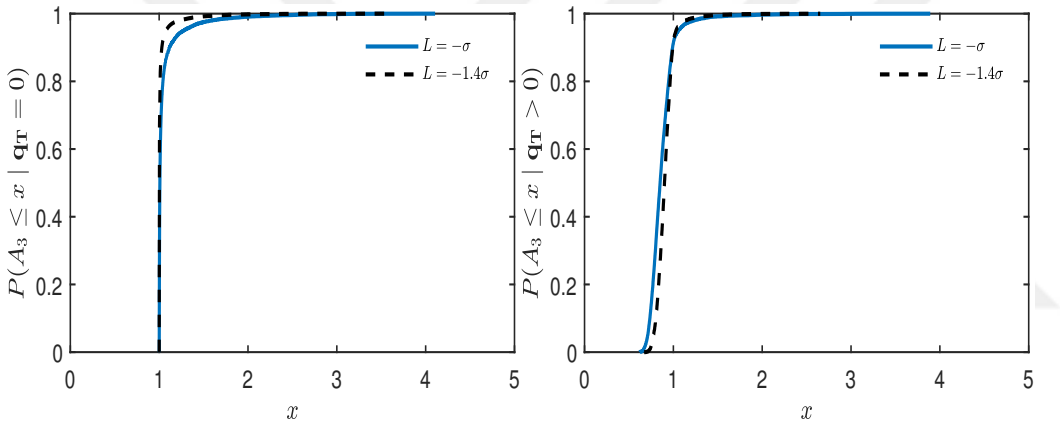


Figure 6.46: Graphs of $P(A_3 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_3 \leq x \mid \mathfrak{q}_T > 0)$ for $L = -\sigma$ and $L = -1.4\sigma$

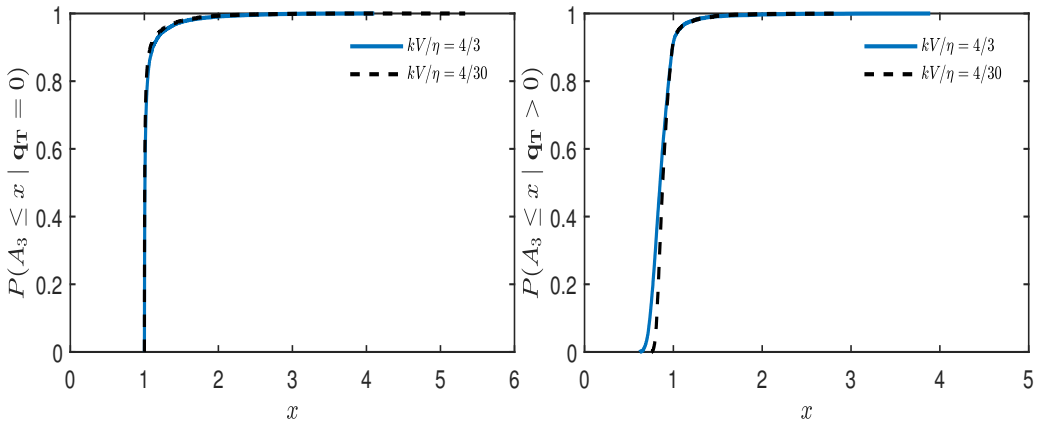


Figure 6.47: Graphs of $P(A_3 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_3 \leq x \mid \mathfrak{q}_T > 0)$ for $\kappa V/\eta = 4/3$ and $\kappa V/\eta = 4/30$

The novel parameter of the algorithm of the present section is $b > 0$ which is the buffer put on the price process before trading restarts; in the following figures we examine how this parameter impacts the distributions of A_2 and A_3:

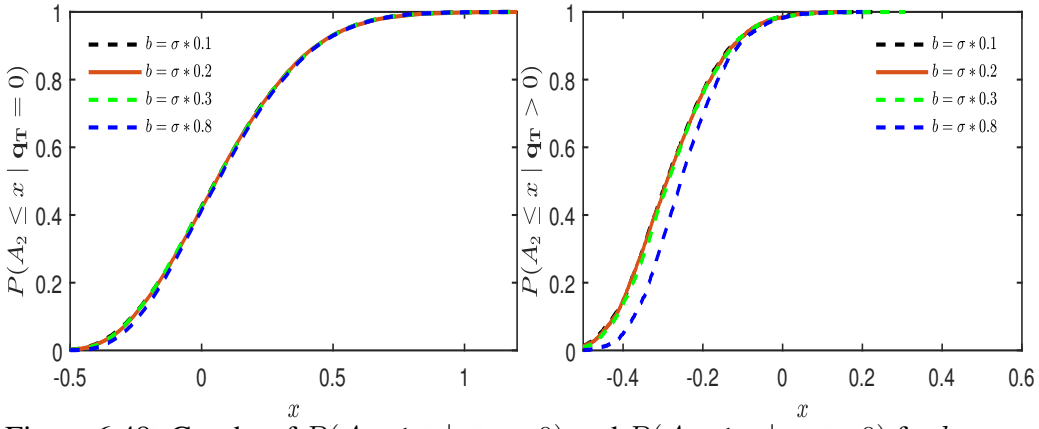


Figure 6.48: Graphs of $P(A_2 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_2 \leq x \mid \mathfrak{q}_T > 0)$ for $b = \sigma * 0.1$, $b = \sigma * 0.2$, $b = \sigma * 0.3$ and $b = \sigma * 0.8$

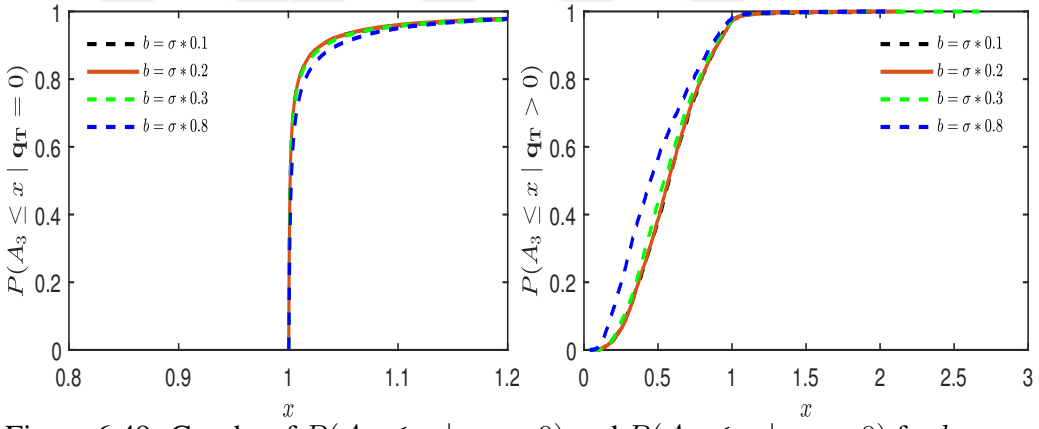


Figure 6.49: Graphs of $P(A_3 \leq x \mid \mathfrak{q}_T = 0)$ and $P(A_3 \leq x \mid \mathfrak{q}_T > 0)$ for $b = \sigma * 0.1$, $b = \sigma * 0.2$, $b = \sigma * 0.3$ and $b = \sigma * 0.8$

We note that b has limited impact on the distribution of A_2 and A_3 when the position is liquidated fully (i.e., the case $\mathfrak{q}_T = 0$). When conditioned on $\mathfrak{q}_T > 0$, we see that the distribution of A_2 shifts to the right and that of A_3 shifts to the left with increasing b.

CHAPTER 7

CONCLUSION

In this thesis we modified the standard IS to allow its behavior to depend more on the behavior of the price of the asset being traded. To achieve this we added two more parameters to the stochastic optimal control formulation of the model: a measurable set $\boldsymbol{S}$ and a process I taking values $\{0, 1\}$. The set $\boldsymbol{S}$ determines when full liquidation is required and I determines when trading takes place. We give four examples((3.14),(3.15),(3.16) and (3.17)) for $\boldsymbol{S}$ and I in which are all based on a lower bound L specified for the price process. We provided a solution to the modified stochastic optimal control problem via its BSDE representation. For the case when the price process is Markovian we derived representations of the value functions for $(I^{(i)}, \boldsymbol{S}^{(i)}$ where $i = 1, 2, 3, 4)$ as the solution of related PDE.

To measure the financial performance of the modified IS order we define A 6.5, which is the percentage deviation from the target price S_0 of the average price at which the position is (partially) closed in the time interval $[0, T]$. We note that A can be divided into three pieces: one corresponding to permanent price impact (A_1), one corresponding to random fluctuations in the price (A_2) and one corresponding to transaction/bid-ask spread costs (A_3). A_1 turns out to be a linear function of $1 - q_T/q_0$; therefore, its distribution is fully determined by that of q_T/q_0. We provide a numerical study of the distribution of q_T/q_0 and the conditional distributions of A_2 and A_3 given q_T/q_0 under the assumption that $\bar{S}_t = \sigma W_t$ for the all four cases indicating how these distributions change with model parameters.

An important assumption in the present work is the monetary representation of the terminal position q_T, for which we used $q_T S_T$. Many other choices are obviously

possible depending on how the events taking place after time T are modelled. These choices will give rise to different stochastic optimal control problems probably requiring new tools and ideas. This is a natural direction for future research. Another natural direction is to try to compute the performance of the algorithms of the present work (especially the joint distribution of $(\mathfrak{q}_T, A)$ on real trading data.

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