CONSTRUCTION OF EXOTIC MANIFOLDS USING CORKS AND PLUGS

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ABSTRACT

CONSTRUCTION OF EXOTIC MANIFOLDS USING CORKS AND PLUGS

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If two manifolds are homeomorphic but not diffeomorphic then we call these manifolds exotic copies of each other. In this thesis, we will find examples of such manifolds in dimension 4. In order to find exotic copies, we make use of Kirby diagrams and we frequently use handle moves such as handle creation, handle cancellation and handle slide which are isotopies.

Keywords: Exotic manifold, Cork, Plug, Handle Decomposition

ÖZ

EGZOTİK ÇOK KATLILARIN KORKLAR VE PLAKLAR YOLUYLA İNŞASI

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Yüksek Lisans, Matematik Bölümü
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Eğer iki çok katlı birbirine homeomerfik ama difeomorfik değilse bu çokkatlıları birbirinin ekzotik kopyası olarak adlandırıyoruz. Bu çalışmada 4 boyutlu ekzotik çokkatlılar bulacağız. Ekzotik çokkatlıları bulurken Kirby diagramlarını ve sık olarak kulp yaratma, kulp yok etme ve kulp kaydırma hareketlerini kullanacağız. Bu hareketler bize çokkatlılar arasında isotopi verirler.

Anahtar Kelimeler: Ekzotik çokkatlı, Kork, Plak, Kulp Ayrışımı

To my family

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CHAPTER 1

INTRODUCTION

Let Y be a smooth manifold which is homeomorphic to a manifold X but not diffeomorphic to it. Then we say that Y is an exotic copy of X. Over the last century mathematicians tried to find exotic copies (especially exotic $\mathbb{R}^n$'s and S^n's). Here are some results which can be considered as milestones:

- If X is homeomorphic to Y, then it is diffeomorphic to Y when X (and so Y) is a 2- or 3-dimensional manifold ([15], Moise 1952). This also holds when dimension is one [14].
- Milnor [11] give the first examples of exotic S^7.
- There exists a unique smooth structure on the $\mathbb{R}^n$ when $n \neq 4$ [19].
- There exist uncountably many exotic smooth structures on $\mathbb{R}^4$ [5], [7], [8].

In this thesis, we find concrete examples of exotic manifolds using their Kirby diagrams in the light of the following theorem:

Theorem 1.0.1. *([3], [17], [10]) When we have two homeomorphic but non-diffeomorphic simply connected closed 4-manifolds, we can obtain one from the other by removing a contractible 4-manifold and then, by gluing it via an involution on the boundary. This contractible 4-manifold is called a cork.*

In Chapter 2, we introduce the tools we used in this thesis.

In Chapter 3, we define the structures called corks and plugs, which are the main objects of this thesis when detecting exoticity, and then we provide some examples for both objects. Moreover, we give examples of these structures in elliptic surfaces.

In Chapter 4, we define the rational blow-down and logarithmic transform operations and see a relation between them. After that with the help of these operations we find more examples of corks and plugs. Additionally, at the end of the chapter we see that how corks and plugs can be knotted.

CHAPTER 2

PRELIMINARIES

In this section, we will give some definitions and results which are going to be used in following chapters.

2.1 Kirby Diagrams

In order to construct manifolds throughout this thesis we will use handles. For more about the content of this chapter see [1], [16]. Because we are interested in 4-manifolds, we will always use 4-dimensional handles, however, general definition is as follows:

Definition 2.1.1. If $0 \leqq k \leqq n$, we define *n-dimensional k-handle* h as a copy of $D^k \times D^{n-k}$ such that it is attached to the boundary of an n-dimensional manifold X along $\partial D^k \times D^{n-k}$ with the help of an embedding $\phi : \partial D^k \times D^{n-k} \mapsto \partial X$.

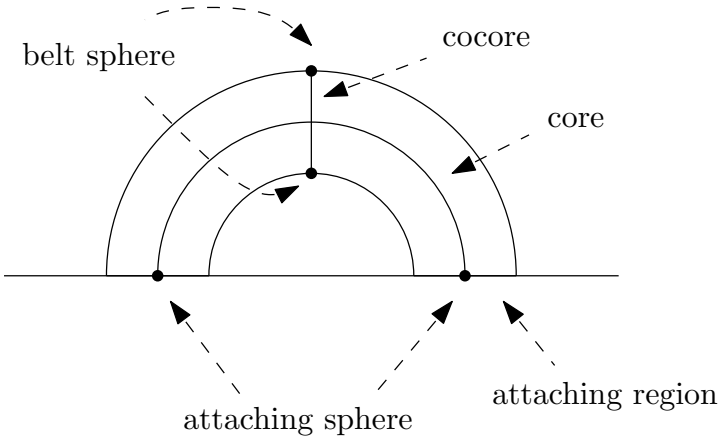


Figure 2.1: Anatomy of a handle

Observe the fact that there is a deformation retraction of $X \cup_\phi h$ onto $X \cup_{\phi|\partial D^k \times 0} D^k \times 0$, which means that up to homotopy, attaching k-handle is the same as attaching a k-cell. As in the Figure 2.1, $D^k \times 0$ is the *core of the handle*, $0 \times D^{n-k}$ is the cocore, ϕ is the *attaching map*, $\partial D^k \times D^{n-k}$ is the *attaching region*, $\partial D^k \times 0$ is the *attaching sphere* and $0 \times \partial D^{n-k}$ is the belt sphere. Also we call the number k as *index* of the handle.

Note that $X \cup_\phi h$ is determined by two pieces of data:

- an embedding $\phi|\partial D^k \times 0 : S^{k-1} \to \partial X$ with trivial normal bundle,
- an identification of the normal bundle $\nu\phi|\partial D^k \times 0(S^{k-1})$ with $S^{k-1} \times \mathbb{R}^{n-k}$.

In order to understand framings on a sphere S^{k-1} in ∂X^n with trivial normal bundle, we first select a framing f_0 for each $p \in S^{k-1}$ and compare it with any framing f which gives an element of $GL(n-k)$. We can also assume that this element gives $I_{n-k \times n-k}$ at a choosen point $p_0 \in S^{k-1}$. Therefore, we get an element of $\pi_{k-1}(GL(n-k))$, and $\pi_{k-1}(GL(n-k)) = \pi_{k-1}(O(n-k))$ by the Gram-Schmidt procedure. Thus, by above we get a bijection between isotopy classes of framings of S^{k-1} in ∂X^n and $\pi_{k-1}(O(n-k))$.

In this thesis, n is always 4. Moreover, $\pi_0(O(3))$, $\pi_2(O(1))$ are trivial groups and we only need to care about framings of two handles which are parametrized by $\pi_1(O(2)) = \pi_1(SO(2)) = \pi_1(S^1) = \mathbb{Z}$.

Definition 2.1.2. A *handle decomposition* of X is an identification of X with a manifold obtained by attaching handles to $I \times \partial_- X$ so that $\partial_- X$ corresponds to $0 \times \partial_- X$ in the obvious way. A manifold X with a given handle decomposition is called a *relative handlebody* built on $\partial_- X$. When $\partial_- X = \emptyset$ it is called a *handlebody*. By Morse theory ([12], [13]) we know that every smooth, compact manifold pair $(X, \partial X)$ admits a handle decomposition.

When we are given a pair of handles h_1 and h_2 of the same index k attached to a manifold X, we can isotope the attaching sphere for h_1 in $\partial(X \cup h_2)$, sliding along a disk $D^k \times \{pt\}$ and returning it to ∂X. Then we can extend this isotopy to whole

manifold. Because these moves induce isotopies on corresponding manifolds they do not change the smooth type of the original manifold X.

If we have a pair of handles h_1 and h_2 of the same index k attached to a manifold X, we can isotope the attaching sphere of h_1 in $\partial(X \bigcup h_2)$ by sliding it along a disk $D^k \times pt \in \partial h_2$ and then by returning it to ∂X.

In dimension 4, 2-handles are attached along the framed knots K_1 and K_2. A parallel curve K_2' determining the framing on K_2 will bound a disk $D^2 \times pt \in \partial(X \cup h_2)$. We slide h_1 by isotoping K_1 over one such disk. In practice, this means we form a band-sum of K_1 and K_2'. As we may precede the slide by an isotopy, we are allowed to use any band disjoint from the rest of the link, and the choice will affect the resulting link. If K_1 and K_2 are oriented, we call the move *handle addition* if the sum respects the orientations of K_1 and K_2', and a *handle subtraction* otherwise. If we let n_1 be the initial framing of h_1 and n_2 be the framing of h_2, then the new framing of h_1 which we call n_1' is given by the formula:

$$n_1' = n_1 + n_2 \pm 2lk(K_1, K_2)$$

where $lk(K_1, K_2)$ is the linking number between the knots K_1 and K_2. Also we use the sign $+$ if it is a handle addition and $-$ if it is a handle subtraction. Note that handle slides induce diffeomorphisms between corresponding smooth manifolds. The following figure is an example of how we perform the handle slide operation.

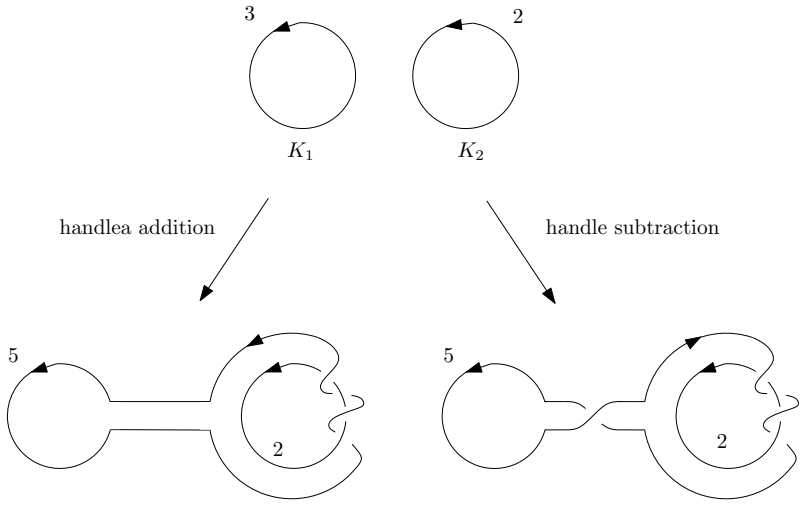


Figure 2.2: Handle slide operation

Definition 2.1.3. The symmetric bilinear form

$$Q_X : H^2(X, \partial X; \mathbb{Z}) \times H^2(X, \partial X; \mathbb{Z}) \to \mathbb{Z}$$

defined by $Q_X(a, b) =< a \cup b, [X] >= a \cdot b \in \mathbb{Z}$ (where $[X]$ is the fundamental class of X) is called the *intersection form* of X. By Poincarè duatlity we can define Q_X on $H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z})$ as $Q_x(a, b) =< PD(a) \cup PD(b), [X] >= PD(a) \cdot PD(b) \in \mathbb{Z}$ where $PD(x)$ is the Poincarè dual of x for any $x \in H_2(X; \mathbb{Z})$.

With the help of intersection forms and the theorem of Freedman below, the problem of showing two manifolds are homeomorphic becomes much simpler.

Theorem 2.1.4. *(Freedman [4], [9]) If simply connected smooth 4-manifolds have the same intersection forms, then they are homeomorphic.*

2.2 Seiberg Witten Invariants

Definition 2.2.1. Let X and Y be two smooth manifolds. We say that X (respectively Y) is an exotic copy of Y (respectively X) if X is homeomorphic to Y but it is not diffeomorphic to Y.

It is generally very hard to say that two manifolds are not diffeomorphic when they are homeomorphic. However, Seiberg Witten invariants are very useful in distinguishing diffeomorphism types of two manifolds. Therefore, we will review Seiberg Witten invariants. For more deatails see [1] Chapter 13. Seiberg Witten invariants are diffeomorphism invariants and they make use of Seiberg Witten equations.

Let X be a closed oriented smooth 4-manifold. Then X has a $spin^c$ structure. Now let $Spin^c(X)$ be the set of all $spin^c$ structures on X. Then we have

$$Spin^c(4) = U(1) \times Spin(4)/ \pm 1$$

which means that we have a projection map $det : Spin^c(4) \mapsto U(1)$ and this map determines a complex line bunde $L \mapsto X$. Also note that any $s \in Spin^c(X)$ can be thought as two principal $U(2)$-bundles $S^{\pm} \mapsto X$.

Now recall that a $U(1)$-connection on L is an operator such that $d_A : \Gamma(L) \mapsto \Gamma(L)$ which can be written locally as $d_A\sigma = d\sigma + A\sigma$. Then let A be a $U(1)$-connection on L and $\phi \in \Gamma(S^{\pm})$ be a self-dual spinor field. Now the *Sieberg Witten equations* are

$$D^A\phi = 0,\, F_A^+ = \sigma(\phi)$$

where D^A is the Dirac operator of A and $F_A^+ = F_A = \star F_A$ is the self-dual of the curvature 2-form of A and σ is a squaring map. Note the fact that Seiberg Witten equations are invariant under the action of gauge group G.

Now define the *moduli space* of solutions of Seiberg Witten equations as

$$M(s) = \{(A, \phi) \text{ which solve SW equations}\}/G$$

Definition 2.2.2. The *Seiberg Witten invariant* is the map

$$SW_X : Spin^c(X) \mapsto \mathbb{Z}$$

which is defined as the signed count of $M(s)$ for each $s \in Spin^c(X)$.

CHAPTER 3

CORKS AND PLUGS

In this section, we will define corks and plugs and see basic examples of them.

3.1 Corks and Plugs

Definition 3.1.1. ([18], [1]) A pair (C, τ) is called a *cork* if C is a compact Stein 4-manifold with boundary and $\tau : \partial C \to \partial C$ an involution such that τ extends to a self-homeomorphism of C, however, it does not extend to a self-diffeomorhism of C. A cork (C, τ) is called a *cork of a smooth manifold* X if replacing C via the involution τ changes the diffeomorphism type of X. (Clearly, by regluing a cork with the map τ we get a manifold which is homeomorphic X.)

Definition 3.1.2. ([18], [1]) A pair (P, τ) is called a *plug* if P is a compact Stein 4-manifold with boundary and $\tau : \partial P \to \partial P$ an involution such that τ does not extend to a self-homeomorphism of P. A plug (P, τ) is called a *plug of a smooth manifold* X if replacing P via the involution τ changes diffeomorphism type of X while preserving its homeomorphism type.3.1

Definition 3.1.3. ([18], [1]) Let $W_{m,n},\ W_n,\ \overline{W}_n$ be as in Figure 3.1. Let

$$f_n : \partial W_n \to \partial W_n, \quad \overline{f}_n : \partial \overline{W}_n \to \partial \overline{W}_n, \quad f_{m,n} : \partial W_{m,n} \to \partial W_{m,n}$$

be the maps obtained by replacing the dotted circle and 0-framed 2-handle which corresponds to first surgering $S^1 \times B^3$ to $B^2 \times S^2$ and then surgering $B^2 \times S^2$ to $S^1 \times B^3$. These maps are diffeomorphisms on the ∂W_n and $\partial \overline{W}_n$ because doing surgery on the interior of a manifold does not change the boundary of the manifold.

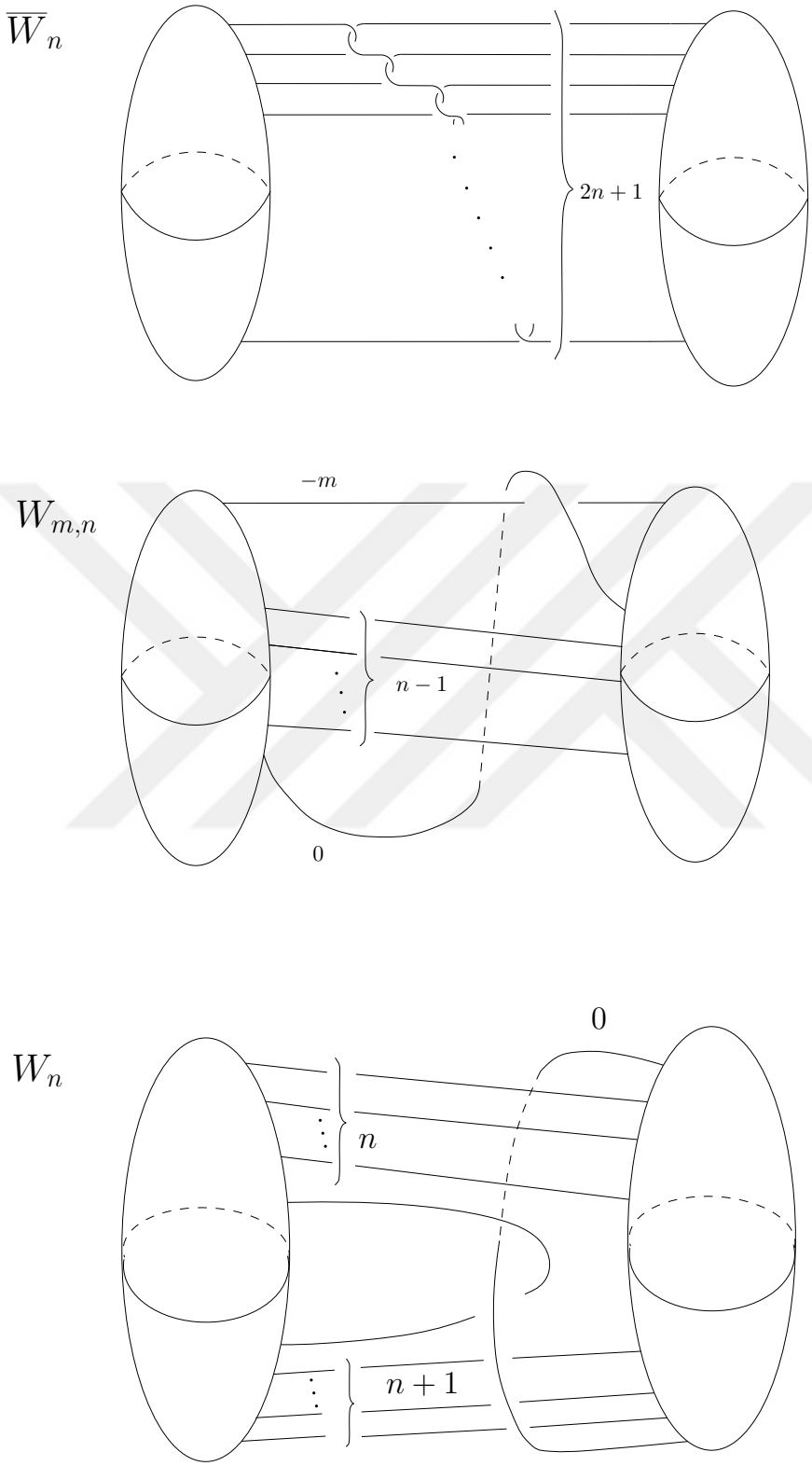


Figure 3.1: $\overline{W}_n$, $W_{m,n}$ and W_n in standard positions

In dotted notation we have:

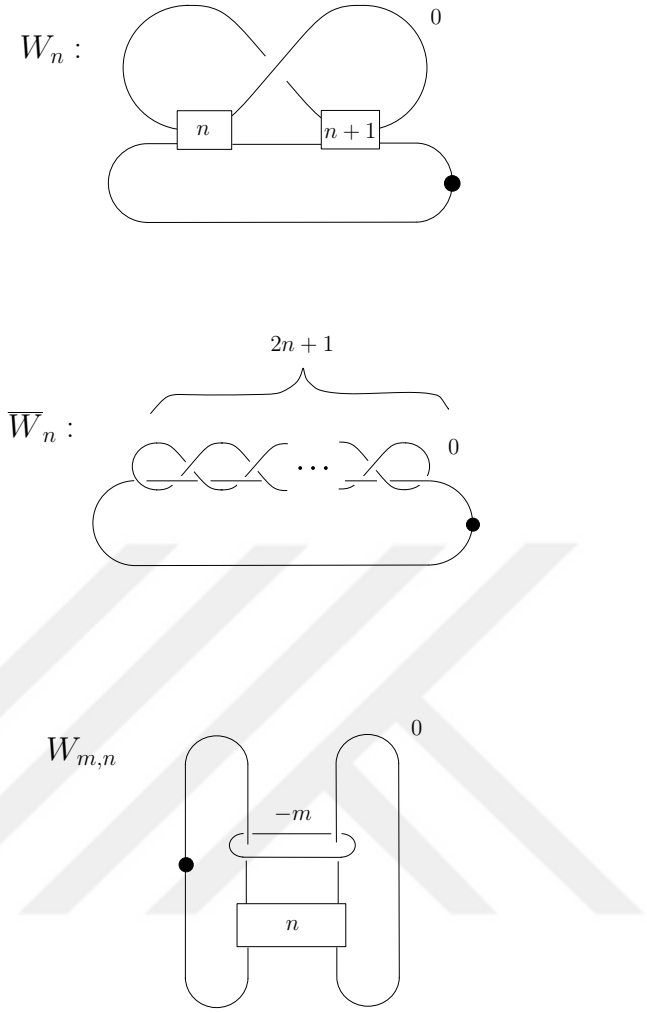


Figure 3.2: $\overline{W}_n$, $W_{m,n}$ and W_n in the dotted circle notation

Lemma 3.1.4. W_n and $\overline{W}_n$ are contractible.

Proof. When we are building the handle bodies, up to homotopy we are attaching k-cell when we glue a k-handle as we can contract the handle to its core. Therefore, both of the manifolds are homotopy equivalent to a CW-complex which consist of one 0-cell, one 1-cell and one 2-cell. There is a unique way to attach a 1-cell to a zero cell, and attaching map of 2-cell is homotopic to the generator of the group $\pi_1(S^1)$ as it goes around the one cell algebraically once. If we attach the 2-handle with the generator of the group $\pi_1(S^1)$ we get D^2. Thus, both of the manifolds are homotopy equivalent to D^2, so they are contractible. □

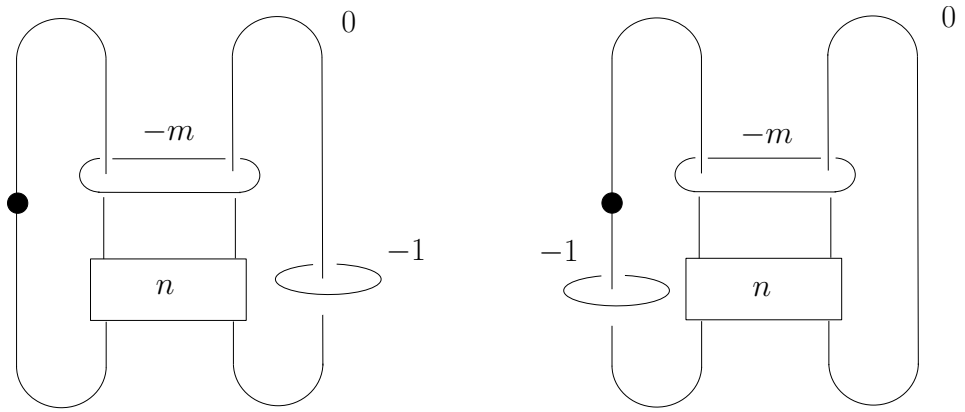


Figure 3.3: Two non-homeomorphic manifolds

Lemma 3.1.5. (1) For $n \geqq 1$, we can extend the involution $f_n : \partial W_n \to \partial W_n$ to a self homeomorphism of W_n.

(2) For $n \geqq 1$, we can extend the involution $\overline{f}_n : \partial \overline{W}_n \to \partial \overline{W}_n$ to a self homeomorphism of $\overline{W}_n$.

(3) For $n \geqq 2$, $m \geqq 1$, the involution $f_{m,n} : \partial W_{m,n} \to \partial W_{m,n}$ cannot extended to a self homeomorphism of $W_{m,n}$.

Proof. (1) and (2) follow from the fact that W_n and $\overline{W}_n$ are contractible and ∂W_n, $\partial \overline{W}_n$ are homology spheres, thus the claim follows from [2] Corollary 0.9.

(3) Follows from the fact that smooth 4-manifolds in Figure 3.3 have different intersection forms (the right one does not contain an element with self intersection -1) given by

$$\begin{pmatrix} -2n - mn^2 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -2n - mn^2 & -1 - mn \\ -1 - mn & -1 - m \end{pmatrix},$$

respectively. □

In order to see the intersection forms of the manifolds, we slide (subtraction) 0-framed 2-handle over $(-m)$-framed 2-handle n times. We can find the last framing of the 0-framed 2-handle from the recursive formula:

$$x_2^n = x_1 + x_2^{n-1} - 2l^n$$

where l^n is the linking number between 0-framed 2-handle and $(-m)$-framed 2-handle after sliding(subtraction) 0-framed 2-handle over $(-m)$-framed 2-handle n times, so

$$l^n = 1 + mn$$

and x_1 is the framing of the $(-m)$-framed 2-handle so

$$x_1 = -m$$

x_2^n is the framing of the 0-framed 2-handle after sliding it over $(-m)$-framed 2-handle n many times, clearly $x_2^0 = 0$.

Therefore, summing the following equations:

$$\begin{aligned} x_2^n &= x_1 + x_2^{n-1} - 2l^n \\ x_2^{n-1} &= x_1 + x_2^{n-2} - 2l^{n-1} \\ x_2^{n-2} &= x_1 + x_2^{n-3} - 2l^{n-2} \\ &\vdots \\ x_2^1 &= x_1 + x_2^0 - 2l^0 \end{aligned}$$

we get

$$x_2^n = x_2^n - x_2^0 = -nm - 2(n + mn(n-1)/2) = -2n - mn^2.$$

In this way as it can be seen in Figure 3.4, we can resolve the linking between 0-framed 2-handle and the dotted circle, and then we can just cancel the $(-m)$-framed 2-handle and the dotted circle pair so that we have only two 2-handles left. Then from the resulting diagram we can easily compute the above intersection form.

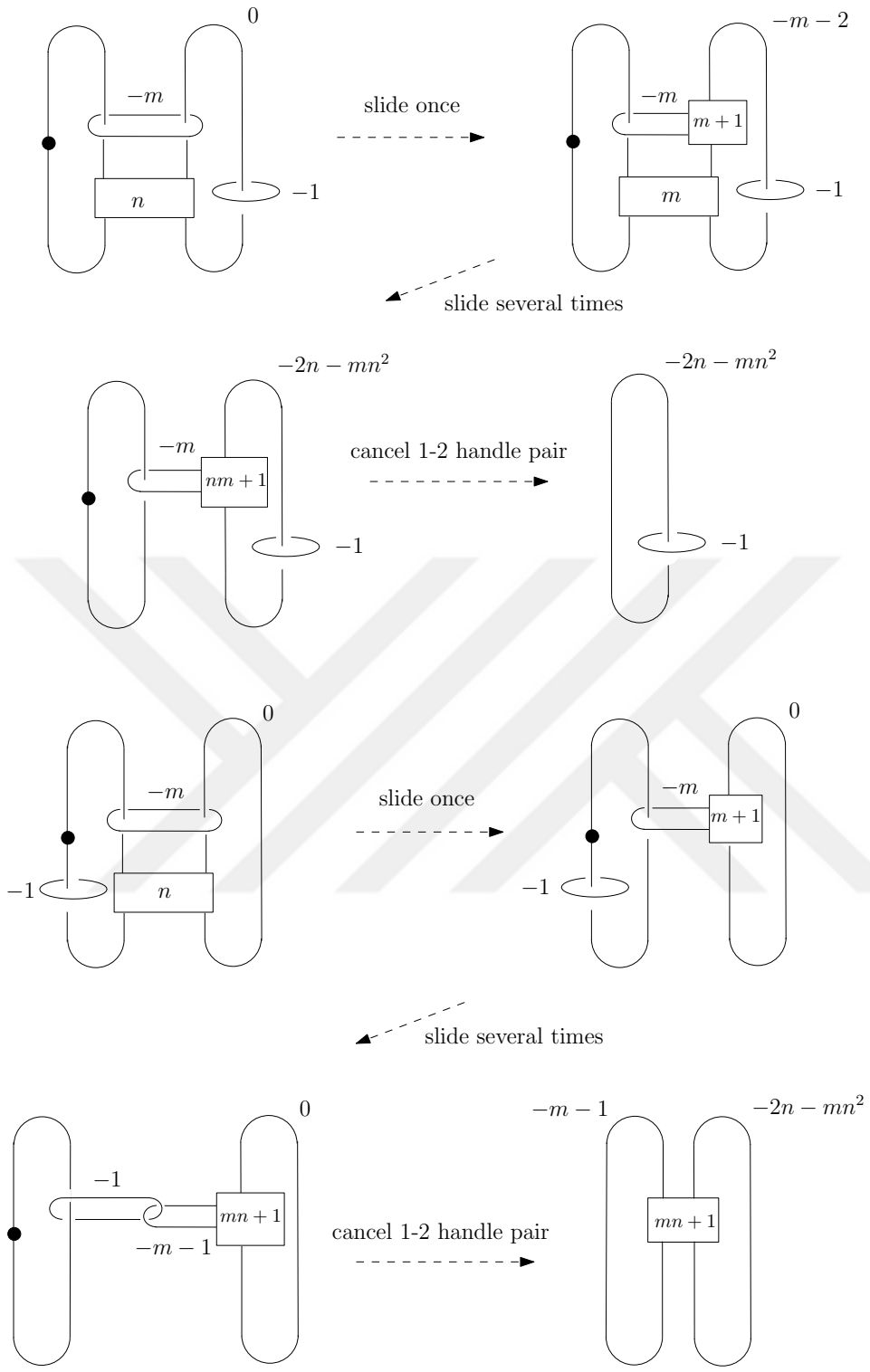


Figure 3.4: Handle slides in the proof Lemma 3.1.5

Now let us see an example of a cork in a smooth 4-manifold.

Theorem 3.1.6. *([18]) (1) Let $n \geqq 1$ and W_n^1 , W_n^2 be the smooth 4-manifolds in Figure 3.5. Then, W_n^1 and W_n^2 are exotic copies of each other. Thus, the involution $f_n : \partial W_n \to \partial W_n$ cannot be extended to a self diffeomorphism of W_n. In this case W_n^2 is obtained from W_n^1 by regluing W_n via f_n.*

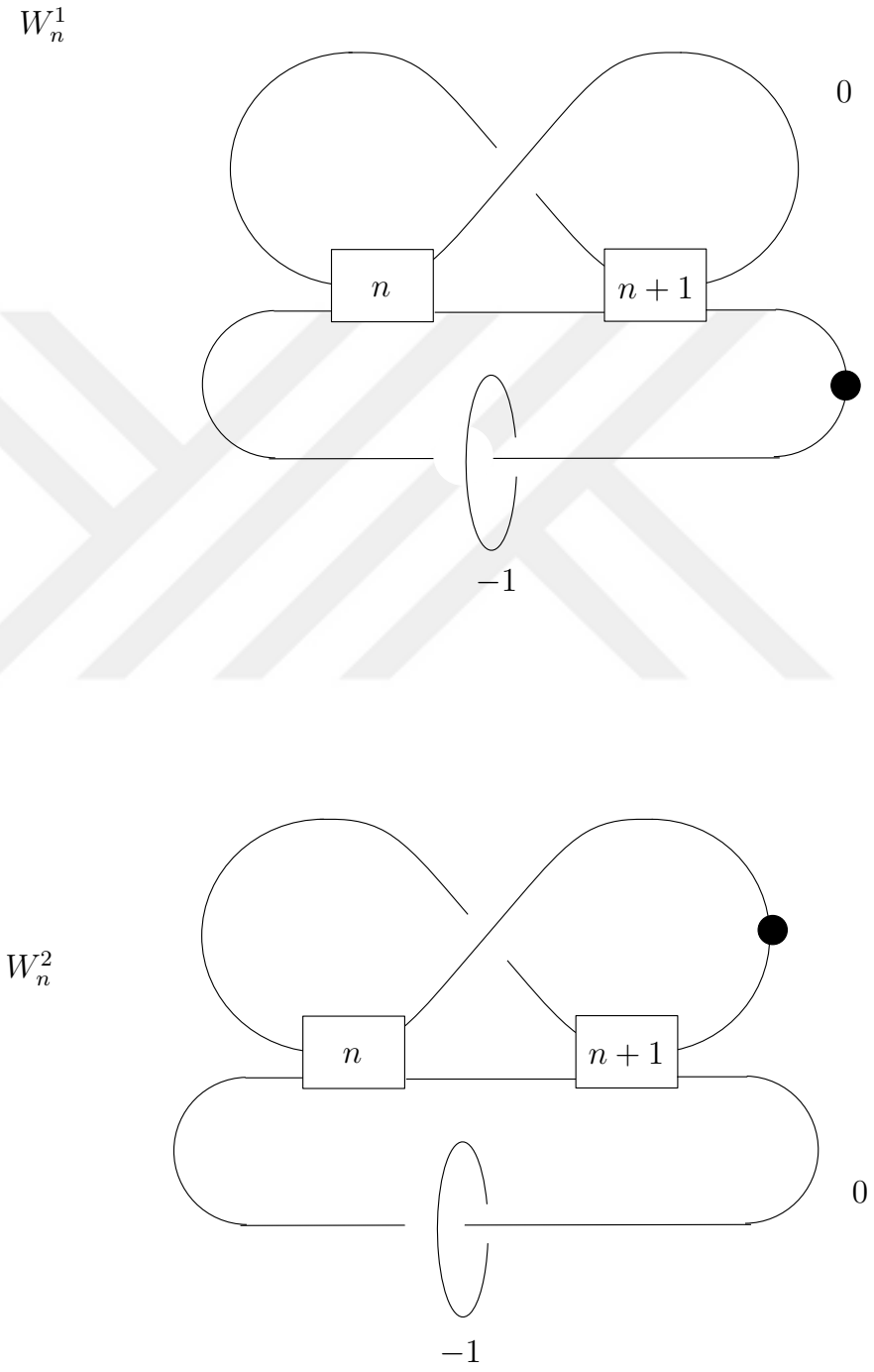


Figure 3.5: W_n^1 and W_n^2

([18]) (2) Let $n \geqq 1$ and $\overline{W}_n^1$, $\overline{W}_n^2$ be the smooth 4-manifolds in Figure 3.6 . Then, $\overline{W}_n^1$ and $\overline{W}_n^2$ are exotic copies of each other. Thus, the involution $\overline{f}_n : \partial\overline{W}_n \to \partial\overline{W}_n$ cannot be extended to a self diffeomorphism of $\overline{W}_n$. In this case $\overline{W}_n^2$ is obtained from $\overline{W}_n^1$ by regluing $\overline{W}_n$ via $\overline{f}_n$.

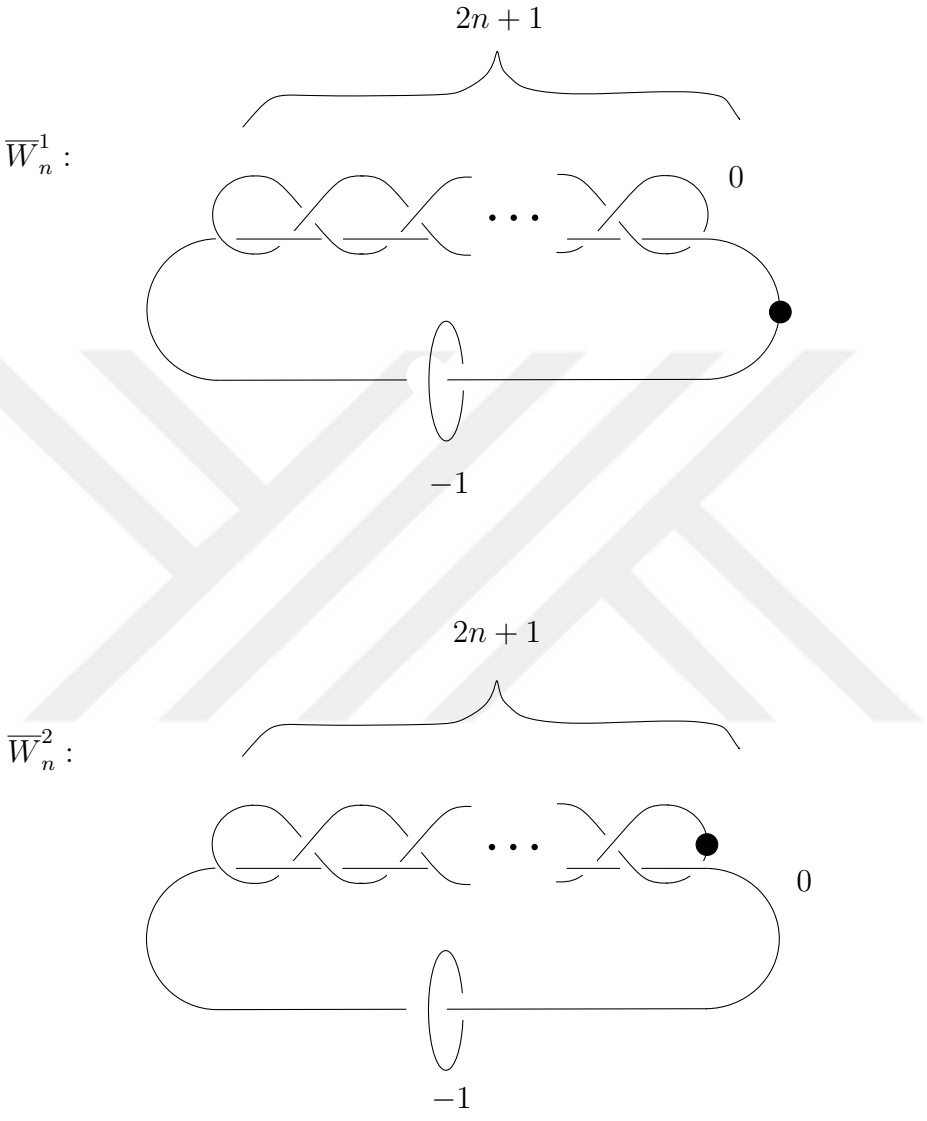


Figure 3.6: $\overline{W}_n^1$ and $\overline{W}_n^2$

([18]) (3) Let $n \geqq 2$, $m \geqq 1$ and $W_{m,n}^1$, $W_{m,n}^2$ be the smooth 4-manifolds in Figure 3.7 . $W_{m,n}^1$ and $W_{m,n}^2$ are homeomorphic but not diffeomorphic. Thus, the involution $f_{m,n} : \partial W_n \to \partial W_{m,n}$ cannot be extended to a self diffeomorphism of $W_{m,n}$. In this case $W_{m,n}^2$ is obtained from $W_{m,n}^1$ by regluing $W_{m,n}$ via $f_{m,n}$.

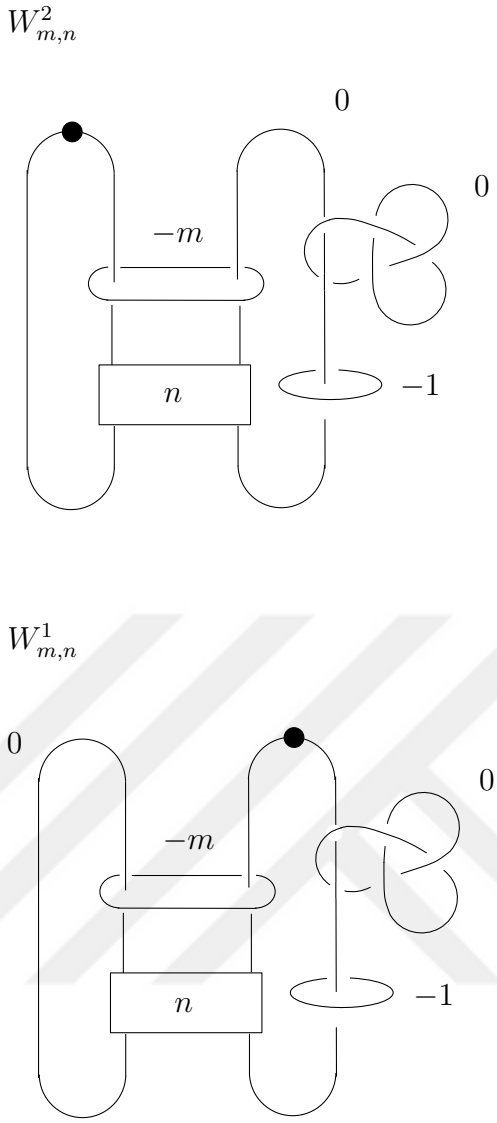


Figure 3.7: $W^2_{m,n}$ and $W^1_{m,n}$

Proof. (1) Lemma 2.7 implies that W^1_n and W^2_n are homeomorphic. We use Eliashberg criterion and see that Thurston-Benequin number (which is defined the as writhe minus the number of right(or left) cusps in the front projection of a given Legendrian knot) of the attaching circle of each 2-handle is strictly greater than corresponding framing (see the Figure 3.8), and so we conclude that W^1_n is a Stein 4-manifold. However, W^2_n cannot be Stein because it has an embedding of S^2 which has self intersecion -1 and every compact Stein manifold can be embedded into a minimal symplectic 4-manifold. (For more discussion see [6].)

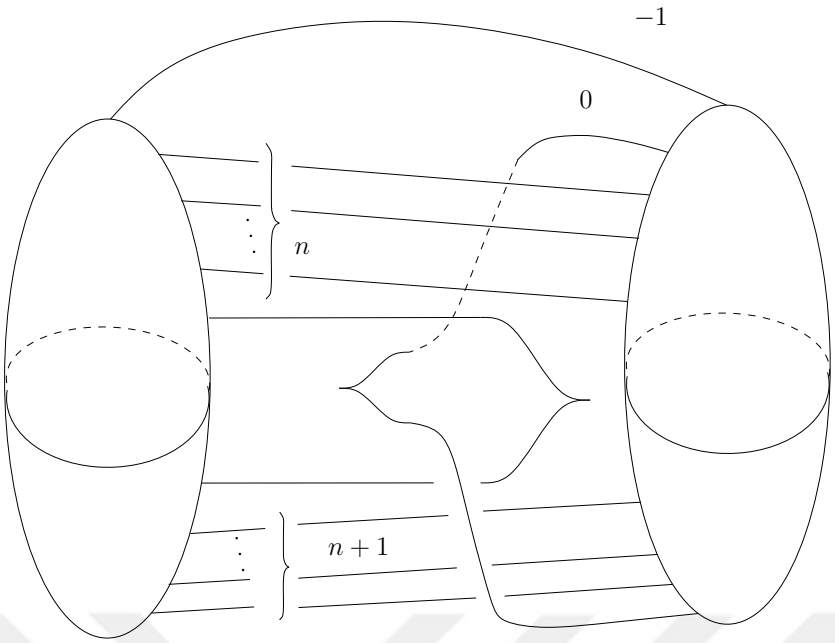


Figure 3.8: W_n^1 in Legendrian position

(2) Similar to case (1) we see that $\overline{W}_n^1$ is a compact Stein 4-manifold whereas $\overline{W}_n^2$ is not.

(3) Similarly, $W_{m,n}^1$ is a compact Stein 4-manifold but $W_{m,n}^2$ is not. We can argue that they are homeomorphic by looking at their intersection forms. Indeed,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -m & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -2n - n^2m & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

are the intersection forms of $W_{m,n}^1$ and $W_{m,n}^2$, respectively.

If we let $k = 2n + n^2m$ and

$$S = \begin{pmatrix} 1/\sqrt{k+1} & 1/\sqrt{k+1} & 0 \\ 0 & -\sqrt{k+1} & 0 \\ 1/\sqrt{k+1} & 1/\sqrt{k+1} & 1 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{m} & 1/\sqrt{m} \\ 0 & 0 & \sqrt{m} \end{pmatrix}$$

and also if we let

$$B = \begin{pmatrix} -2n - n^2m & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -m & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then we obtain

$$S^T \cdot B \cdot S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$P^T \cdot A \cdot P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By direct computation we also see that $det(A) = det(B) = 1$, so they are both unimodular forms. By looking at their diagonalization we see that they are also indefinite. As they have same rank parity and signature, by classification of indefinite unimodular forms ([16] Theorem 1.2.14) we see that they are isomorphic.

In order to see how these intersection forms are obtained, slide the handles as in the Figure 3.9 and Figure 3.10. Note that in the first slide of $W^2_{m,n}$ we argue as in the case of $W_{m,n}$ to get the framing $-2n - mn^2$. Now, applying Freedman's result we get the claim. □

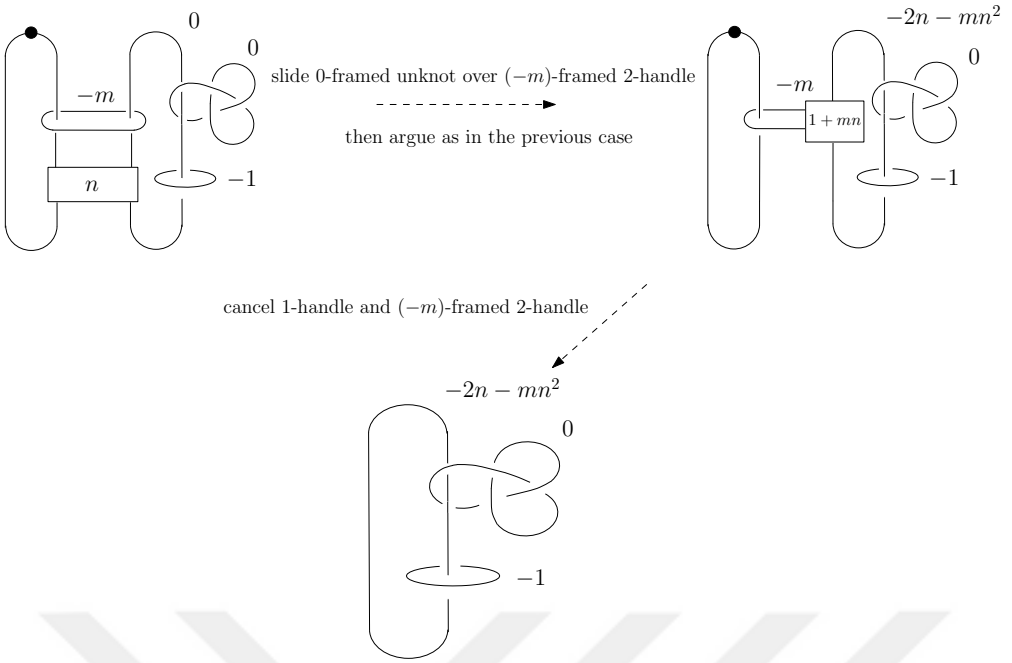


Figure 3.9: Slides for the second intersection form

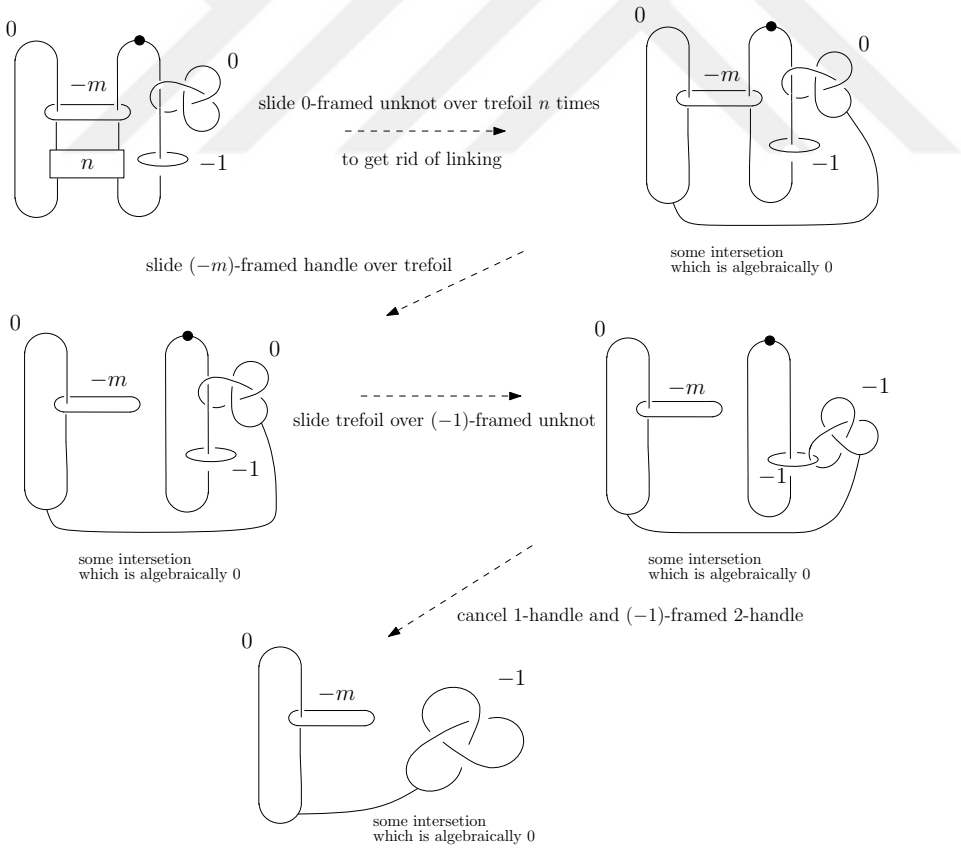


Figure 3.10: Slides for the first intersection form

3.2 Examples

In this part we will provide some examples of closed manifolds which contain corks W_n and plugs $\overline{W}_n$.

Here we borrow the following result form [16].

Theorem 3.2.1. *If $n \geqq 1$ Figure 3.11 is the handle decomposition of the elliptic surface $E(n)$.*

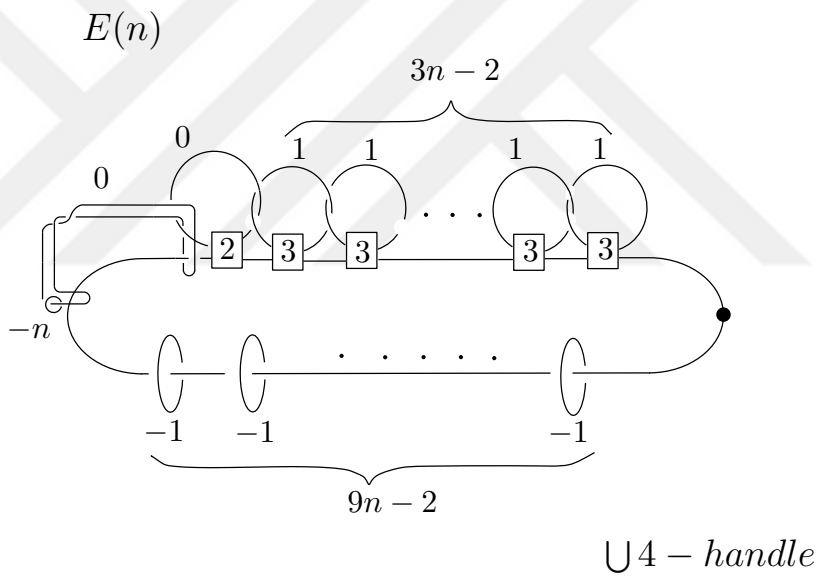


Figure 3.11: $E(n)$

Then we have an easy corollary:

Theorem 3.2.2. *If $n \geqq 1$ Figure 3.12 is the handle decomposition of the elliptic surface $E(n)$.*

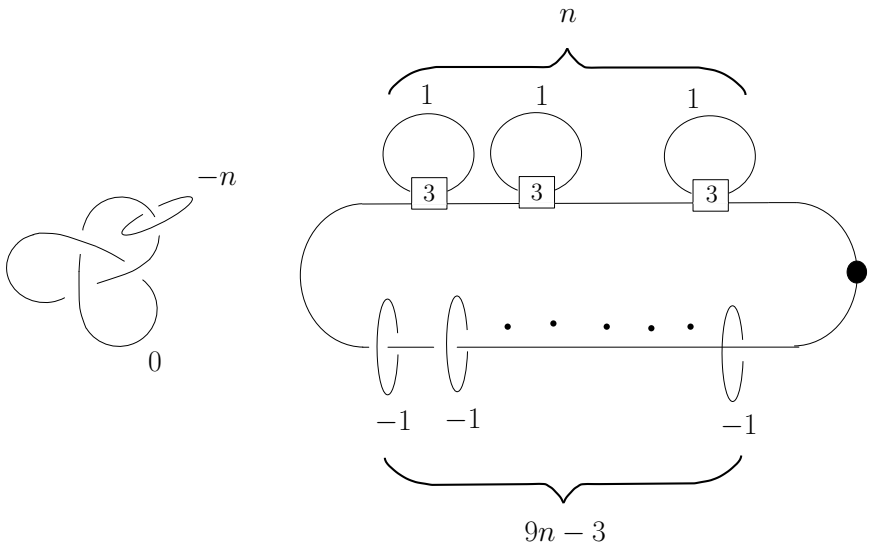


Figure 3.12: $E(n)$

Proof. Pull of the left-most 0-framed unknot by sliding over (-1)-framed knot. □

Theorem 3.2.3. *Letting $n \geqq 1$ and $m \geqq 1$ we have the following facts:*

1) $E(2n)\#\overline{\mathbb{C}P}^2$ contains the manifold in Figure 3.13.

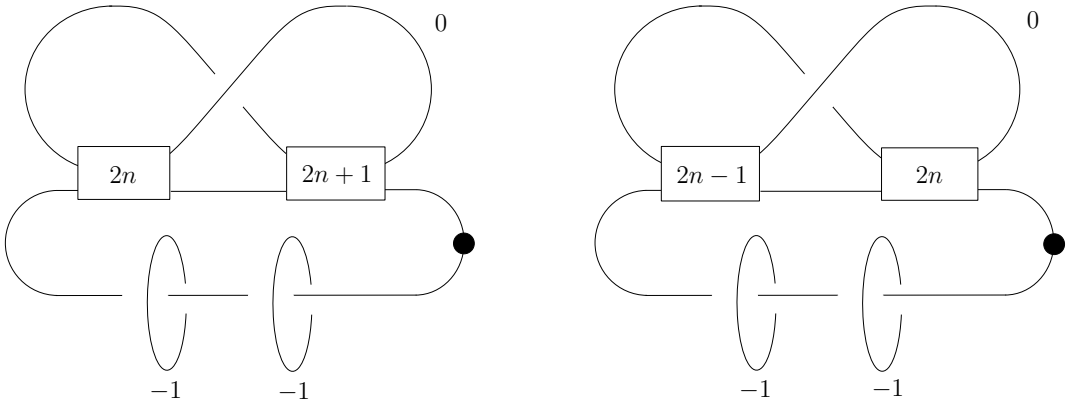


Figure 3.13: A manifold contained in $E(2n)\#\overline{\mathbb{C}P}^2$

Proof. We slide as in Figure 3.14 to get Figure 3.15. By similar isotopies we can get the diagram of $E(2n)$ as in Figure 3.16. Then we blow up to get Figure 3.17 .

By sliding left 0-framed 2-handle over the other in Figure 3.17 we get left figure in Figure 3.13. □

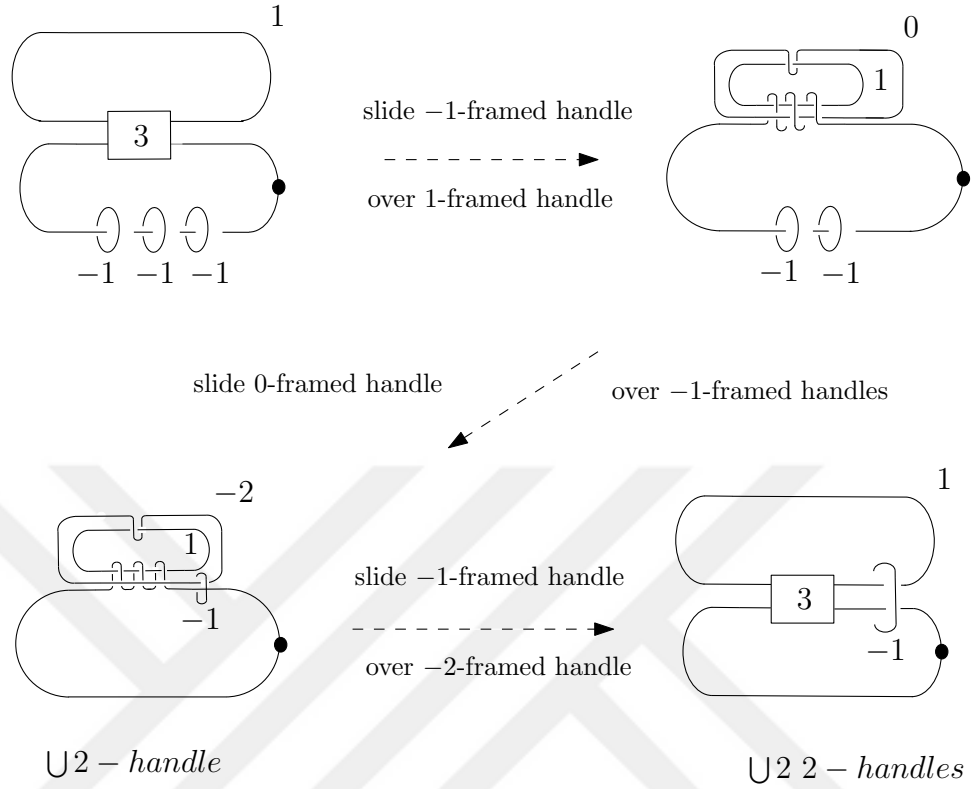


Figure 3.14

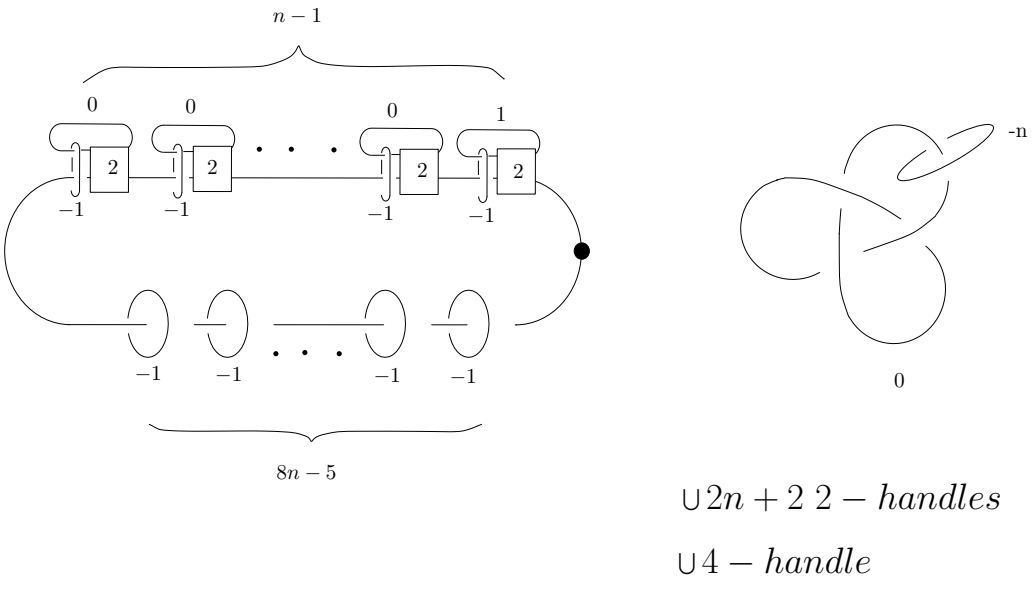
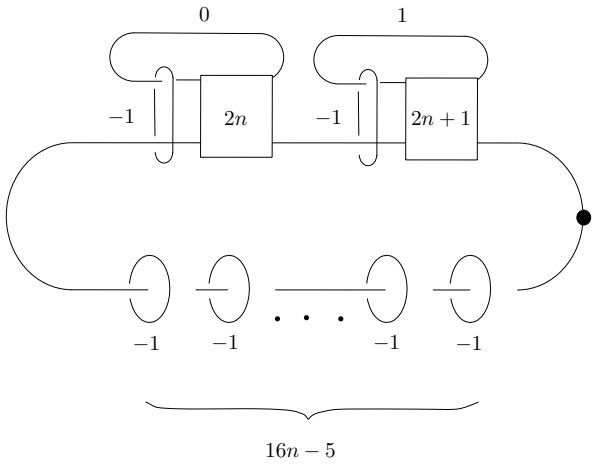


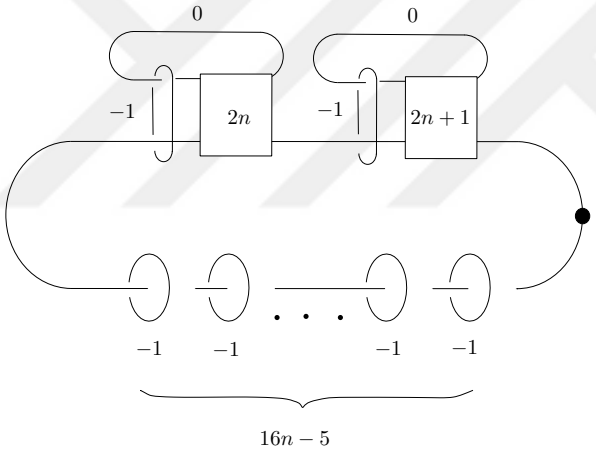
Figure 3.15: $E(n)$



$\bigcup 8n\ 2-handles$

$\bigcup 4-handle$

Figure 3.16: $E(n)$



$\bigcup 8n+1\ 2-handles$

$\bigcup 4-handle$

Figure 3.17: $E(n)\#\overline{\mathbb{C}P}^2$

Theorem 3.2.4. *2)* $E(2n)\#m\overline{\mathbb{C}P}^2$ *contains the manifold in Figure 3.18.*

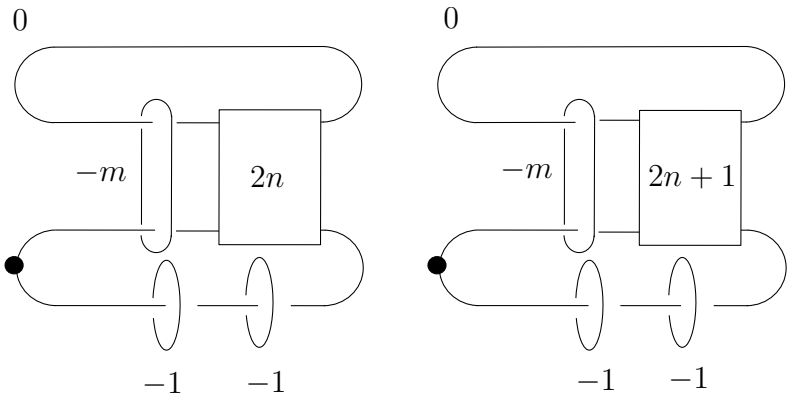


Figure 3.18: A manifold contained in $E(2n)\#m\overline{\mathbb{C}P}^2$

Proof. Blowing up the right (left) middle (-1)-framed 2-handle $m-1$ times in Figure 3.17 gives the Figure 3.18 □

Theorem 3.2.5. *3)* $E(2n+1)\#\overline{\mathbb{C}P}^2$ *contains the manifold in Figure 3.19.*

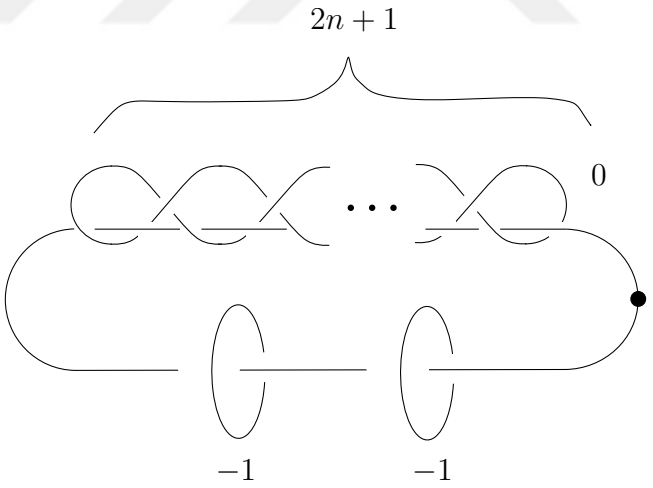


Figure 3.19: A manifold contained in $E(2n+1)\#\overline{\mathbb{C}P}^2$

Proof. Using the case 1 and using Figure 3.15 of $E(2n+1)$ we get the Figure 3.20 as an another diagram of $E(2n+1)$. By sliding handles as in Figures 3.20-3.25 and blowing up the Figure 3.25 we get Figure 3.19.

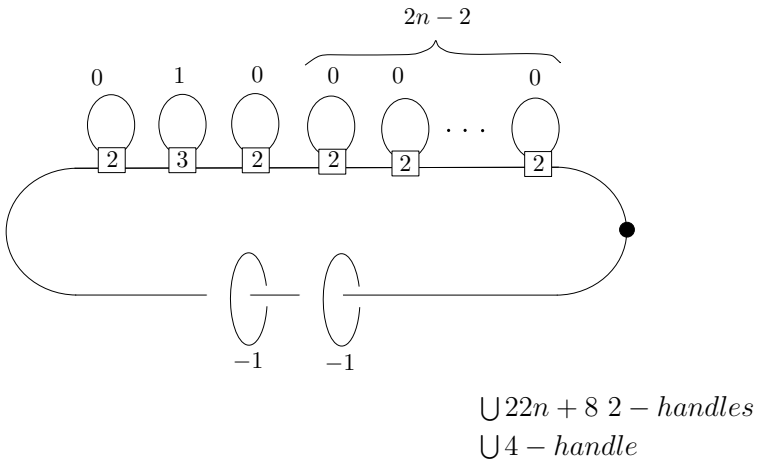


Figure 3.20: $E(2n+1)$

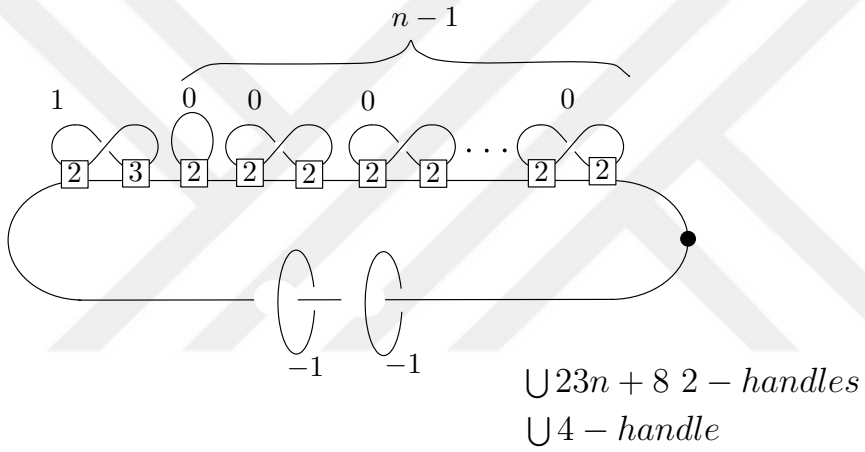


Figure 3.21: $E(2n+1)$

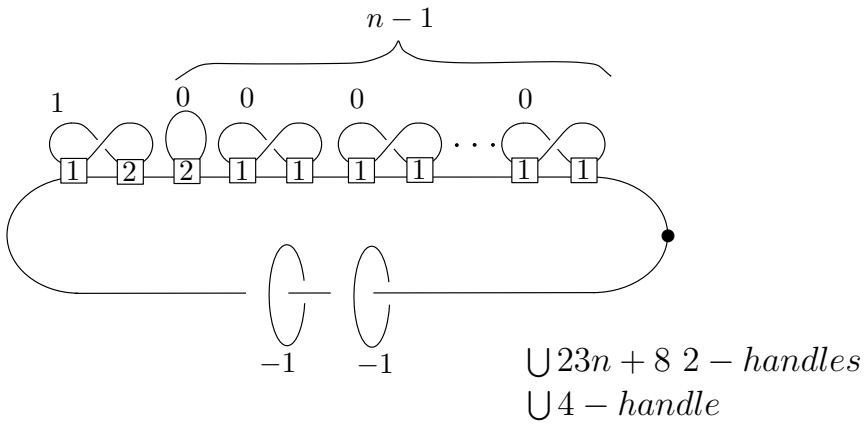


Figure 3.22: $E(2n+1)$

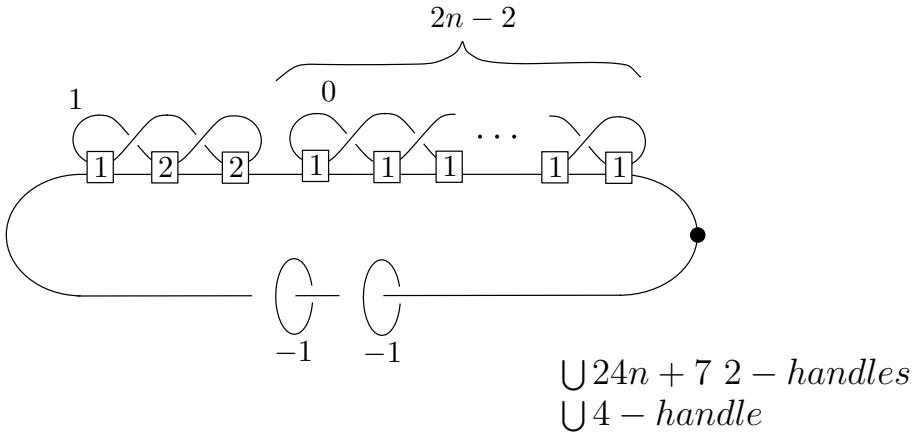


Figure 3.23: $E(2n+1)$

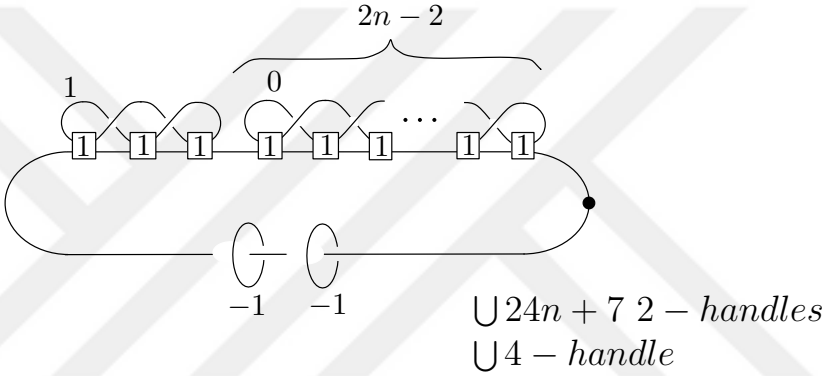


Figure 3.24: $E(2n+1)$

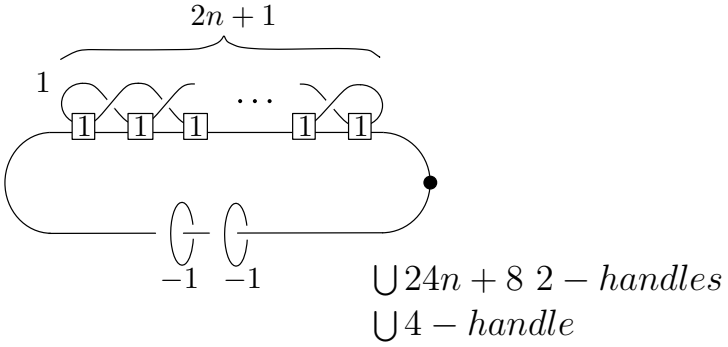


Figure 3.25: $E(2n+1)$

□

By using this theorem we can easily prove the following:

Theorem 3.2.6. *Letting $n \geqq 1$ and $m \geqq 1$ we have the following facts:*

1) W_{2n-1} and W_{2n} are corks of $E(2n)\#\overline{\mathbb{C}P}^2$

2) $W_{m,2n}$ and $W_{m,2n-1}$ are plugs of $E(2n)\#m\overline{\mathbb{C}P}^2$

3) $\overline{W}_{2n-1}$ is a cork of $E(2n+1)\#\overline{\mathbb{C}P}^2$

Proof. Theorem follows from the fact that by regluing corks and plugs with obvious maps splits off $2\overline{\mathbb{C}P}^2$. However, $E(2n+1)\#\overline{\mathbb{C}P}^2$, $E(2n)\#m\overline{\mathbb{C}P}^2$ and $E(2n)\#\overline{\mathbb{C}P}^2$ can not split of $2\overline{\mathbb{C}P}^2$ by blow-up and the Seiberg-Witten formula. □

CHAPTER 4

RATIONAL BLOW-DOWN AND LOGARITHMIC TRANSFORM

4.1 Rational Blow-down and Logarithmic Transform

In this section, we will review logarithmic transform and rational blow-down operations introduced by Fintushel and Stern [21].

Let B_p and C_p be the smooth 4-manifolds in Figure 4.1 and Figure 4.2. Note that ∂C_p and ∂B_p are both diffeomorphic to lens space $L(p^2, p-1)$.

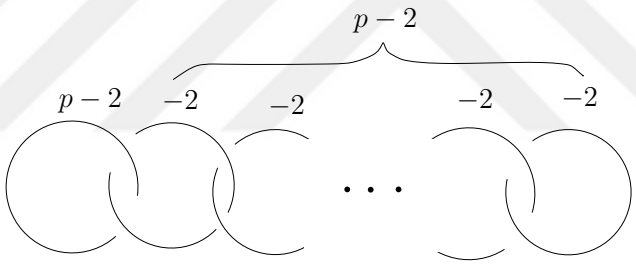


Figure 4.1: C_p

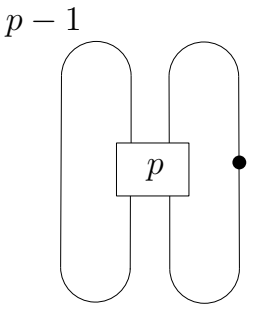


Figure 4.2: B_p

Definition 4.1.1. Suppose that a smooth 4-manifold X contains an embedding of C_p. Then *rational blow-down* of X *along* C_p is the smooth 4-manifold obtained by removing C_p and gluing B_p instead, and we denote the rational blow-down of X as $X_{(p)}$.

Now we have the following lemma:

Lemma 4.1.2. The following figure is another diagram of C_p.

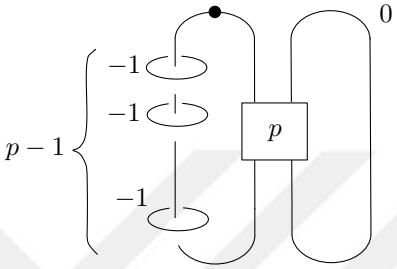


Figure 4.3: C_p

Proof. By the slides indicated in the below figure we get the result.

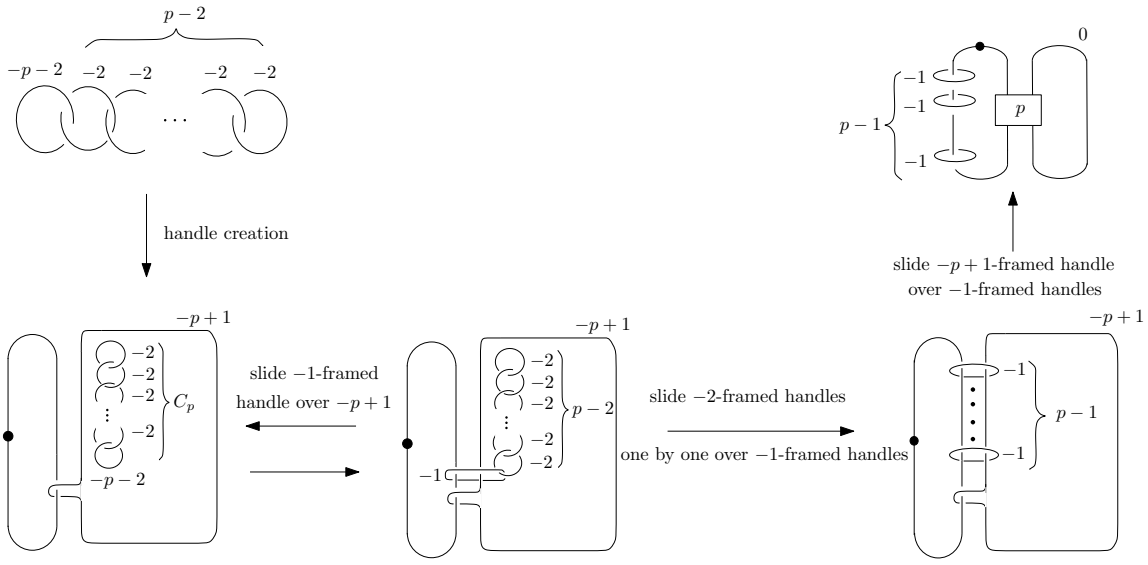


Figure 4.4: Proof of Lemma 4.1.1

□

Because it changes C_p to B_p, the following operation is the rational blow-down operation:

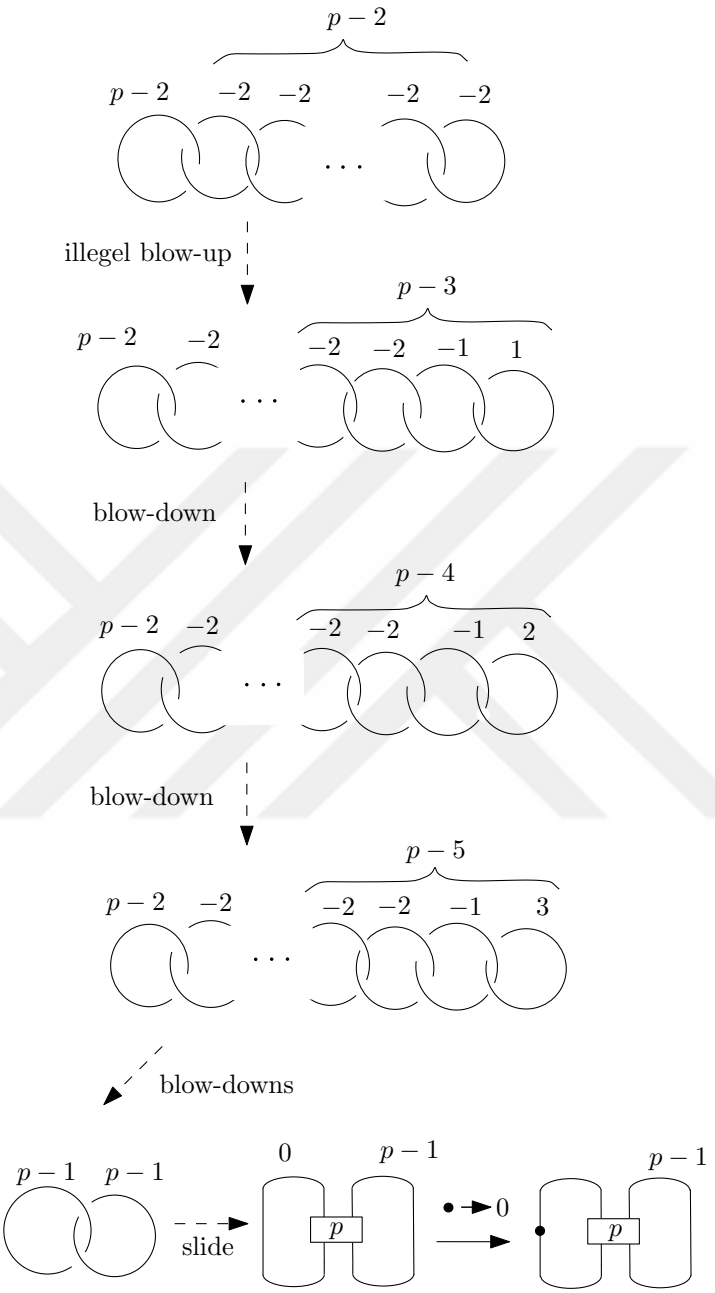


Figure 4.5: Rational Blow-down operation

Using the other diagram of C_p we see that following is also rational blow-down operation:

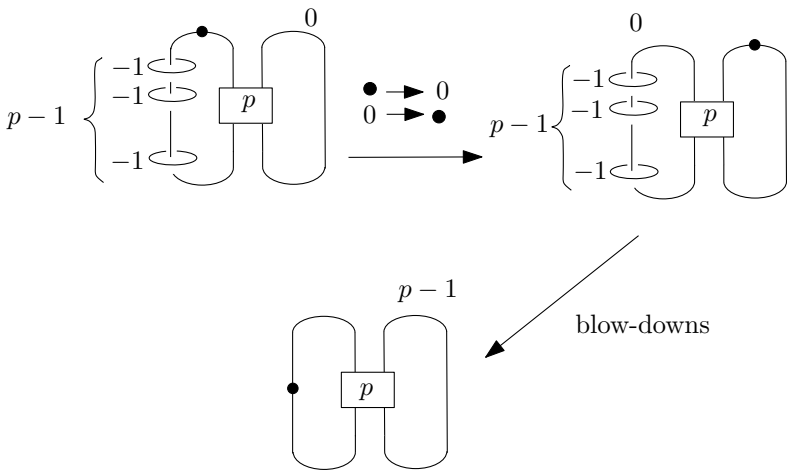


Figure 4.6: Rational Blow-down operation

After rational blow-down operation we will discuss logarithmic transform operation. Using the standart basis of $H_1(T^3) = H_1(S^1;\mathbb{Z}) \oplus H_1(S^1;\mathbb{Z}) \oplus H_1(S^1;\mathbb{Z})$, let ϕ_p $(p \geqq 0)$ be the self diffeomorphism of T^3 given by the automorphism $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & p \end{pmatrix}$.

Now suppose that a smooth 4-manifold X contains T^2 which has trivial normal bundle $\nu(T^2) = T^2 \times B^2$. Then we have the following definition.

Definition 4.1.3. A *logarithmic transform* of X *with multiplicity* p is defined to be the smooth 4-manifold obtained by removing $int\nu(T)$ and gluing it back by the map $\phi_p : T^2 \times S^1 \mapsto \partial\nu(T)$

The following Figure 4.7 shows how we obtain the logarithmic transform of X. In the figure we see the trivial normal bundle of the torus $T = \nu(T^3)$ in X. During the procedure we will keep track of the circles on the boundary of $\partial\nu(T)$ labelled by α, β, γ. First we replace the dot of the lower circle by a 0 and 0 of the middle circle with a dot. Then we blow up and then slide and blow down again. By repeating this procedure we get the 8th figure. Finally, we replace the 0-framed 2-handle with a dotted circle. We see that this operation corresponds to a logarithmic transform of X with multiplicity p by inspecting the positions of the circles α, β, γ.

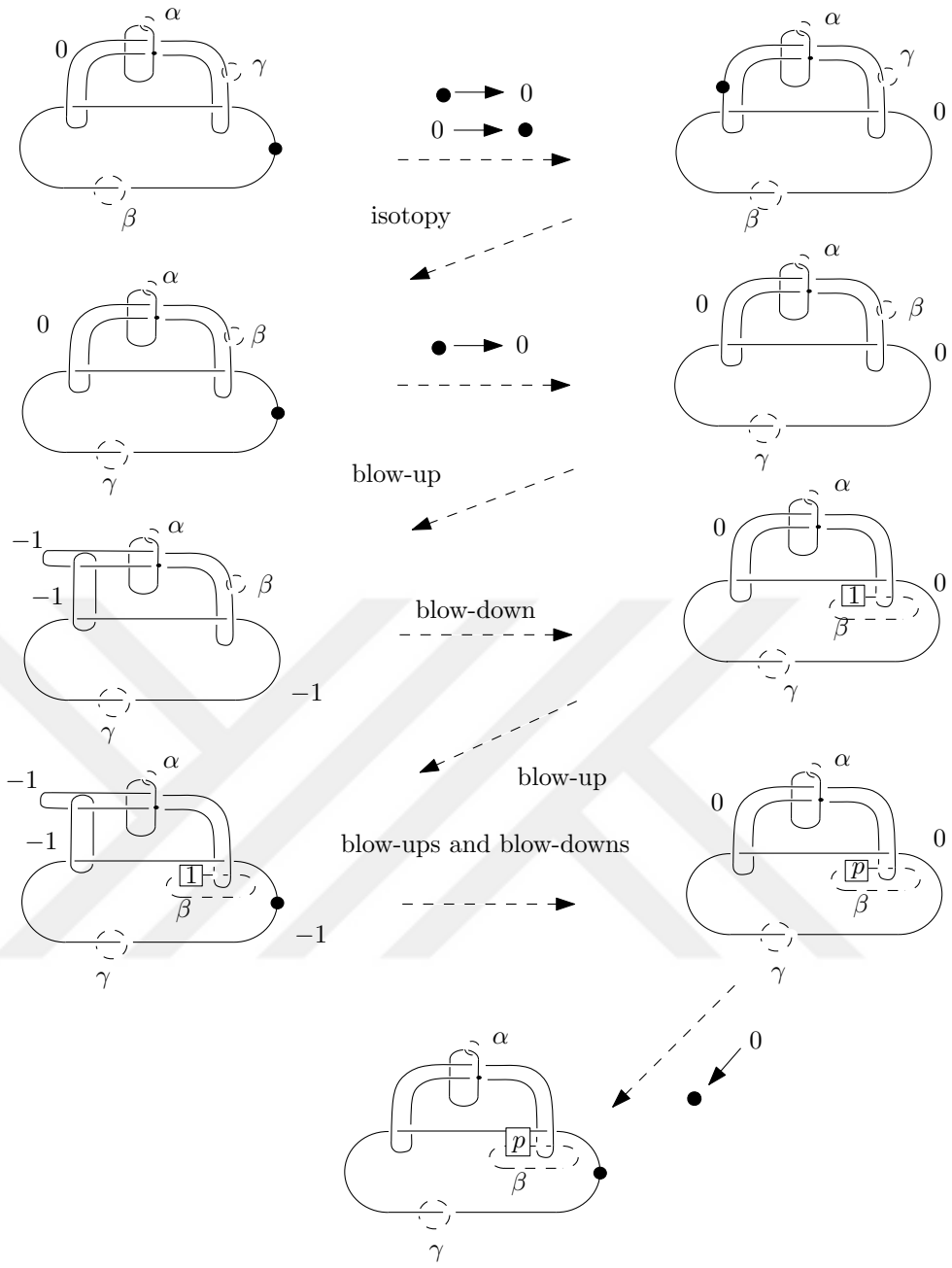


Figure 4.7: Logarithmic Transform Operation

Theorem 4.1.4. *Let a smooth 4-manifold X has a fishtail neighborhood which is drawn in Figure 4.8. Suppose $X_p(p \geqq 2)$ is the smooth 4-manifold obtained from X by logarithmic transform with multiplicity p in this fishtail neighborhood. Then there exist C_p in $X\#(p-1)\overline{\mathbb{C}P}^2$ so that the rational blowdown of $X\#(p-1)\overline{\mathbb{C}P}^2$ along the mentioned C_p is diffeomorphic to X_p.*

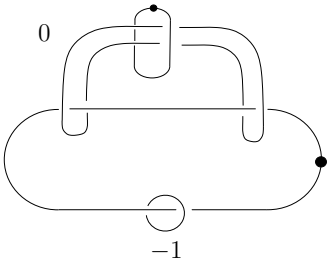


Figure 4.8: Fistail Neighborhood

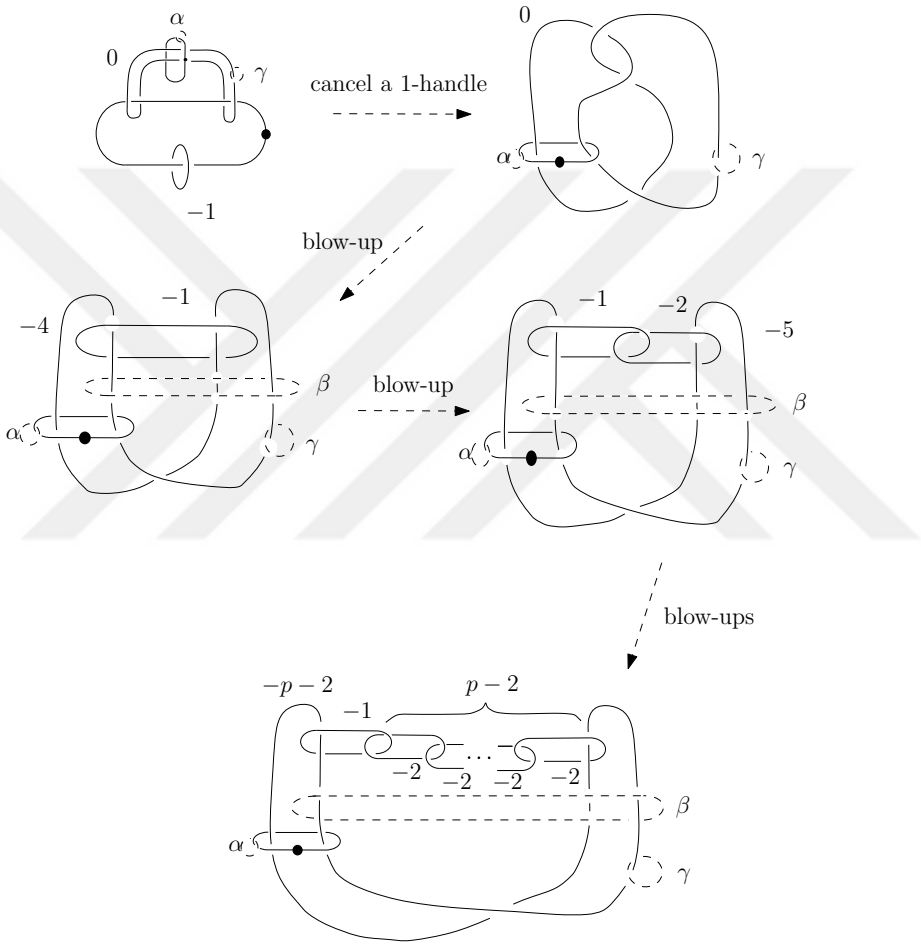


Figure 4.9: Obtaining C_p with blow-ups

Proof. In the Figure 4.9, we see how we can obtain $C_p \subset X\#(p-1)\overline{\mathbb{C}P}^2$. During this operation we again keep track of circles α, β, γ. By applying the procedure in Figure 4.5to $X\#(p-1)\overline{\mathbb{C}P}^2$ along the C_p we constructed in Figure 4.9, we see that the theorem holds by inspecting the positions of the circles α, β, γ. □

4.2 Corks, Plugs and Rational Blow-down

In this section, we will explore a relation between rational blow-down operation and regluing corks and plugs.

Let $T_{p,q}$, $U_{p,q}$ and $V_{p,q}$ be as in Figure 4.10. Note the fact that all these manifolds contain C_p.

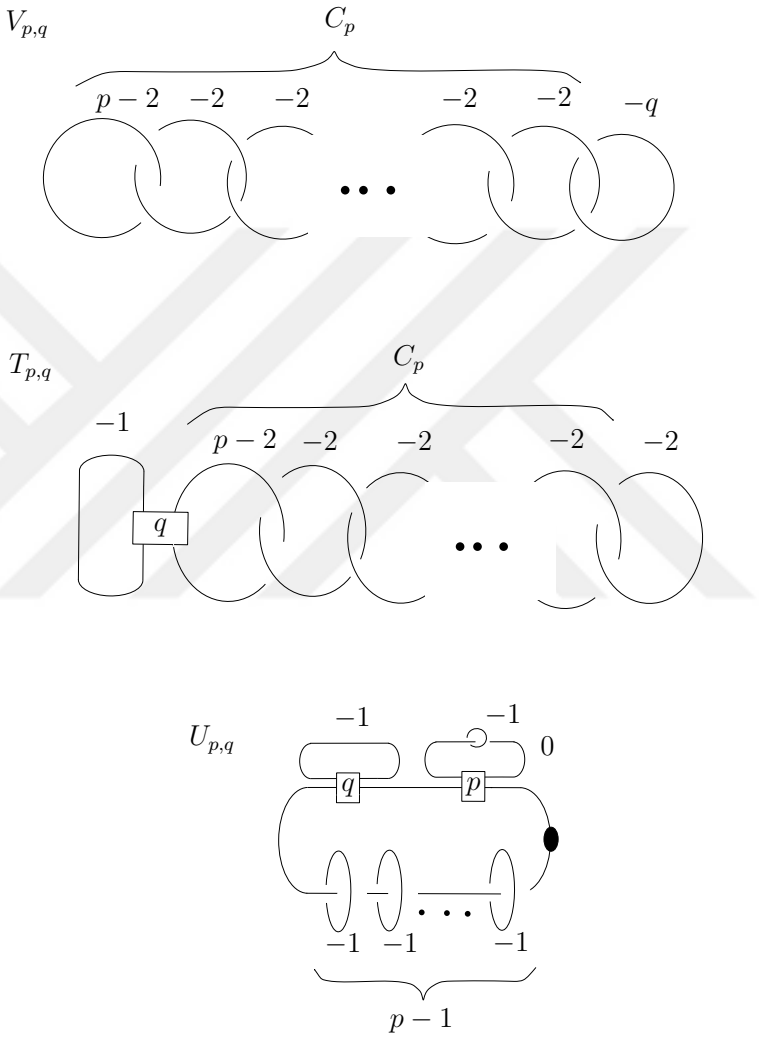


Figure 4.10: $T_{p,q}$, $U_{p,q}$ and $V_{p,q}$

Theorem 4.2.1. *(1) Suppose that a smooth 4-manifold X contains $T_{p,p-1}$ (respectively $T_{p,p+1}$). Then X contains W_{p-2} (respectively W_{p-1}). Also if we let X_p be the rational blow-down of X along C_p, then by regluing W_{p-2} (respectively W_{p-1}) via f_{p-2} (respectively f_{p-1}) we get $X_p\#(p-1)\overline{\mathbb{C}P}^2$.*

Proof. (1) For the case $q = p + 1$ we have the decomposition of $T_{p,p+1}$ as in the 6th figure in Figure 4.11. After obtaining this decomposition we change the 0 and dot of the upper circles. This operation does not change the diffeomorphism type of our manifold as it amounts to removing a 4-ball and regluing it. After applying this operation we can easily find W_{p-1} in our original manifold X. By removing and gluing W_{p-1} again by the map f_{p-1} we get the bottom diagram in Figure 4.12. We can also easily check that $X_p\#(p-1)\overline{\mathbb{C}P}^2$ is given by replacing dot and 0 before regluing the 4-ball. Thus, these two operations give the same smooth 4-manifold at the end.

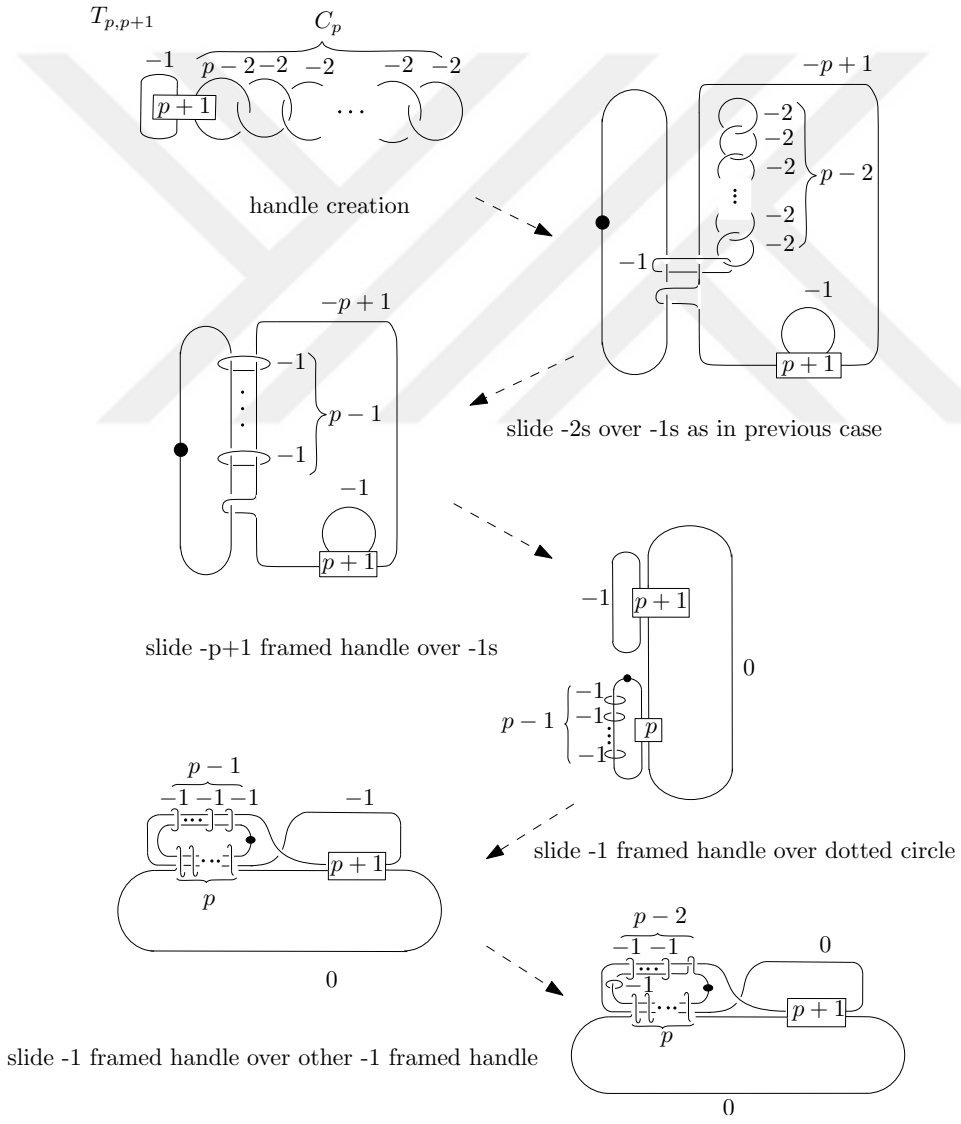


Figure 4.11

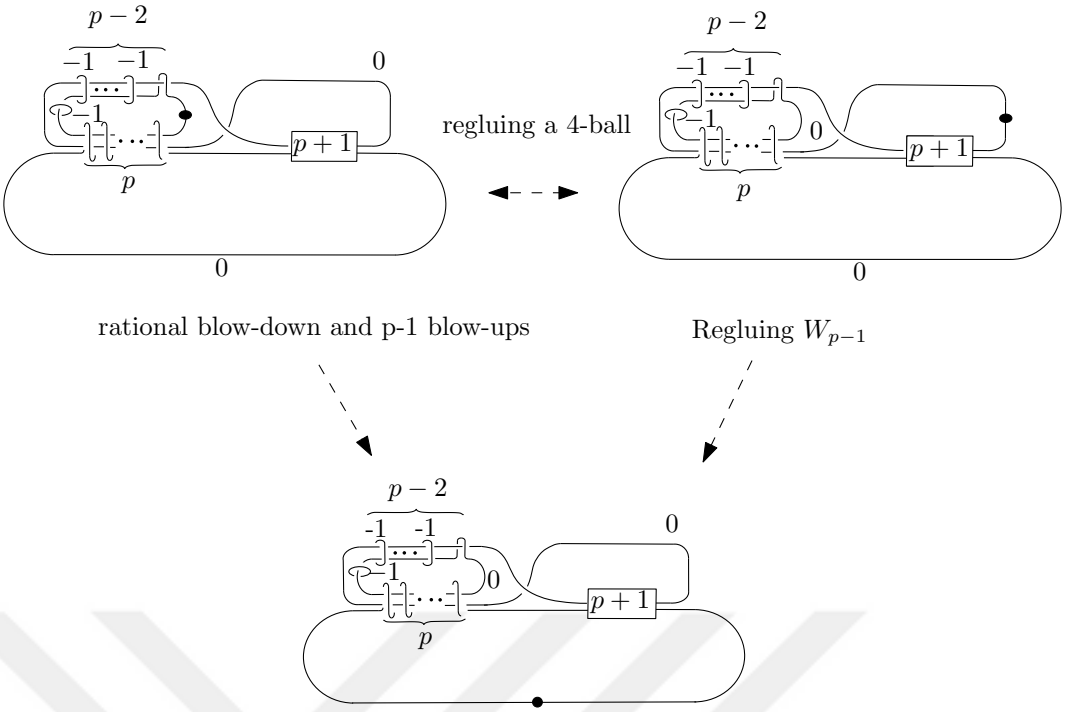


Figure 4.12

□

(2) Suppose that a smooth 4-manifold X contains $U_{p,p-1}$ ($U_{p,p+1}$). Then X contains W_{p-2} (W_{p-1}). Also if we let X_p be the rational blow-down of X along C_p, then by regluing W_{p-2} (W_{p-1}) via f_{p-2} (f_{p-1}) we get $X_p\#(p-1)\overline{\mathbb{C}P}^2$.

Proof. (2) Sliding 0-framed 2-handle over the dotted circle as shown in the Figure 4.13 we get a similar case to (1). Then arguing as in the case (1) again we get the result.

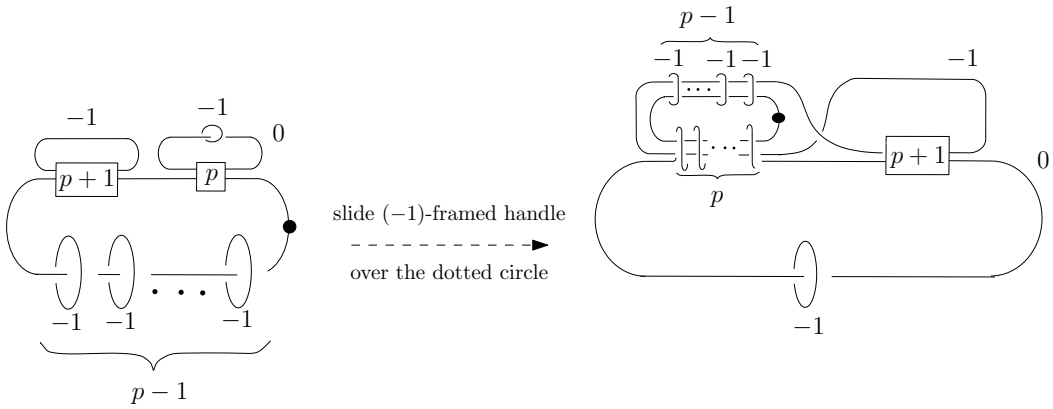


Figure 4.13

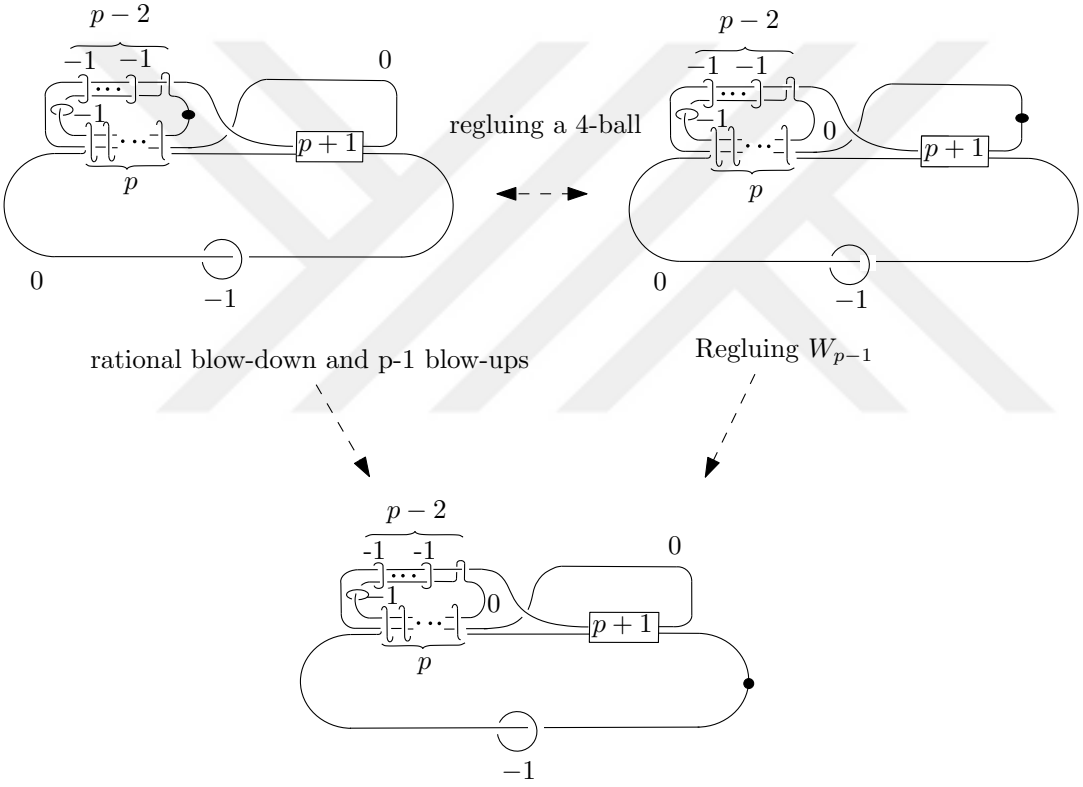


Figure 4.14

□

(3) Suppose that a smooth 4-manifold X contains $V_{p,q}$. Then X contains $W_{q-1,p}$. Also if we let X_p be the rational blow-down of X along C_p, then by regluing $W_{q-1,p}$ via $f_{q-1,p}$ we get $X_p\#(p-1)\overline{\mathbb{C}P}^2$.

Proof. (3) The claim follows from the below Figure 4.15.

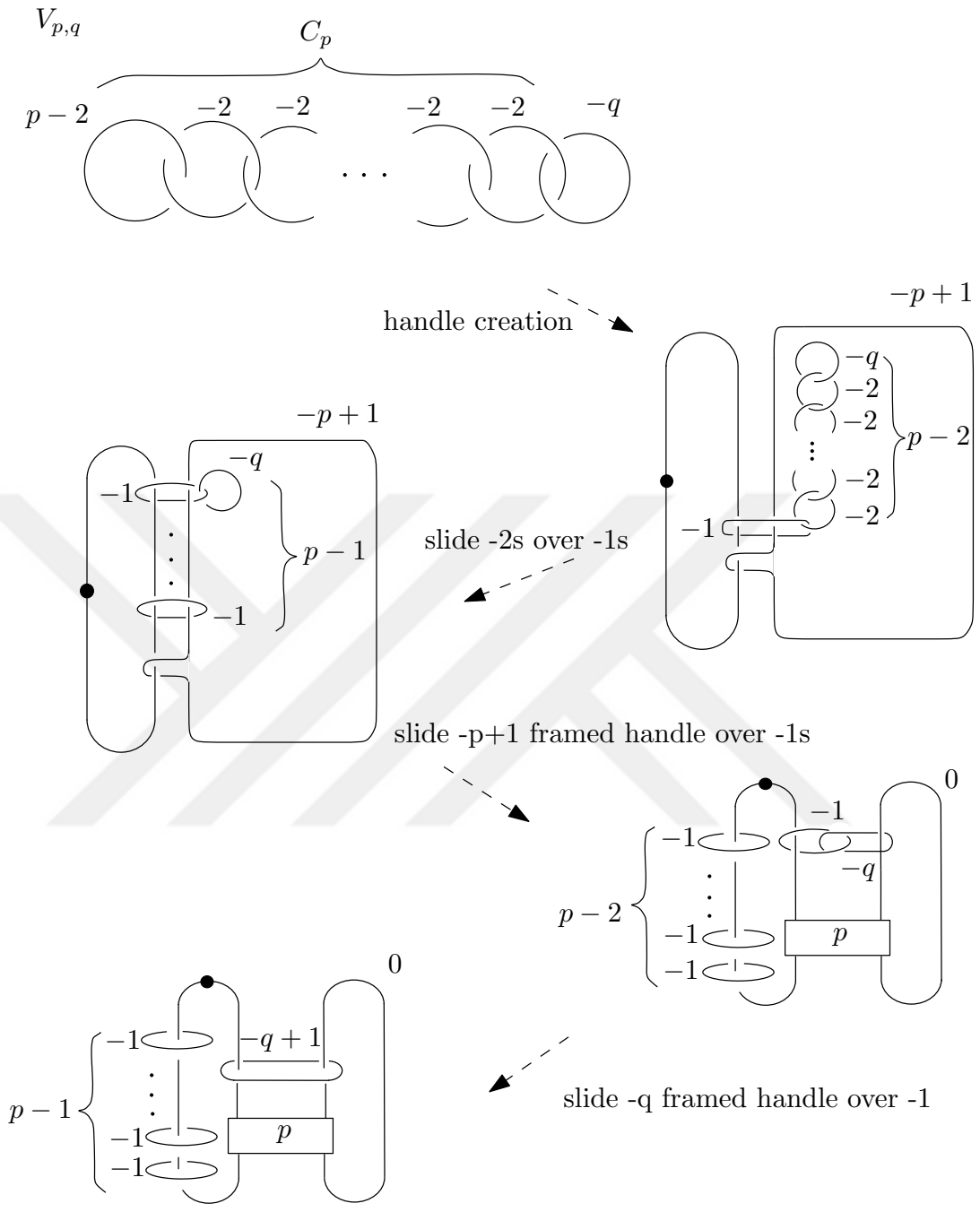


Figure 4.15

□

Let E_3 be the smooth 4-manifold constructed by Yasui [22]. Then by using the theorem above we have the following examples:

Example. (1) By removing a cork W_1 in $\mathbb{C}P^2\#11\overline{\mathbb{C}P}^2$ and regluing via the map f_1 we get $E_3\#\overline{\mathbb{C}P}^2$. Observe the fact that $\mathbb{C}P^2\#11\overline{\mathbb{C}P}^2$ is homeomorphic but not diffeomorphic to $E_3\#2\overline{\mathbb{C}P}^2$.

(2) By removing a plug $W_{1,3}$ in $CP^2\#11\overline{\mathbb{C}P}^2$ and regluing via the map $f_{1,3}$ we get $E_3\#\overline{\mathbb{C}P}^2$.

Proof. (1) and (2) follows from the fact that E_3 can be obtained from rationally blowing down C_3 in a submanifold $T_{2,3}$ or $V_{2,3}$ of $\mathbb{C}P^2\#11\overline{\mathbb{C}P}^2$ (see [22]) and applying the Theorem 4.2.1.

□

Example Let $(E(2n)\#2\overline{\mathbb{C}P}^2)_{2n}\#(2n-1)\overline{\mathbb{C}P}^2$ be the smooth 4-manifold obtained from $E(2n)\#2\overline{\mathbb{C}P}^2$ by rational blow-down operation along the C_{2n}. Then by removing W_{2n-1} from $E(n)\#(2n+1)\overline{\mathbb{C}P}^2$ and regluing it back via the map f_{2n-1} we get the elliptic surface $(E(2n)\#2\overline{\mathbb{C}P}^2)_{2n}\#(2n-1)\overline{\mathbb{C}P}^2$. Observe that $E(n)\#(2n+1)\overline{\mathbb{C}P}^2$ is homeomorphic but not diffeomorphic to $(E(2n)\#2\overline{\mathbb{C}P}^2)_{2n}\#(2n-1)\overline{\mathbb{C}P}^2$.

Proof. Using Figure 3.16 of $E(2n)$ we can easily find $U_{2n,2n+1}$ inside $E(2n)\#2\overline{\mathbb{C}P}^2$. Then the claim easily follows from Theorem 4.2.1. □

Example. Let $n, p \geqq 2$, then by removing a plug $W_{n,p}$ in $E(n)\#(p-1)\overline{\mathbb{C}P}^2$ and regluing it back via $f_{n,p}$ we get the elliptic surface $E(n)_p\#(p-1)\mathbb{C}P^2$

Proof. Note that $E(n)$ has a cusp neighbourhood.

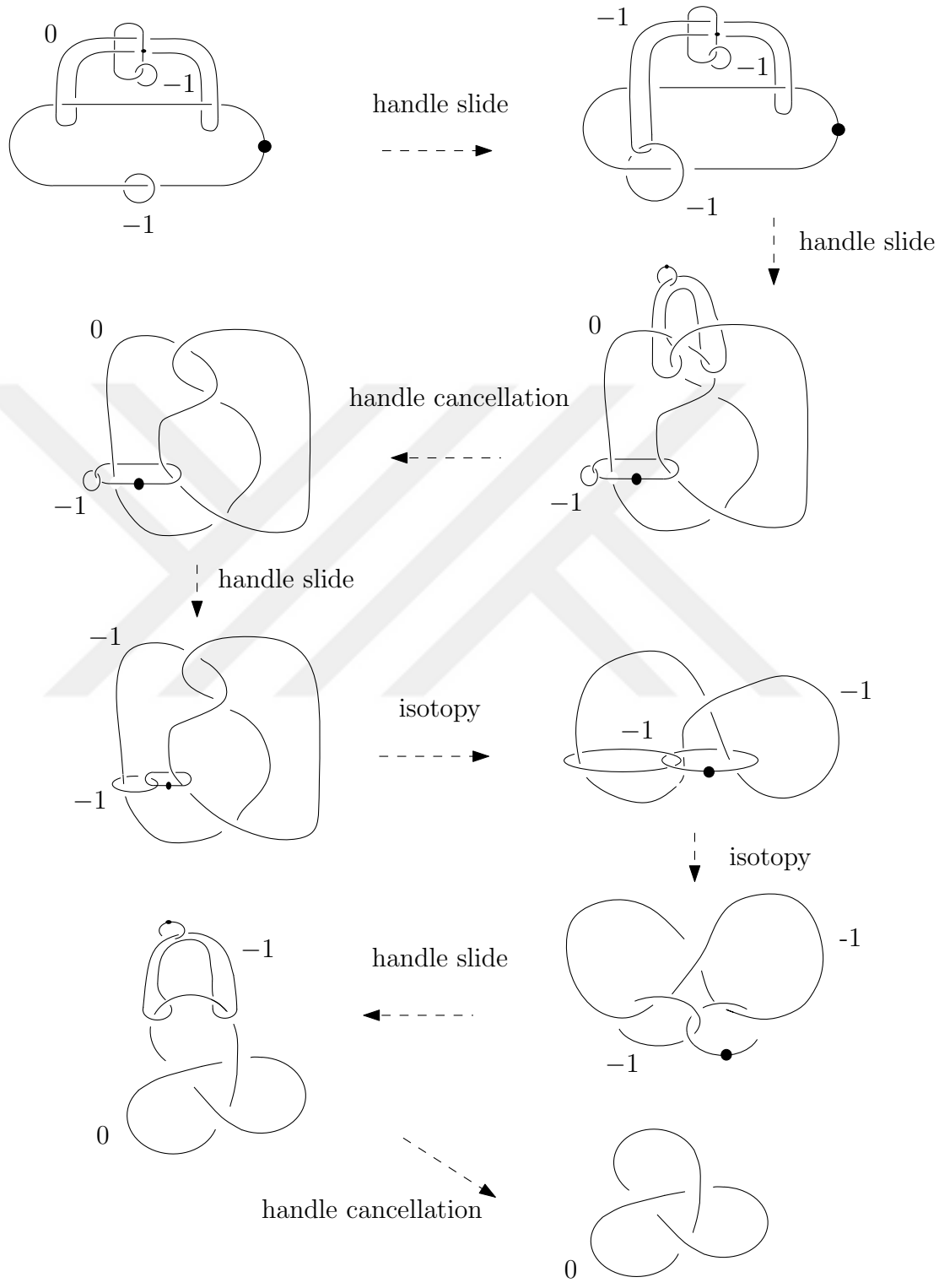
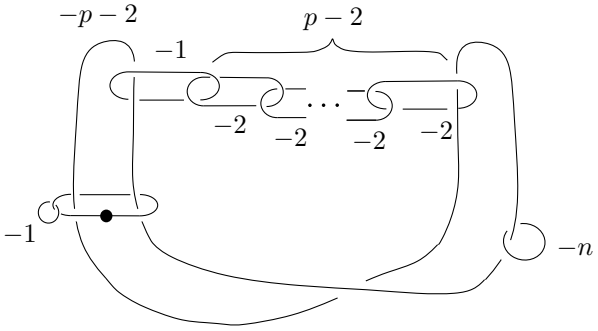


Figure 4.16: Finding Cusp Neighbourhood in $E(n)$

So blowing up cusp neighborhood of $E(n)$ we can construct $C_p \subset E(n)\#(p-1)\overline{\mathbb{C}P}^2$ as in the Figure 4.9. By this way we can find $V_{p,n+1} \subset E(n)\#(p-1)\overline{\mathbb{C}P}^2$ as follows:



slide -n framed handle over -1 framed handle

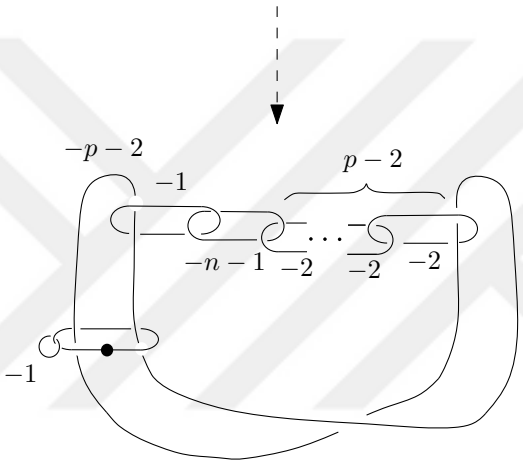


Figure 4.17: Finding $V_{p,n+1} \subset E(n)\#\overline{\mathbb{C}P}^2$

First slide as in the Figure 4.9 (in this case we have γ as $-n$-framed handle and α as -1-framed handle). Then slide $-n$-framed handle over -1-framed handle. □

4.3 More examples

Theorem 4.3.1. *$\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ is homeomorphic but not diffeomorphic to the manifold in Figure 4.18. Also the manifold in Figure 4.18 has the same Seiberg-Witten invariant with the elliptic surface $E(n)_{2,3}$. Therefore, the mentioned manifold does not contain any 2-sphere with -1 self intersection number ,i.e, it is minimal. Also W_1 is a cork and $W_{1,3}$ is a plug of this minimal 4-manifold.*

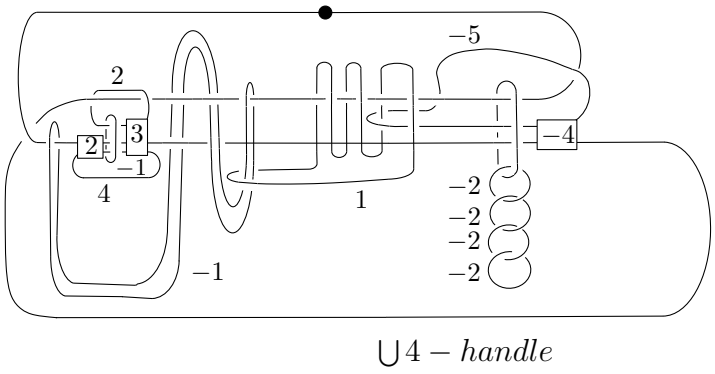


Figure 4.18: An exotic copy of $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$

Proof. By sliding handles as in [22] (figure 8-17) we get Figure 4.20. After blow-ups we get Figure 4.21. By an isotopy we get Figure 4.22. Sliding as indicated in Figure 4.22 and Figure 4.23 we obtain Figure 4.24. After a blow-up we get Figure 4.25. Note the fact that Figure 4.25 contains C_5. Applying rational-blowdown operation to this C_5 we get the smooth 4-manifold E_3' whose diagram is given in Figure 4.18. The same arguments in [22] shows that $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ is homeomorphic to E_3'. Furthermore, elliptic surface $E(1)_{2,3}$ has the same Sieberg-Witten invariant with E_3', so we conclude that E_3' is minimal. Clearly, $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ is not minimal. Thus, we obtain the first part of the claim.

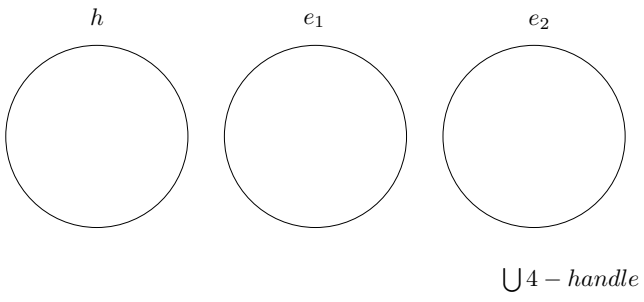


Figure 4.19: $\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2$

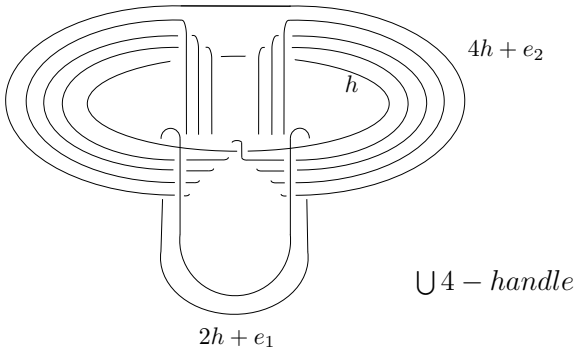


Figure 4.20: $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$

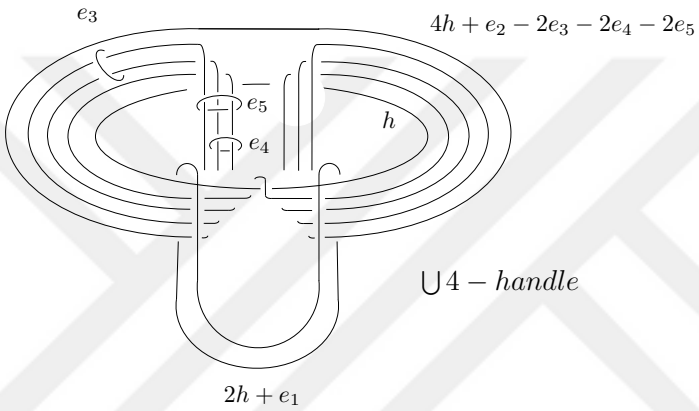


Figure 4.21: $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P}^2$

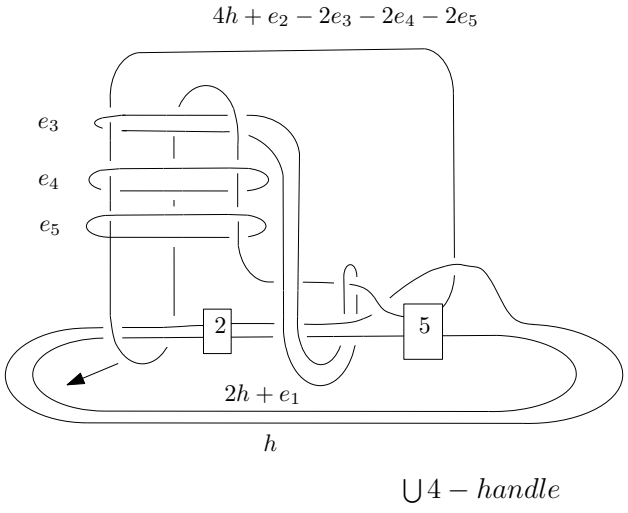


Figure 4.22: $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P}^2$

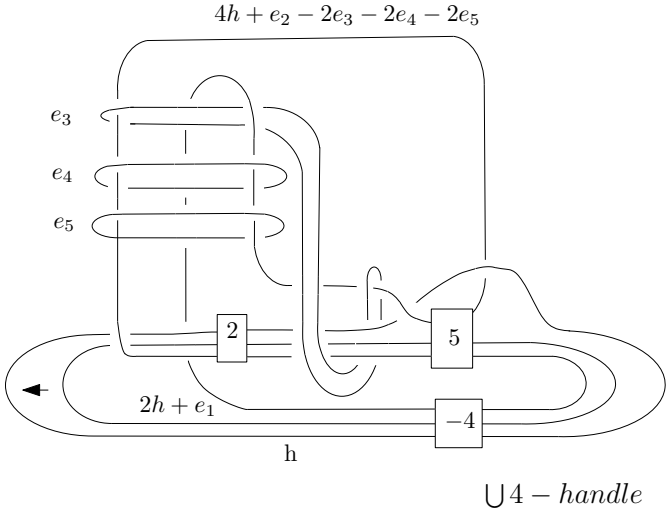


Figure 4.23: $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P}^2$

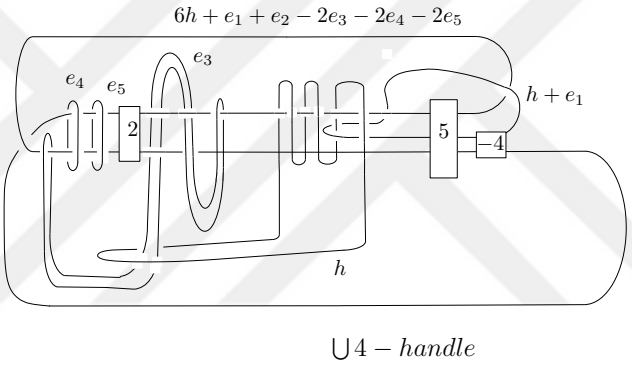


Figure 4.24: $\mathbb{C}P^2 \# 5\overline{\mathbb{C}P}^2$

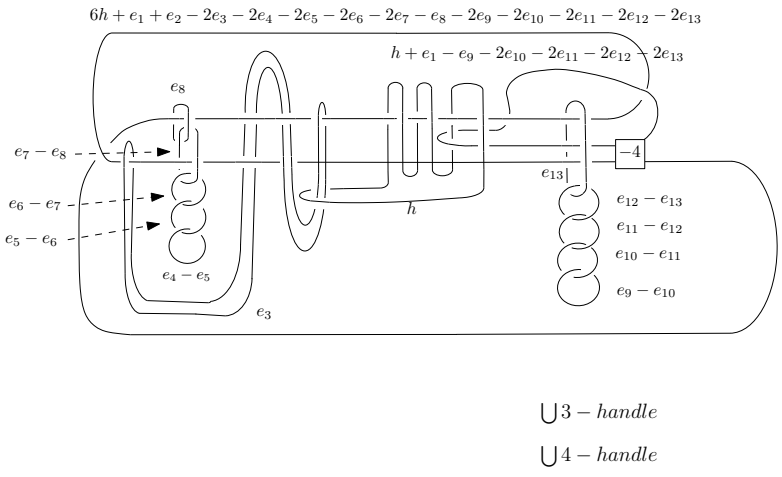


Figure 4.25: $\mathbb{C}P^2 \# 13\overline{\mathbb{C}P}^2$

By the handle slides an in Figure 4.29 we obtain W_1. Performing cork twist we obtain a 2-sphere whose self intersection number is -1, thus by cork twist we obtain a non-minimal surface. Therefore, W_1 is a cork of E_3'. (By creating handle pair in Figure 4.19 we obtain Figure 4.26. Then similar to the first case we can get Figure 4.27 as handle decomposition of $\mathbb{C}P^2\#13\overline{\mathbb{C}P}^2$ from Figure 4.26. We can see an embedding of C_5 inside Figure 4.27. Because we did not slide any handles over the new 2-handle (coming from handle creation) this copy of C_5 is isotopic to the one in Figure 4.25. Therefore, by rational blow-down operation we get Figure 4.28 of E'. First diagram in Figure 4.29 is obtained from Figure 4.28 when we ignore handles. Note that in the last diagram of Figure 4.29 we have W_1. By applying cork twist to this cork we get a -1 intersecting 2-sphere. Thus, W_1 is a cork of E'.)

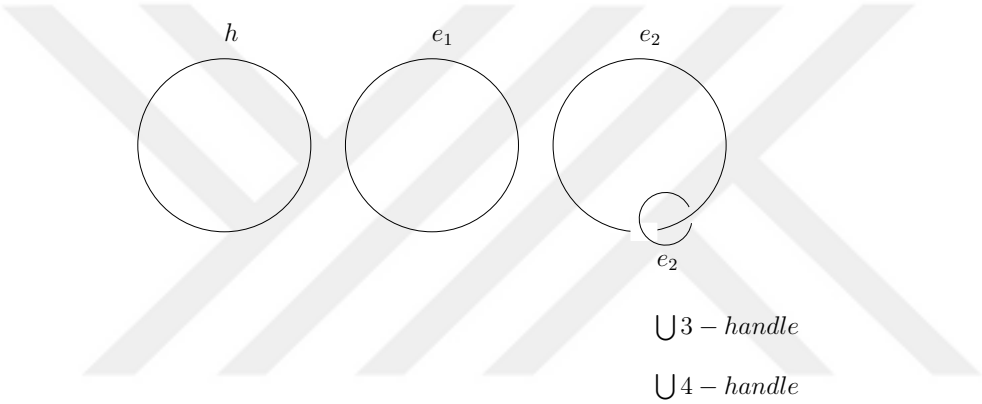


Figure 4.26: $\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2$

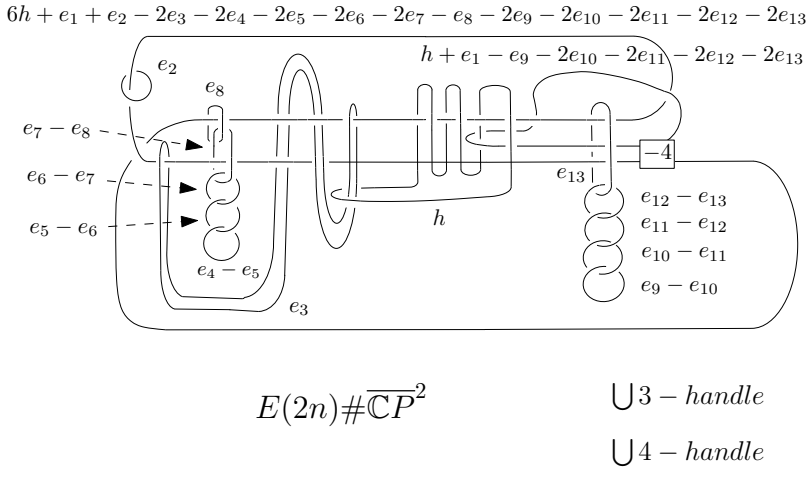


Figure 4.27: $\mathbb{C}P^2\#13\overline{\mathbb{C}P}^2$

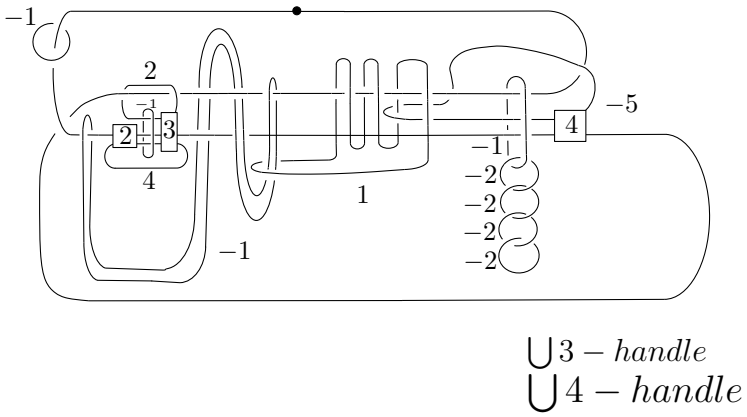


Figure 4.28: $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$

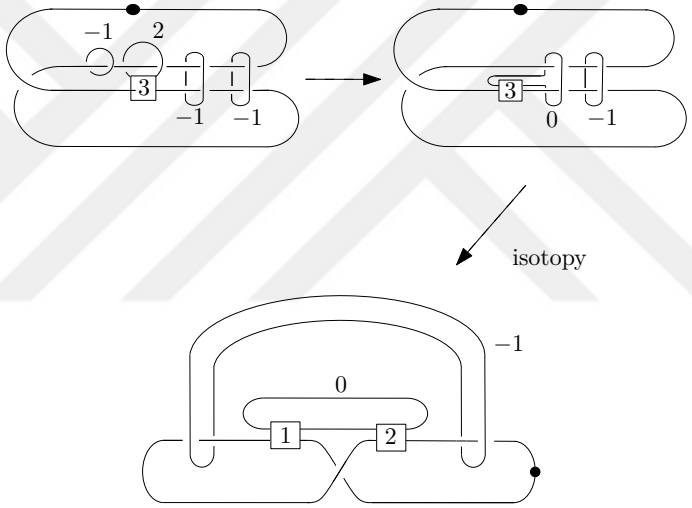


Figure 4.29: Finding a cork

By the handle slides as in Figure 4.30 we obtain $W_{1,3}$. Performing cork twist we obtain a 2-sphere whose self intersection number is -1, thus by cork twist we obtain a non-minimal surface. Therefore, $W_{1,3}$ is a plug of E_3'. (By similar moves as in the second case we can obtain the first diagram in Figure 4.30 from Figure 4.28. Then by handle moves we get the last diagram in Figure 4.30 and we get $W_{1,3}$ inside E_3'. By a plug twist we again get a -1 intersecting sphere as in the case of 2. Therefore, $W_{1,3}$ is a plug of E_3'.)

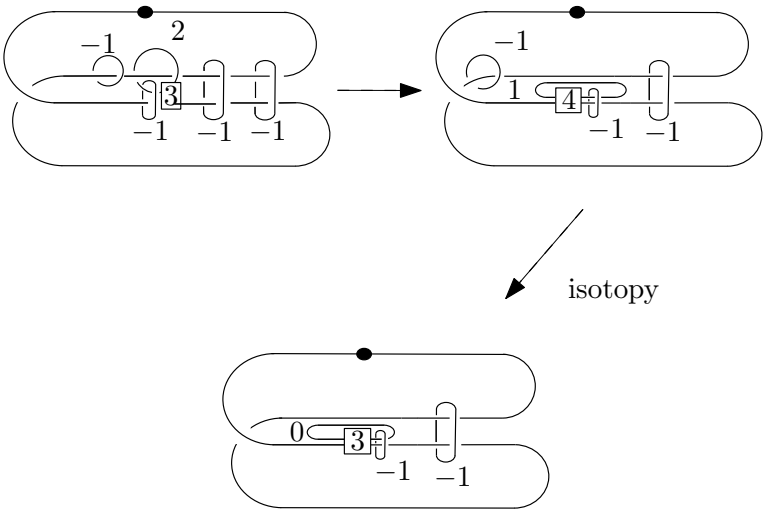


Figure 4.30: Finding a plug

□

4.3.1 Knotted corks and plugs

Theorem 4.3.2. *1) The manifold in Figure 4.31 is isotopic to* $\mathbb{C}P^2\#13\overline{\mathbb{C}P}^2$.

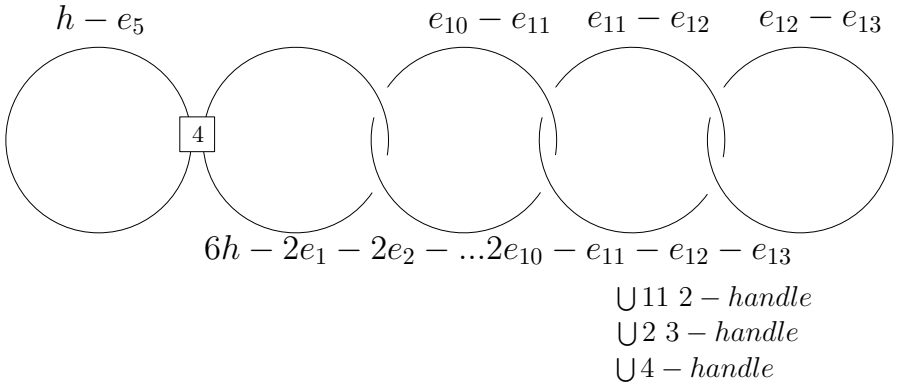


Figure 4.31: $\mathbb{C}P^2\#13\overline{\mathbb{C}P}^2$

2) The manifold in Figure 4.32 is isotopic to $\mathbb{C}P^2\#13\overline{\mathbb{C}P}^2$

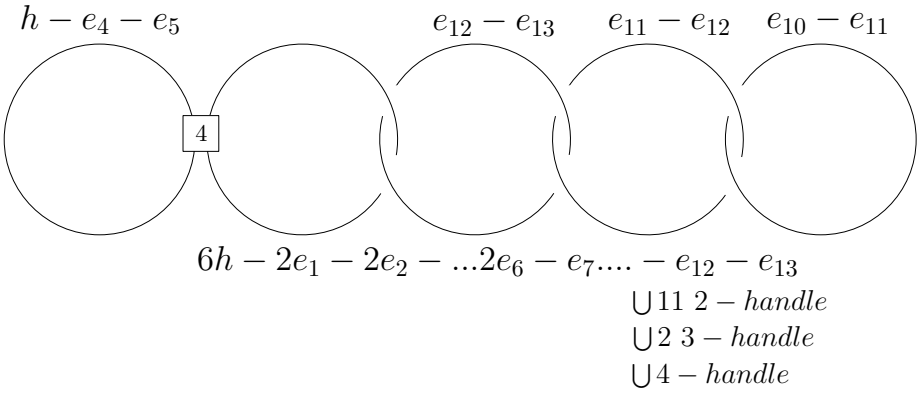


Figure 4.32: $\mathbb{C}P^2\#13\overline{\mathbb{C}P}^2$

Proof. (1) The moves similar to the ones in Figures 4.19-4.24 gives Figure 4.33 of $\mathbb{C}P^2\#3\overline{\mathbb{C}P}^2$. After that applying a isotopy give Figure 4.34. Further handle slide give Figure 4.35. Blowing up this figure gives Figure 4.36.Then blowing-ups as in Figure 4.9 give Figure 4.31

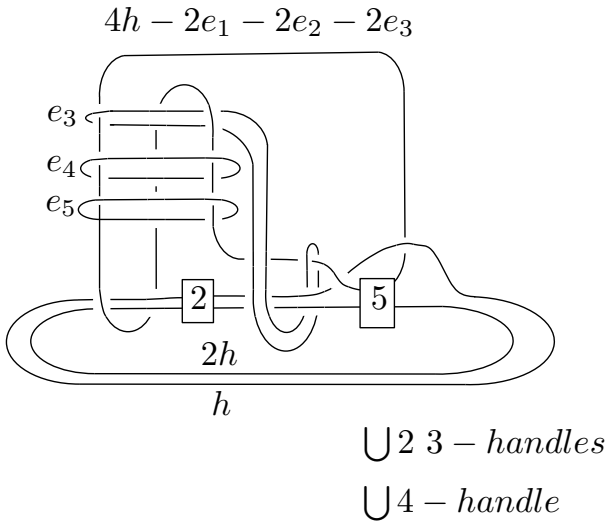


Figure 4.33: $\mathbb{C}P^2\#3\overline{\mathbb{C}P}^2$

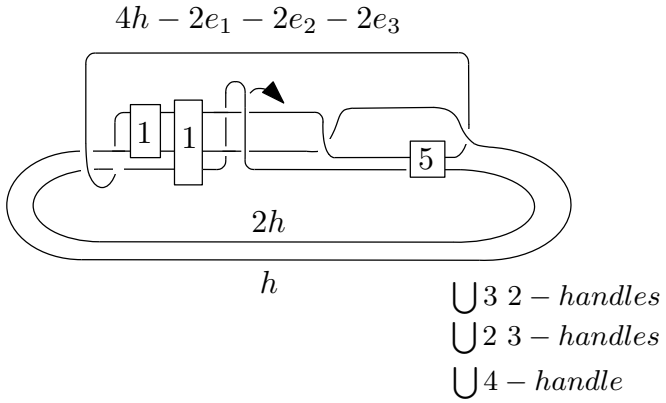


Figure 4.34: $\mathbb{C}P^2\#3\overline{\mathbb{C}P}^2$

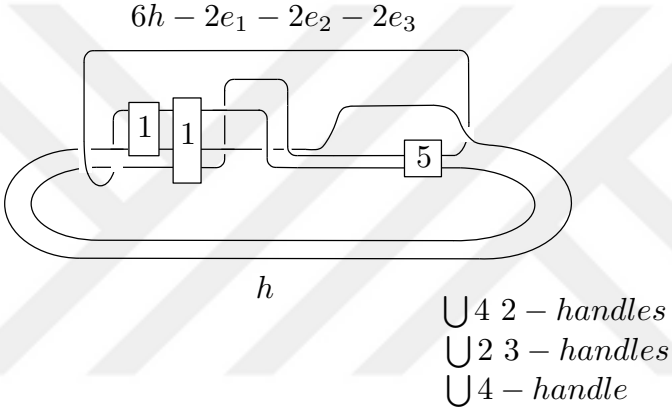


Figure 4.35: $\mathbb{C}P^2\#3\overline{\mathbb{C}P}^2$

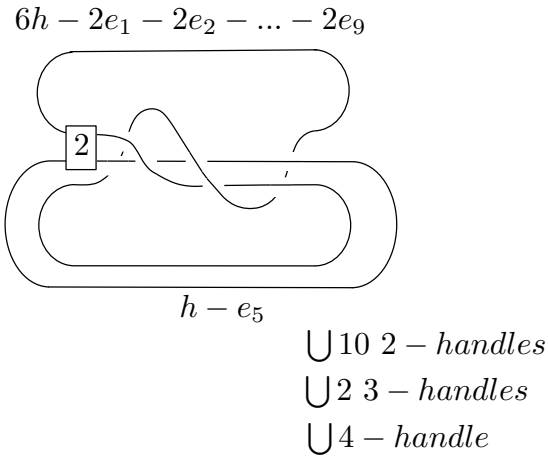


Figure 4.36: $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$

(2) Blowing up the Figure 4.34 gives Figure 4.37 of $\mathbb{C}P^2\#5\overline{\mathbb{C}P}^2$. With more blow-ups we obtain Figure 4.38. Then we have Figure 4.32 of $\mathbb{C}P^2\#13\overline{\mathbb{C}P}^2$ by similar moves in [22] (Figure19-25).

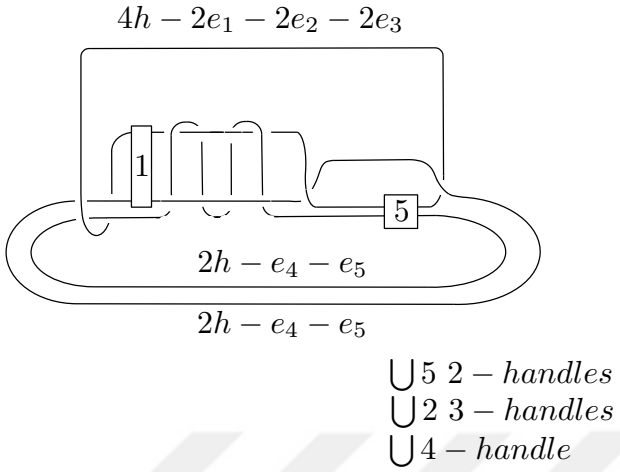


Figure 4.37: $\mathbb{C}P^2\#5\overline{\mathbb{C}P}^2$

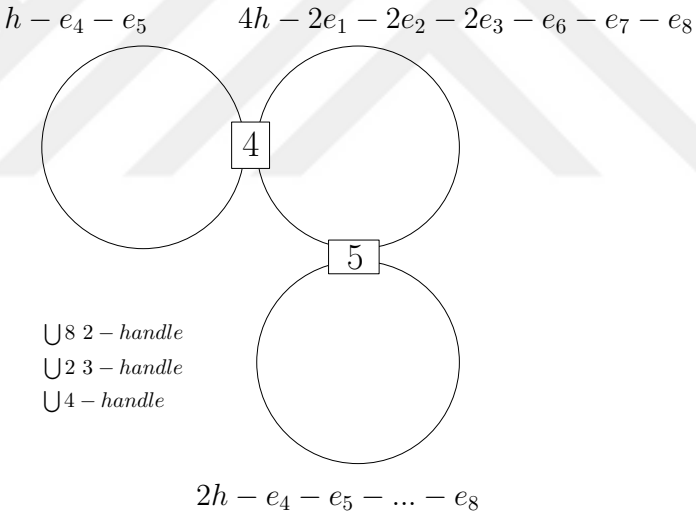


Figure 4.38: $\mathbb{C}P^2\#8\overline{\mathbb{C}P}^2$

□

Definition 4.3.3. X_5 is the smooth 4-manifold obtained by rational blow-down operation along the copy of C_5 in the theorem above in part 1.

Definition 4.3.4. X_3 is the smooth 4-manifold obtained by rational blow-down operation along the copy of C_5 in the theorem above in part 2.

Then we have an easy corollary:

Corollary 4.3.5. 1) By performing rational blow-down operation to $\mathbb{C}P^2\#14\overline{\mathbb{C}P}^2$ along the C_5 in Figure 4.39, we get $X_5\#\overline{\mathbb{C}P}^2$.

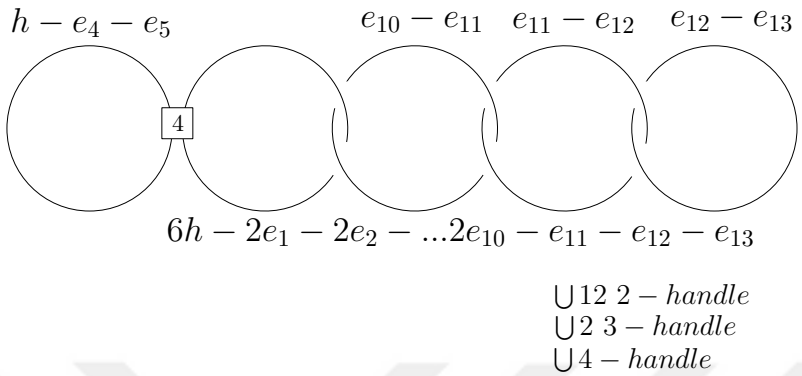


Figure 4.39: $\mathbb{C}P^2\#14\overline{\mathbb{C}P}^2$

2) By performing rational blow-down operation to $\mathbb{C}P^2\#14\overline{\mathbb{C}P}^2$ along the C_5 in Figure 4.40, we get $X_5\#\overline{\mathbb{C}P}^2$.

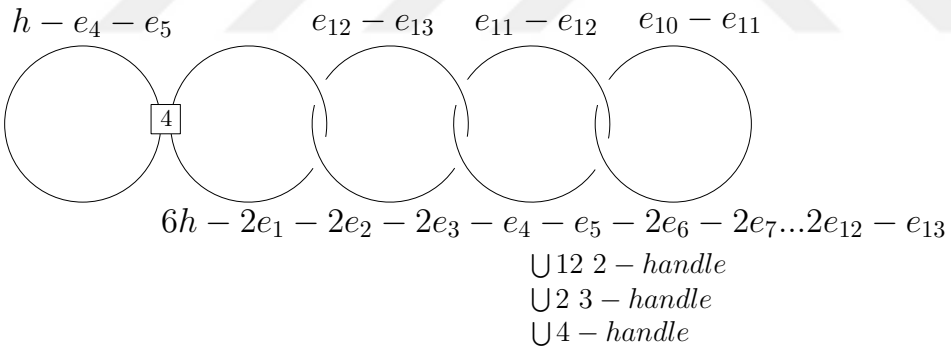


Figure 4.40: $\mathbb{C}P^2\#14\overline{\mathbb{C}P}^2$

Corollary 4.3.6. 1) By removing W_2 and regluing it with the map f_3, we obtain the smooth 4-manifold $X_5\#5\overline{\mathbb{C}P}^2$.

2) By removing W_2 and regluing it with the map f_3, we obtain the smooth 4-manifold $X_3\#5\overline{\mathbb{C}P}^2$.

Proof. Follows easily from Theorem 4.2.1 and Corollary 4.3.5. □

Theorem 4.3.7. *1) X_3 has same Seiberg-Witten invariant with $E(1)_{2,3}$. Thus it is diffeomorphic to $E(1)_{2,3}$ and it is not diffeomorphic to $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$.*

2) X_5 has same Seiberg-Witten invariant with $E(1)_{2,5}$. Thus it is diffeomorphic to $E(1)_{2,5}$ and it is not diffeomorphic to $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$.

3) For $k \geqq 0$, no two of the manifolds $X_3\#k\overline{\mathbb{C}P}^2$, $X_5\#k\overline{\mathbb{C}P}^2$ and $\mathbb{C}P^2\#(9+k)\overline{\mathbb{C}P}^2$ are diffeomorphic.

Proof. We see that X_3 and X_5 are homeomorphic by corrollary. Using Freedman's result 2.1.4 and Rochlin's theorem we also see that both manifolds are homeomorphic to $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$. Using the results in [22] X_3 has the same Sieberg-Witten invariant as $E(1)_{2,3}$, X_5 has the same Sieberg-Witten invariant as $E(1)_{2,5}$. Therefore, using the arguments similar to [20] we get the claim. □

With the help of this result we have the following theorems:

Theorem 4.3.8. *(Knotting corks) There are two copies of W_3 inside $\mathbb{C}P^2\#14\overline{\mathbb{C}P}^2$ such that the manifolds obtained by performing cork twist to these copies of W_3, and $\mathbb{C}P^2\#14\overline{\mathbb{C}P}^2$ are all homeomorphic but no two of them are diffeomorphic.*

Proof. Follows easily from Corollary 4.3.6 and Theorem 4.3.7. □

Theorem 4.3.9. *(Knotting plugs) There are two copies of $W_{4,5}$ inside $\mathbb{C}P^2\#16\overline{\mathbb{C}P}^2$ such that the manifolds obtained by performing plug twist to these copies of $W_{4,5}$, and $\mathbb{C}P^2\#16\overline{\mathbb{C}P}^2$ are all homeomorphic but no two of them are diffeomorphic.*

Proof. Blowing down C_5 inside $V_{5,5}$ in the manifold $\mathbb{C}P^2\#16\overline{\mathbb{C}P}^2$ we obtain the smooth 4-manifold $E_3'\#3\overline{\mathbb{C}P}^2$. Thus, Theorem 4.2.1 shows that we obtain $E_3'\#7\overline{\mathbb{C}P}^2$ from $\mathbb{C}P^2\#16\overline{\mathbb{C}P}^2$ by a plug twist to $W_{4,5}$ and $E_5\#7\overline{\mathbb{C}P}^2$ from $\mathbb{C}P^2\#16\overline{\mathbb{C}P}^2$ by a plug twist to $W_{4,5}$. Likewise, [22] (prop 3.2 (2)) shows that $E_5\#7\overline{\mathbb{C}P}^2$ is obtained by removing $W_{4,5}$ inside $\mathbb{C}P^2\#16\overline{\mathbb{C}P}^2$ and regluing with the map $f_{4,5}$. Thus, similar to Theorem 4.3.7 we see that $E_5\#7\overline{\mathbb{C}P}^2$, $E_3'\#7\overline{\mathbb{C}P}^2$ and $\mathbb{C}P^2\#16\overline{\mathbb{C}P}^2$ are all homeomorphic, but no two of them are diffeomorphic. □

REFERENCES

[1] Selman Akbulut. *4-Manifolds*. Oxford University Press, Oxford, 1990.

[2] Steve Boyer. Simply-connected 4-manifolds with a given boundary. *Transactions of the American Mathematical Society*, 298:331–357, 1986.

[3] W.C. Hsiang R. Strong C.L Curtis, M.H Freedman. A decomposition theorem for h-cobordant smooth simply-connected compact 4-manifolds. *Invent. Math.*, 123:343–348, 1996.

[4] M. Freedman. The topology of four-dimensional manifolds. *J. Diff. Geom.*, 17:357–453, 1990.

[5] Taylor Laurence Freedman Michael H. A universal smoothing of four-space. *J. Diff. Geom.*, 24:69–78, 1986.

[6] R.E. Gompf. Handlebody construction of stein surfaces. *Annals of Mathematics*, 148:619–693, 1998.

[7] Robert Gompf. An infinite set of exotic $\mathbb{R}^4$'s. *J. Diff. Geom.*, 21:283–300, 1962.

[8] Taubes Clifford H. Gauge theory on asymptotically periodic 4-manifolds. *J. Diff. Geom.*, 25:363–430, 1987.

[9] F. Quinn M. Freedman. *The topology of 4-dimensional manifolds*. Princeton University Press, Princeton, 1990.

[10] R. Matveyev. A decomposition of smooth simply-connected h-cobordant 4-manifolds. *J. Differential Geom.*, 44:571–582, 1996.

[11] John Milnor. On manifolds homeomorphic to the 7-sphere. *Annals of Mathematics*, 64:399–405, 1956.

[12] John Milnor. On manifolds homeomorphic to the 7-sphere. *Ann. Math*, 64:399–405, 1956.

[13] John Milnor. *Morse Theory*. Princeton University Press, Princeton, 1963.

[14] John Milnor. *Topology from the Differentiable Viewpoint*. University Press of Virginia, Virginia, 1965.

[15] T. Rado. Über den begriff der riemannschen fläche. *Acta Litt. Scient. Univ. Szegd*, 2:101–121, 1925.

[16] A.I. Stipsicz R.E. Gompf. *4-Manifolds and Kirby Calculus*. American Mathematical Society, Berlin, 1999.

[17] R. Matveyev S. Akbulut. A convex decomposition theorem for 4-manifolds. *Internat. Math. Res. Notices*, 7:371–381, 1998.

[18] Kouichi Yasui Selman Akbulut. Corks, plugs and exotic structures. *Journal of Gökova Geometry Topology*, 2:40–82, 2008.

[19] John R. Stallings. The piecewise-linear structure of euclidean space. *Proceedings of the Cambridge Philosophical Society*, 58:481–488, 1962.

[20] Z. Szabo. Exotic 4-manifolds with $b^+ = 1$. *Math. Res Lett*, 6:731–741, 1996.

[21] Z. Szabo. Rational blowdowns of smooth 4-manifolds. *J. Differantial Geom.*, 2:181–235, 1997.

[22] K. Yasui. Exotic rational elliptic surfaces without 1-handles. *Algebr. Geom. Topol.*, 8:971–996, 2008.