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A teacher's enactment of responsive teaching moves: the concept of the mean

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Abstract
This study examines how a mathematics teacher, whose noticing skills developed through a 2-year online professional development program, implemented responsive teaching moves in a classroom setting while teaching the concept of the mean. Accordingly, the study was built on the responsive teaching model, which stems from noticing students' mathematical thinking. After discussing various scenarios involving student strategies for alternative mathematics content with 30 other participating teachers in an online environment, a teacher with eight years of professional experience implemented two tasks related to the concept of the mean in two sixth-grade classrooms. The analysis revealed that the teacher employed responsive teaching moves, albeit in varying orders, with different frequencies, and at varying durations. This led us to conclude that a sustained professional development program focused on developing teachers' noticing of students' thinking appeared to improve teachers' in-the-moment responses regarding the concept of the mean.

Keywords Mean · Online professional development program · Responsive teaching · Teacher noticing

Introduction

It is widely recognized among mathematics educators and researchers that effective mathematics instruction should be grounded in students' understanding (Lee & Cross Francis, 2018). This understanding serves as a valuable resource for teachers, enabling them to make informed instructional decisions and enhance their teaching practices (National Council of

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Teachers of Mathematics [NCTM], 2014). Thus, in recent years, mathematics educators have stressed that teachers need to have professional competence, defined as teacher noticing skills, to explore students' understanding (Kaiser et al., 2017), which is an ongoing process rather than a "static category of knowledge" (Sherin et al., 2011, p. 5). Teacher noticing skill can be regarded as a springboard that enables teachers to translate their knowledge into practice in a real teaching environment (Blömeke et al., 2015) and necessitates analyzing complex teaching environments (Jacobs et al., 2010; van Es & Sherin, 2002). Recent reviews further confirm that teacher noticing continues to be conceptualized as a central and evolving professional competence in mathematics education, with increasing attention to its relationship with instructional decision-making and classroom practice (Weyers et al., 2024).

Given the significance of the skill as a professional competence, numerous studies have investigated the extent to which teachers demonstrate noticing skills in different mathematics topics and have explored ways to enhance these skills through various approaches (Tekin-Sitrava et al., 2025; Doğan-Coskun et al., 2023; Girit Yildiz et al., 2023; Warshauer et al., 2021). Nevertheless, how teachers enact their noticing skills in classroom practice remains a critical but largely underexamined area. This gap is underscored by recent theoretical work emphasizing that noticing is highly contextual and shaped through classroom interactions, pointing to the need for research that examines noticing as enacted in instructional settings (van Es & Sherin, 2021, 2025; van Es et al., 2022). From this perspective, teachers need to create an interactive classroom environment that focuses on actively involving students in discussing and solving the tasks and on teachers' use of evidence of students' thinking. With this goal in mind, Jacobs and Empson (2016) identified core elements of practice that support a teaching approach focused on and responsive to students' mathematical thinking, conceptualizing it as responsive teaching.

Responsive teaching is rooted in noticing, a key professional practice that precedes the enactment of teachers' observable instructional moves (Jacobs et al., 2024). Drawing on this theoretical perspective and in response to long-standing evidence of students' conceptual difficulty with the notion of the mean (Jacobbe & Carvalho, 2011), this study examines how a teacher enacts her noticing skills in classroom practice when teaching this challenging topic. Specifically, the paper investigates how a mathematics teacher, whose professional noticing had been developed through a two-year online professional development program (PDP) (Çatman Aksoy et al., 2022; Tekin-Sitrava et al., 2025), transforms those skills into responsive teaching within a classroom environment. In doing so, this study responds to recent calls for research that investigates how teacher noticing is enacted in classroom contexts rather than in decontextualized or simulated settings (van Es et al., 2022; Weyers et al., 2024). Accordingly, the study is guided by the following research question.

How does an in-service mathematics teacher, whose noticing skills were developed through an online PDP, implement responsive teaching moves in a classroom setting while teaching the mean concept?

Responsive teaching

Responsive teaching focuses on identifying noteworthy issues related to children's understanding and how these issues can be addressed in response to their evolving understanding, which changes continuously during instruction (Jacobs & Empson, 2016). The relationship between noticing and responsive teaching has been conceptualized in two complementary views. Some researchers argue that noticing students' mathematical thinking serves as the foundation for teachers' responsive teaching practice. Attention to and interpretation of students' strategies increase the likelihood that subsequent teacher moves will be adaptive to support or extend students' understanding (Jacobs et al., 2011; Levin et al., 2013). Others emphasize that responsive teaching can create opportunities for students to elicit and render their thinking visible. From this perspective, noticing is not only a process that happens before a teacher responds but also a practice that can be enhanced through the use of specific teaching strategies. In this sense, teaching moves are not regarded as only teachers' responses to students but also provide teachers with opportunities to attend to students' strategies and interpret their understanding based on students' responses (Haverly et al., 2020). Taken together, these perspectives highlight the mutually reinforcing nature of noticing and responsive teaching: noticing equips teachers to be responsive, while responsive teaching simultaneously cultivates richer opportunities for noticing. Recent reviews and theoretical syntheses further reinforce this reciprocal relationship, positioning noticing and responsive teaching as dynamically interconnected practices in mathematics classrooms (Alsina et al., 2025).

Conceptualizing noticing students' mathematical thinking as foundational to responsive teaching, Jacobs et al. (2015) proposed a framework of teaching moves intended to create a learning environment responsive to students' mathematical thinking. In this framework, Jacobs et al. (2015) identified five teaching moves that demonstrate how teachers can conduct mathematics instruction to support and extend students' mathematical understanding. These moves are 1) ensuring the student makes sense of the problem (making sense move-MSM), 2) exploring details of the student's existing strategy (exploring details move-EDM), 3) encouraging the student to consider different strategies (encouraging strategies move-ESM), 4) connecting the student's thinking to symbolic notation (connecting move-CM), and 5) adjusting the problem to match the student's understanding (posing problems move-PPM).

The *making sense* move (MSM) entails engaging students in a discussion of the problem to clarify what they understand and identify areas of confusion. In this move, the teacher may ask students, "What do you think after reading the problem?", "What are the givens?", and "Could you explain the problem in your own words?". Jacobs et al. (2015) specified that this move (MSM) usually exists at the beginning of the interaction between the teacher and students. In the *exploring details* move (EDM), the teacher aims to explore the details of students' thinking, making them develop a solution strategy before building on their existing thinking. To do this, Jacobs et al. (2015) state four subcategories: posing general starter questions to the student, pressing the student for an explanation of specific parts of their problem-solving process, linking the student's representation and the problem context, and expanding the students' understanding of quantities used during problem-solving. The teacher carries out the *exploring details* move once students have solved the problem, initiating a discussion about their understanding and solution strategies through general or

content-specific questions. For this move (EDM), the questions that the teacher may ask are "Why did you draw such a representation for this problem context?" or "What do these numbers mean in this problem context?" The student's responses to these questions enable the teacher to enter the student's mind and capture the student's existing understanding, embedded in her strategy.

The aim of implementing *encouraging strategies* move (ESM) is to enable students to evaluate the problem from a different and broader perspective, or to compare their strategy with others' strategies. To employ the *encouraging strategies* move, the teacher may ask, "Is there another way to solve the problem?" or "Your friend solved the problem with another strategy. What do you think about his strategy?". With the *connecting* move (CM), Jacobs and Empson (2016) emphasized the importance of linking symbolic notation to a problem situation, and they asserted that teachers should encourage students to connect their thinking to symbolic notation. To encourage the students to connect their solution to the symbolic notation, the teacher may ask, "Can you write down the equation corresponding to your drawing?" or "Can you write the general rule of the situation using the table that you generated?". The remarkable point of *the connecting* move is that the teacher wants the students to construct tables, representations, equations, and expressions that they did not generate while solving the problem (Jacobs & Empson, 2016).

The last teaching move is *posing* a more comprehensive problem or adjusting the problem to the student's level of thinking (PPM). Jacobs et al. (2015) reported that students' struggles while solving the problem indicate that it may not be within their zone of understanding. On the other hand, students who can solve the problem need to meet a more challenging problem that aligns with their area of understanding. In both cases, the teachers should adjust the problem or pose a more comprehensive one to bring the problem within the student's zone of understanding. For example, the teacher may modify only the number or context of the problem to make it easier and pose more challenging problems to encourage the student to think more deeply and in more detail.

Beyond Jacobs and Empson's responsive teaching moves framework, Ellis et al. (2019) presented a framework of teacher moves particular to inquiry-oriented instruction, called Teacher Moves for Supporting Student Reasoning (TMSSR). This framework categorizes teacher moves according to teachers' roles in supporting student reasoning during classroom discourse (eliciting, responding, facilitating, and extending) and further differentiates sub-moves along a continuum reflecting their potential to support students' mathematical thinking. From this perspective, teaching moves vary in the extent to which they invite students to express and develop their own mathematical ideas, and thus treating sub-moves as analytically equivalent may obscure meaningful variation in teachers' responsiveness (Ellis et al., 2019). However, TMSSR conceptualizes teacher moves primarily as instructional actions that support student reasoning, without explicitly grounding these moves in teachers' noticing of students' mathematical thinking. In contrast, Jacobs et al.'s framework is explicitly anchored in teacher noticing, viewing teaching moves as emerging from teachers' in-the-moment attending to, interpreting, and responding to students' ideas. Because the present study is concerned not only with how teachers support students' reasoning but also with how noticing is transformed into instructional action, Jacobs et al.'s (2015) framework provides a stronger theoretical alignment with our research aims.

At the same time, Jacobs and Empson's framework conceptualizes responsive teaching moves at a relatively general level. As it includes only five teaching moves, it provides

limited insights into capturing variation within each move. To address this limitation, we refined their framework by introducing hierarchically organized sub-moves. This refinement is theoretically motivated by recent research emphasizing that teachers' instructional responses differ not only in type but also in their depth of engagement with students' mathematical thinking. Studies on teacher noticing and in-the-moment decision-making highlight that instructional moves vary in how much they elicit, interpret, and build on student reasoning, and that these qualitative differences shape the instructional trajectory of classroom discourse (Kooloos et al., 2022; Scheiner, 2021). Recent reviews of teacher noticing further stress the need for fine-grained analytic frameworks that capture the dynamic and context-sensitive nature of teachers' interpretive work in classrooms (Weyers et al., 2024). From this perspective, organizing teaching moves into sub-moves enabled us to examine not only *which* moves were used, but also *how* they were enacted in the classroom, particularly how the teacher orchestrated whole-class discussion, an aspect that is not explicitly captured in Jacobs and Empson's (2016) framework. In doing so, we address recent calls for research examining how such frameworks are implemented in classroom (Haverly et al., 2020; Weyers et al., 2024), with particular attention to the concept of mean as a domain-specific context.

Domain specificity: the mean

A literature review by Jacobbe and Carvalho (2011) indicated that both students and teachers have a lack of conceptual understanding, particularly of the mean. More specifically, the studies (Enisoglu & Isiksal-Bostan, 2017; Karaca & Ay, 2024; Watson & Moritz, 2000) underline the issue of students heavily relying on procedural methods, such as the add-and-divide algorithm, in calculating the mean without a deep understanding of its conceptual base, e.g., mean as a balance point and mean as a representative value. A limited procedure-based understanding of the mean can hinder the application of the concept meaningfully in statistical reasoning and real-world contexts.

The concept of the mean has been part of the school mathematics curriculum long before the recent movements around the world to increase the emphasis on statistics in K-12 education (Jacobbe & Carvalho, 2011). We encounter the mean (or average) in many situations in our daily lives, as it is a widely used statistical measure in various fields, including economics, sports, media, meteorology, and so on. Therefore, understanding the mean is essential for statistical literacy to make data-driven decisions (Gal, 1995).

The PreK–12 Guidelines for Assessment and Instruction in Statistics Education II: A Framework for Statistics and Data Science Education (GAISE II) emphasizes developing students' data and statistical literacy through three developmental levels (A, B, and C) at the school level (Bargagliotti et al., 2020). At Level A, students are introduced to the statistical problem-solving process by working with categorical and numerical data with a focus on understanding the key features of distributions for quantitative variables (center, variability, shape). At Level B, students analyze more complex datasets, applying statistical measures such as the mean, median, and spread to identify patterns and draw conclusions from the data. At Level C, students engage in formal statistical inference and modeling, often using computer simulations to explore variability and uncertainty in data. The report describes two methods for introducing the concept of the arithmetic mean as students prog-

ress through levels A and B. The fair share (also known as leveling off) interpretation of the mean, introduced at level A, refers to distributing the total of data points equally across all cases. This equal share value represents the mean of the distribution. Then, at level B, students are expected to progress to thinking about the mean as the balance point. In this interpretation, the mean is the point on a number line around which the distribution is balanced. In other words, the total distance between the mean and the data values below the mean value is equal to the sum of the distances between the mean value and the data values above the mean. These two approaches provide a rich conceptual understanding of the mean for students, as the former has a connection to the algorithm of the mean, and the latter elucidates that the distributions with different variability can have the same mean (See van de Walle et al., 2019). Such conceptual understanding is crucial for developing students' correct use of the mean algorithm in solving problems and their understanding of the mean concept as a statistical tool for summarizing and comparing data sets (Cai, 1998).

In classrooms, however, teachers appear to confront certain difficulties in teaching and learning statistics, including students' tendency to understand the mean only procedurally and their struggle to interpret it (Koparan, 2015). Hence, it is essential to develop teachers' knowledge not only of statistical concepts but also of noticing students' statistical thinking to support the development of students' conceptual understanding (Pfannkuch & Ben-Zvi, 2011). A recent study by Shin (2020) revealed that pre-service mathematics teachers who completed statistics and pedagogy courses performed better in noticing differences in students' reasoning about the mean concerning the two interpretations mentioned above (fair share and balance point) than in students' understanding of variability. However, none of them could customize their instructional decisions for individual students by taking their solutions into account and envisioning the next strategies based on the level of understanding of each student. A small number of studies with mathematics teachers also indicated similar difficulties, particularly in responding to a specific misunderstanding in student thinking with appropriate intervention strategies in teaching statistics (Gökce&Kazak, 2017; Callingham & Watson, 2011; Watson et al., 2008). These research findings suggest the need to shift teaching practice from imposing teachers' knowledge and understanding to being responsive to students' understanding when teaching statistics. To address this need, we examined how a teacher whose noticing skills had been developed through a two-year online professional development program (PDP), responded to her students' statistical thinking in a classroom setting using a responsive teaching framework refined with hierarchically organized sub-moves.

Methods

Research design and participants

This research is part of a 2-year professional development design experiment (Zawojewski et al., 2008) focusing on developing mathematics teachers' noticing and responsive teaching skills. The paper specifically examines how a mathematics teacher, following participation in an online PDP, enacted responsive teaching moves in the classroom while teaching the concept of mean.

In the larger study, 31 middle school mathematics teachers (Grades 5–8) with less than 15 years of experience working in a public middle school participated in the online PDP. In this paper, we mainly focus on one teacher, Alicia, through an in-depth exploration of her in-the-moment classroom responses following her involvement in the online PDP. Alicia was selected among the participants because her noticing skill was graded as very low in the pre-test administered at the beginning of the study and improved throughout her participation in the online PDP (see Çatman Aksoy et al.,2022) for the development of Alicia's noticing skills). She was among the most attentive teachers, consistently articulating her interpretations of student solutions, posing challenging questions, and commenting on other teachers' reflections throughout the PDP implementation. In addition, Alicia volunteered to have her classroom teaching observed by the researchers following her participation in the online PDP. Based on these criteria, she was therefore selected as a strong candidate for further investigation of her classroom practice.

Professional development program and research process

The Professional Development Program (PDP) consisted of an online teacher learning platform in which teachers collaborated via Microsoft Teams. The PDP was designed to provide a learning environment that enabled teachers to share and reflect on their interpretations of students' understanding. To achieve this aim, the related literature on improving noticing skills and responsive teaching in mathematics education was first analyzed. In addition, a need analysis for the online PDP that supports teacher development of noticing skills and responsive teaching, and the related literature on teaching and learning mathematical topics, was conducted. Then, the content of the online PDP was prepared by the design research team composed of mathematics education experts (for details, see Tekin-Sitrava et al., 2025) in line with tentative design principles (Bakker, 2018). During the longitudinal project, 17 scenarios involving written scenarios and videos focusing on students' solution strategies related to numbers, algebra, geometry, measurement, data analysis, and probability (for details, please see Tekin-Sitrava et al., 2025) were posted over two years, and teachers were expected to share their understanding/ideas or comment on other teachers' responses.

Before the online PDP discussions, the researchers administered a pre-test to assess the participating teachers' noticing skills, as depicted in Fig. 1. Subsequently, pre-interviews were conducted with each participant to clarify their written responses and to gain an in-depth understanding of their noticing skills of students' thinking. Then, the teachers engaged

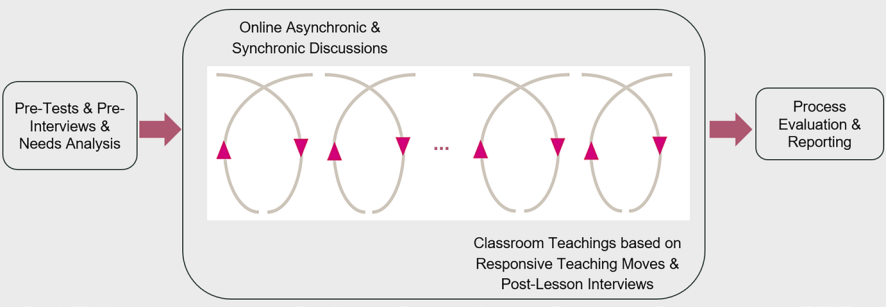


Fig. 1 The research process

in online discussions on various students' solution strategies, as shown in the first two cycles of Fig. 1. These discussions were often asynchronous, with occasional synchronous sessions when needed. Following these discussions, volunteer teachers implemented some of the problems or activities in their classrooms. This paper focuses on Alicia's teaching moves during two tasks (Basketball Problem and Balance Point Activity) related to the mean concept from the last two cycles in Fig. 1. These tasks, posted on the online PDP for 4 weeks, are described in detail below.

Scenario 1 (Basketball Problem). The participating teachers were given the basketball problem and two student strategies, typically observed in data comparison tasks, to assess their noticing skills in the pre-test. Since most teachers struggled to identify these strategies, the research team decided to discuss the problem on the platform. Adapted by Gökce (2019) from McGatha et al. (2002), the problem required students to compare the scores of two players across ten games and decide which player the coach should select. Solving the problem required analyzing the data using measures of central tendency and variability and interpreting the results in context. As shown in Fig. 2, Student 1's (S1) reasoning was partially aligned with the intended statistical ideas in the task, since S1 selected Barış (the second player) by considering average scores but did not attend to variability. Student 2's (S2) reasoning was not aligned with these ideas, as S2 focused only on certain scores and disregarded the rest.

The research team provided the strategies of S1 and S2 along with guiding questions on attending to student solutions, interpreting student understanding, and responding to students' thinking. Participating teachers were expected to comment on these questions and engage in asynchronous discussions by responding to one another's reflections. To deepen the discussion, the research team posed prompts, such as "Do you agree with Participant X's

Basketball problem

The coach will select one player from the school's basketball team to play in the all-star tournament. Below is a listing of points scored by the two candidates, which the coach thinks choosing, for the last ten games of the season.

Players	Points Scored in Games									
Arda	8	14	11	13	15	12	10	10	9	13
Barış	4	15	8	17	18	9	19	6	18	5

Your students, Student 1 and Student 2, responded as follows:

Student 1 (S1)'s response: He should choose Barış because while Arda's average is 11.5, Barış's average is 11.9.

Student 2 (S2)'s response: He should choose Barış because he scored 15 or more in half of the games that he played.

Guiding questions

a. Explain in detail how the student(s) solved the given problem. Do you think that students' reasoning is correct? Why?

b. By focusing on the student(s)'s answers, explain in detail what you noticed about the student(s)' understanding.

c. Suppose you decide in class how to enhance the student(s)'s understanding through the next instruction steps.

Based on the student(s)'s understanding of the statistical concepts, how would you proceed in your next steps?

(c1) What questions would you ask your student to ensure that s/he understands the problem correctly?

(c2) What questions would you ask your student to explore his/her solution strategy?

(c3) What questions would you ask to guide your student to develop different problem-solving strategies for this question?

(c4) What questions would you ask your student for him to express his/her thinking/solution in the given problem situation symbolically?

(c5) What would you do in the next step to support your student's mathematical understanding or to make them think more comprehensively? For example, what questions would you ask, and what explanations would you make?

Fig. 2 Scenario 1: Basketball problem and the guiding questions for noticing

explanation of the student's response? Why or why not?" These discussions enabled teachers to consider additional student strategies, including the use of median or mode in data comparison or considering the order of data. After two weeks of asynchronous discussion, the whole group engaged in a synchronous session, where the research team raised evaluative questions such as, "Teacher A thinks that … Do you agree? Why?" to highlight key points and synthesize major themes during the asynchronous discussion.

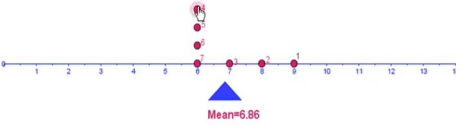
Scenario 2 (Balance Point Activity). The second scenario, the *Balance Point Activity*, assigned on the online platform, also focused on the concepts of mean and variability, but from a different perspective (see Fig. 3). The activity presented in Online Appendix A emphasized the balance point interpretation of mean and highlighted variability by comparing different distributions with the same mean. Considering the limited conceptual understanding of the mean among both students and teachers, and the tendency to overemphasize procedural algorithms in classrooms (Jacobbe & Carvalho, 2011), the research team selected this activity to highlight the conceptual basis of the mean.

To illustrate student thinking and stimulate teacher discussion with the same guiding questions shown in Fig. 2, the research team first conducted an interview with a sixth-grade student using this activity. Video excerpts illustrating the student's reasoning were subsequently shared on the online platform. An example excerpt is shown in Fig. 3.

The activity required students to determine the prices of seven toys with a mean price of 6 euros, with each successive question introducing conditions of increasing difficulty, supported by a GeoGebra application (https://www.geogebra.org/m/qbvtsazx) that represented the seven toys as pink dots on a number line (Fig. 3). The numbers on the dots represented different toys, while the numbers on the line showed their prices. The mean value automati-

Balance point activity
Ali bought seven small toys for himself, his two siblings, and four cousins. When he asked his father about the prices, he told him that the average price was 6 euros. Ali now wonders what the individual prices of the toys could be. Let's help him find possible prices.
(See Appendix A for **questions 1-5**)

6) Using the information we have, can you predict the prices of the seven toys? Show the prices of the toys on the number line below. Explain your reasoning.
Your student responded to this question as follows:
The student's response: When I set the prices of three toys higher than 6 euros, the mean price of all seven toys increased.



To reduce the mean, I set the prices of the other three toys below 6 euros. As a result, the overall mean returned to 6.

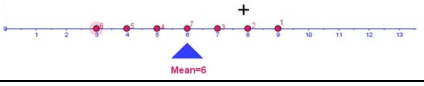


Fig. 3 Scenario 2: balance point activity

cally changed with the positions of the dots, and only whole-number prices were allowed in the app. Initially, the price of each toy was set at 6 euros. Using the GeoGebra app, the balance point activity required to construct distributions with the same mean but different variability according to the given conditions in the questions. In the final step, the toy prices are recorded in a table, and the total distance of values below and above the mean was observed, which illustrates the mean as the balance point of the data.

Data collection tools

After the discussions on the platform, Alicia implemented the balance point activity and the basketball problem in two sixth-grade classrooms. Although the basketball problem had been discussed first on the platform, Alicia chose to begin with the balance point activity in her teaching. This decision reflected the view that a solid understanding of mean and variability is necessary before undertaking data comparison tasks, such as the basketball problem. This was aligned with the conceptual understanding of the mean as a tool to summarize and compare data sets in decision making (Cai, 1998; Gal, 1995). Alicia's first implementation lasted about 80 min and involved three main parts: (a) a whole-class warm-up where the students engaged in a preparatory activity on 'leveling off' interpretation of the mean (10 min); (b) a whole-class discussion of the individual solutions to the first three questions of the activity (40 min); (c) a whole-class discussion of the last three questions aimed at developing the balance point interpretation of the mean and constructing various distributions based on this idea (30 min). In her second implementation, lasting 35 min, Alicia facilitated a whole-class discussion of individual solutions to the basketball problem and the follow-up problems she had prepared beforehand (Online Appendix B).

Following her implementations, the researchers interviewed Alicia to clarify her teaching moves. The interviews also aimed to explore the teacher's reflections on the role of the online PDP *in relation to* her noticing and instructional decisions. Prior to the interview, Alicia wanted to review the videos quickly to refresh her memory on her implementations and noted particular moments she wanted to discuss. The researchers also independently analyzed her implementations in advance to identify the moments for further exploration. Drawing on both Alicia's and our own notes, we asked questions to elicit her reflections on her moves and her noticing behind them, such as: "Did you expect such an answer? Did you notice … in class or while you were watching the video later? What did you think this student meant by … during your implementation? What was the aim of your question/representation?" Both the classroom implementations and the interviews were video-recorded, and field notes were maintained throughout the process. For this study, Alicia's classroom video recordings served as primary data sources, supplemented by post-lesson interview videos and field notes.

Data analysis

To uncover how a mathematics teacher, after participating in an online PDP, implemented responsive teaching moves in a classroom setting, we analyzed classroom videos using Jacobs et al.'s (2015) responsive teaching framework refined with sub-moves. More specifically, Işıksal Bostan et al. (2023) proposed organizing sub-moves to capture meaningful variation within each move. Table 1 illustrates the modified version of the framework.

Table 1 The refined version of the framework for teaching moves

Teaching Moves	Making Sense MSM	Exploring Details EDM	Encouraging Strategies ESM	Connecting CM	Posing Problem PPM
Less demanding	Repeating	Clarifying	Suggesting alternative solutions	Bridging	Just posing an idea/ supporting another's idea/ problem
	Re-voicing	Re-voicing			
Sub-moves	Encouraging student re-voicing	Orienting (or/ with Invitation)			Posing a problem with number/context modification
	Encouraging students' understanding through questioning	Connecting (or/ with Invitation)	Suggesting alternative solutions with mathematical ideas		Posing a/an problem/idea for an extension
	Cuing	Prompting error correction			
More demanding					

This framework not only deepens our understanding of responsive teaching moves but also highlights variations in how teachers enact these moves in practice. In particular, sub-moves are differentiated by the extent to which teachers engage with students' mathematical thinking. For example, related to making sense move, less demanding sub-moves (e.g., repeating or re-voicing) primarily serve to clarify or surface students' thinking, whereas more demanding sub-moves (e.g., cuing) require deeper engagement with students' reasoning and more active orchestration of classroom discourse. Explanations of each teaching move, including the sub-moves and the corresponding instances, are provided in Online Appendix C.

Moreover, a teaching move was conceptualized as a deliberate instructional action performed in response to students' thinking in line with Jacobs et al. (2015). However, our unit of analysis was not the teaching moves themselves but rather a purposeful instructional episode—a segment of classroom interaction where the teacher maintained a consistent instructional focus. Depending on the instructional situation, a unit could consist of a single teacher action or extend across a sequence of comments, questions, or explanations, as long as these actions were directed toward the same instructional purpose. Accordingly, multiple teaching moves could co-occur within each unit, as long as they served the same overarching instructional purpose.

To identify these units, we first segmented the classroom discourse at conceptual breaks, defined as distinct shifts in focus or changes in topic (Jacobs & Morita, 2002). This segmentation constituted an initial analytical step conducted before coding teaching moves. Conceptual breaks were identified at points where teacher shifted from an understanding of a representation to responding to a student's solution. In the following example from the balance point activity, although Alicia had introduced the number line representation earlier in the balance point activity, the students struggled to use it when they moved on to the main questions. In response, Alicia devoted an extended episode again to support students' understanding of the representation (10:45–14.22 in Fig. 4), which was essential for grasping the balance point interpretation. Therefore, this episode was segmented into a separate instructional unit, distinct from earlier units (e.g., 4:00–7:00 or 08:00–10:00 in Fig. 4), during which she asked questions in response to a student's solution. In this representation-focused

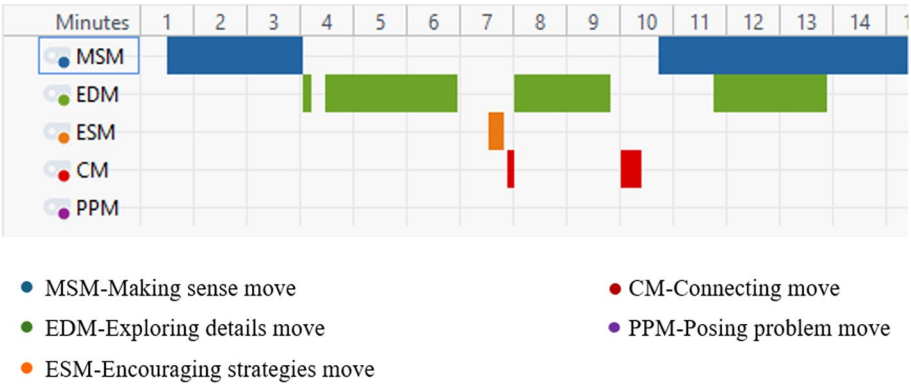


Fig. 4 An example codeline from the first implementation

unit, Alicia used both MSM and EDM to pursue the same overarching purpose—supporting students' understanding of the number line representation. Identifying purpose-based instructional units enabled us to examine how various teaching moves worked together over time to meet specific instructional goals, rather than treating them as isolated actions.

At the beginning of the data analysis, the two authors watched the classroom videos several times to become familiar with the data. They then independently segmented them into instructional units based on observable shifts in instructional focus. The resulting segmentations were compared, and any differences were discussed and refined until agreement was reached. Once the units were finalized, the same two authors independently coded teaching moves using the teaching move descriptors, consulting the interview recordings and field notes as needed. Interrater reliability for teaching moves coding was initially 71% and was improved through multiple rounds of discussion; check-coding was used until a full consensus was reached.

After coding was completed, the codeline feature of MAXQDA, the qualitative data analysis software employed (VERBI Software, 2021), was used to summarize the teaching moves used during classroom implementations. This feature created visuals that display the coded segments in a sequential view, providing more details about how our teacher implemented the teaching moves by indicating their order and frequency. Figure 4 presented a part of Alicia's teaching moves from her first implementation.

The leftmost column in this figure showed all five teaching moves, and the top row displayed each minute during the implementation. Hence, the intersection cell gives information on which move Alicia applied and when. For example, Fig. 4 showed that she applied the EDM between the fourth and sixth minutes of her teaching. This was written in the findings section as EDM in 4.00–6.00 in Fig. 4.

Lastly, the present study is concerned not only with how Alicia implemented responsive teaching moves but also with how her noticing was used in these moves during instruction. Accordingly, our analyses of Alicia's noticing during instruction were grounded primarily on close examination of her in-the-moment instructional actions and questioning during classroom interactions. This analytic stance is aligned with the third approach to studying teacher noticing identified by Sherin et al. (2011), in which researchers infer teachers' noticing on the basis of their visible instructional moves. When available, evidence from

post-lesson interviews was used to support and elaborate these inferences by providing the teacher's explicit articulations of her noticing.

Findings

This study investigates a teacher's responsive teaching moves enacted in the classroom while teaching the concept of mean following her participation in an online PDP. The findings focus on Alicia's teaching moves during the balance point activity and the basketball problem, respectively.

The balance point activity

During the post-lesson interviews, Alicia stated that the Balance Point Activity was the first time she focused on the balance point interpretation of the mean. She began by asking students to read the introduction, which stated that the average price of the seven toys was six euros and that the goal was to determine the price of each toy. After eliciting initial ideas from several students about the givens and the task goals to ensure that students understand the problem situation (Encouraging students' understanding through questioning-MSM), the teacher proceeded with a preparatory activity before moving to the main questions. In this activity, she asked students to form a model of the given situation using the unit cubes. That is, assuming each cube represents 1 euro, Alicia asked students to model the prices of the seven toys so that their average would be 6 euros. S1 formed a model, where he considered all toy prices to be 6 euros. Alicia used this model to transition to the main questions, explaining that although all toy prices were equal in S1's model, this was not the case in the current activity. She emphasized that in the balance point activity, the goal was to determine the possible actual toy prices consistent with the given average price.

This activity was based on the leveling-off interpretation of the mean, which Alicia mentioned in her previous lessons to teach the mean algorithm. The teacher explained that she chose to begin with this activity because she intended to connect this interpretation with the balance point interpretation later in the lesson. Since Alicia's purpose was to activate the students' prior knowledge, her move was identified as cuing in the making sense move (MSM) category.

After completing the preparatory activity, the teacher moved on to the main questions. However, the students still had difficulty understanding the pink dots' numbers in the questions, as exemplified in the following episode (15:40–17:27 in Fig. 7).

Episode 1

1 S3: (She finds the price of the second toy as 4 using the mean algorithm.)
2 T: Can you show it here (on the number line) where the second toy corresponds to?
3 S3: (She shows the dots with the number 4, representing the fourth toy.)
4 T: Do these numbers (the numbers on the dots) show the prices? What did we say about the relationship between each dot and the toys?
5 S4: It should come to 4. Because those dots are the sort of toys, right?
6 T: What are these? (She shows the numbers on the dots.)
7 S6: The order of the toys.

8 T: Yes. Which toy is absent here? What are we asked here?
9 S3: The baby.
10 T: Yes, but which toy?
11 S3: Isn't it the second toy? (She points to the toy with the number 6).
12 T: Is this the second toy?
13 S3: Oh, okay, they didn't show the second toy.
14 T: Yes. We will show it. What did you find as the price of your toy?
15 Students: 4
16 T: You said it was four euros. Where can you represent 4 euros here on the number line? How do you show the prices of toys using the numbers in this model?
17 S3: (She points to 4 on the number line).

When S3 showed the dot with "4" on the fourth toy rather than at 4 euros on the number line (Line 3), the teacher noticed the student's confusion about the toy number and toy price (the issue with the representation in the activity—the number line, which is essential for grasping the balance point interpretation). Then, through questions in lines 4–16, the teacher guided students to articulate what the dots represent and helped her realize her incorrect understanding (Prompting-EDM). In this way, she had a chance to remind the students of the goal of the question and what the different numbers in the given representation refer to throughout the dialogue, which indicates the sub-move encouraging students' understanding through questioning (MSM). Hence, the teacher performed the EDM and MSM simultaneously in this example.

The second question in the balance point activity requires students to find the prices of two toys that exceed the mean. One of the students in the class, S5, tried to solve this question using the mean algorithm. That is, she tried to find the prices of the missing toys—the third and fifth ones—after finding the total price. However, she thought the total price was 36 instead of 42, and the dialogue below occurred between the teacher and S5 (28:25–29:33 in Fig. 7).

Episode 2

18 T: The sixth toy is 2 euros (She places the sixth toy to 2 on the number line). You can place the third and fifth ones here as greater than six. What did you think?
19 S5: As you said at the beginning, the third and fifth toys will be greater than six. I thought the total would be 26 if we didn't include the third and fifth ones.
20 T: How did you find 26?
21 S5: 6,12,18,24,26. (The student shows the toys without the third and fifth ones on the number line and adds their prices.)
22 T: Yes, if we do not consider the third and fifth ones, we have four toys remaining. Since each is 6 euros, we get 24. Since the sixth toy is 2 euros, we have 26.
23 S5: Firstly, I thought of 36 as a multiple of 6. Since it was said to be greater than 6, I tried 36. It should be 10 more for the total to be 36.
24 T: Why did you think the multiples of 6?
25 S5: I do not know.

26 T: Why do you think that she might have said the multiples of 6? (She asks the class.)

When S5 introduced her idea of considering the total of 26 (Line 19), the teacher noticed the emerging student strategy and asked her to clarify it, indicating that she applied the exploring details move (EDM). S5 responded that she computed the total price of the five toys as 26 euros, thinking of the price of the second toy as 2 euros and the others as 6 euros (Line 21), making her strategy visible to Alicia. S5's explanation of her strategy provided an opportunity for Alicia to attend to S5's thinking, which she explicitly articulated in Line 22. Drawing on her attention, she immediately made the instructional decision to make the strategy visible to the whole class by re-voicing S5's ideas with her statements (EDM). Later, to explore the details of her thinking in considering the total of 36 as a multiple of 6 (Line 23), she asked why she had used multiples of 6 (Line 24), a question that reflected the connecting sub-move in the EDM. Since S5 was unable to explain her reasoning, Alicia invited the class to think about S5's idea (Connection with invitation-EDM) (Line 26). Thus, she aimed for other students to comment on the details in the mathematical representation of S5's solution.

In the following parts of the lesson, students usually solved the questions using the mean algorithm, as S5 attempted above. After the students solved the third question similarly, the teacher asked a different question: "We solved the questions up to now using the mean formula. So, we always started with the total prices of the toys. Can we also use the positions of the numbers on the number line?". The balance point interpretation of the mean is about the distances of the data values from the mean value. However, the positions of the data did not attract the students' attention in the first three questions. Accordingly, the teacher first clarified the solution process that the students had followed up to that point by attending to their strategy and, through her noticing, making their frequent use of the mean formula visible to the class. That is, she directed students down a different but specific path because she wanted them to examine the positions of the data on the number line and relate them to the mean. In other words, she suggested an alternative solution with mathematical ideas to encourage the class to consider other strategies (Funneling-ESM).

Subsequently, the teacher invited the class to examine the distances in the previous questions and record them in the table in Fig. 5. As the table was filled with the whole class, the students noticed that the total distance between the mean value and the data values below the mean is equal to the sum of the distances between the mean value and the data values above the mean, as the discussion below showed (3:15–7:00 in Fig. 8):

Episode 3

27 S6: I realized that they (the distances) are always equal.
28 T: That is exactly what I was going to ask. What does it mean that they are equal? I mean, is it a coincidence? Or does it have something to do with average?
29 S3: Does it show how much more or less than 6?
30 T: Let's take the first situation (The first row in the table) and show it on our number line in GeoGebra. (S4 and the teacher places the toys.)
31 T: What did we find when we filled the table?
32 S4: 2

Prices of the Toys							Total Distances to the Mean Price	
1	2	3	4	5	6	7	Total distance of toys whose prices are greater than 6 euros	Total distance of toys whose prices are less than 6 euros
8	4	6	6	6	6	6	2	2
6	6	8	6	8	2	6	4	4
6	6	7	5	9	3	6	4	4
7	3	1	6	6	6	13	8	8

Fig. 5 The table filled in the balance point activity

33 T: We found the distances to be 2, right? How does it relate to the mean? S3 said that they are the same.
34 S4: Should we show the second situation in the table?
35 T: Let's interpret the distances in this case. Let's interpret the values in the table.
36 S4: Both are two units away.
37 T: Does it matter which side the two are on?
38 S7: No.
39 T: You say that the side does not matter. Then, do you say that if we take this (The toy on the left side of the mean) and put it to this side (The right side of the mean.), do we again obtain 6 as the mean? (Some students say yes, and some say no.)
40 T: Maybe, I misstated. Does it matter that these two toys are 2 units more and 2 units less than six?
41 S4: The mean will decrease if it is less than six. If it is more than 6, then the mean increases. However, since they are equal, they balance.
42 T: What do we mean by equal?
43 S4: I mean 2 units both to the right and left side—no more, no less.
44 T: Yes, the distances are the same on both the right and left sides.

After S6's insight about equal distances (Line 27), the teacher immediately noticed the idea of symmetry around the mean as conceptually important in her response (Line 28). Alicia suggested examining one of the situations more closely to draw the students' attention to the position of the data relative to the mean value (Line 30). Her follow-up questions regarding the distances of data points from the mean (Lines 31–42) appeared to guide students to develop emerging balance point interpretation of the mean, which indicated bridging the student's thinking to mathematical concepts move—CM.

Towards the end of her implementation, Alicia asked the class to determine the price of seven toys without providing any specific prices, which allowed them to obtain different distributions with the same mean value. In other words, she posed students with a broad, thought-provoking problem within the boundaries of the initial problem to reinforce the idea

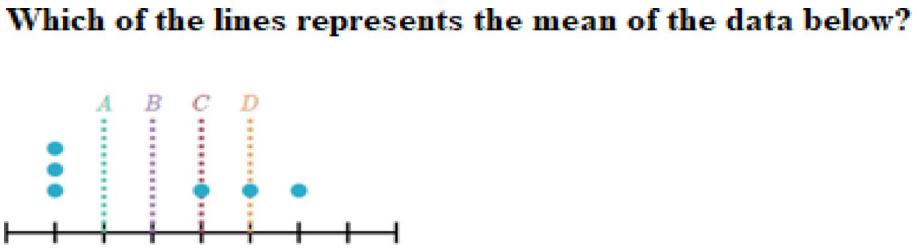


Fig. 6 The follow-up problem

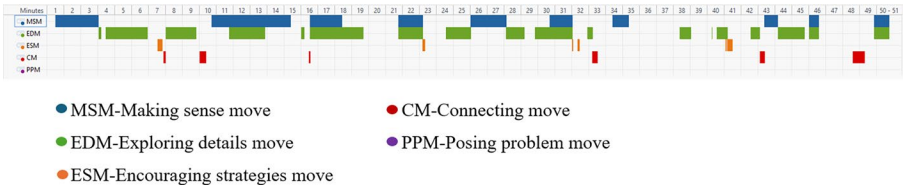


Fig. 7 Teaching moves in the first fifty minutes of the balance point activity

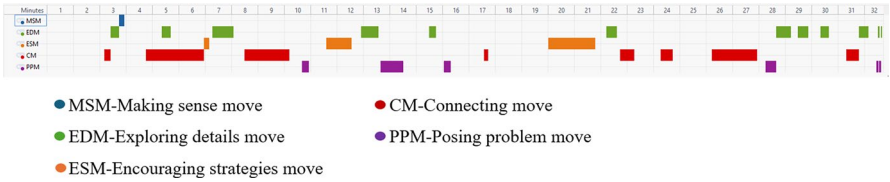


Fig. 8 Teaching moves in the last thirty minutes of the balance point activity

that the mean is the balance point of the data set (PPM). For instance, S3 obtained a symmetrical distribution using the balance point interpretation. After Alicia noticed her distribution, she again bridged this distribution with the mean once more by stating, "This example visually indicates that the mean is the balance point of the data." (CM).

The figure above illustrates another follow-up problem that Alicia posed to the class at the end of the lesson. Instead of providing the mean as in the original activity, this problem presents the data and asks which line in Fig. 6 corresponds to the mean. The absence of data values forces students to use the balance point interpretation, thereby broadening their understanding of this interpretation, which indicated that she used posing a/an problem/idea for extension sub-move in the scope of the PPM once more. The following figures summarize Alicia's teaching moves during the balance point activity. While Fig. 7 shows Alicia's teaching moves in the first fifty minutes of the activity, her teaching moves in the remaining parts are presented in Fig. 8.

Figure 7 showed that Alicia employed all responsive teaching moves in various orders. Moreover, although she conducted whole-class discussions throughout her teaching, the frequency and duration of the moves varied. For instance, she mostly used the MSM and EDM, often simultaneously, in the first 50 min of the activity. (see the blue—MSM—and green boxes—EDM—in Fig. 7). That is to say, while she tried to explore the details of the

students'solutions—the EDM—she needed to return to the MSM to ensure that the students made sense of the problem and the representations in the questions (e.g., Episode 1). In addition, even though Alicia began to perform the ESM and CM at the beginning of the activity, their frequency was significantly lower compared to the EDM and MSM.

On the other hand, the CM was the most frequently used one in the remaining 30 min (see the red boxes in Fig. 8). She used this move to connect the students' thinking in the previous questions with the balance point interpretation of the mean. Also, it was sometimes used in combination with the EDM (e.g., Episode 3). Similar to the beginning of the activity, the frequency of the ESM (orange boxes) was limited. Lastly, Alicia applied the PPM in the second half of the lesson and attempted to enhance her students' level of thinking by posing related problems.

Following the balance point activity, Alicia focused on the range concept and the data comparison tasks, where students needed to use the mean and/or range. The last of these tasks was the basketball problem, which she implemented in a different sixth-grade classroom (approximately 40 min). The teaching moves performed in the basketball problem are explained in the next section.

The basketball problem

The teacher started her lesson by asking students to read and explain the basketball problem. After S1 summarized the problem situation, the students immediately began sharing their decisions on which player they had chosen for the final game. Alicia let some of them explain their decisions without focusing too much on a decision. A few students suggested deciding between the two players by examining the mean or range of the given data, and Alicia noted these values on the board. Upon the teacher asking for the other ideas in general terms, which indicated that she suggested an alternative solution in the scope of the ESM, another student, S3, commented on the variation in the scores, and the following dialogue shows how the teacher applied the exploring details move after the encouraging strategies move—an example of an ESM and EDM cycle—(08:00–08:47 in Fig. 11).

Episode 4

61. S3: I would choose Arda because he is going stable.
62. T: Hmm. His data?
63. S3: Yes.
64. T: How did you decide? Did you look at the scores or the mean and range here? (She shows the mean and range values on the board.)
65. S3: I looked at the scores.
66. T: He looked at the scores.
67. S3: Barış's scores decreased and then increased. Then, they dropped a little more. (He shows the decrease and increase in Barış's scores on the table with his hand.)
68. T: And you want it to be consistent?
69. S3: Yes.
70. T: Why?
71. S3: (He thinks a little.) I have no answer.
72. T: No answer. Ok. Let's hear the other opinions.

When S3 introduced her idea of "going stable" (Line 61), Alicia noticed S3's use of informal notion of variability and first asked him to explain the basis of his decision (Line 64). Upon S3's response in line 67, Alicia interpreted that S3 was reasoning about the fluctuations in scores over time (notion of consistency, Line 68). By asking why he wanted the data to be consistent (Line 70), she oriented him to reconsider his solution (EDM) but S3 could not articulate why consistency matters. Then, Alicia asked for other ideas to support finding alternative solutions (ESM).

In the following parts of the lesson, many students preferred Arda for the season's final game. Some of their rationales were about the range of Barış's scores being greater than the range of Arda's scores or that the scores of Arda were more consistent than those of Barış. Later, the teacher introduced a new criterion to choose players, which was discussed in the online PDP and the following dialogue occurred (9:45–10:25 in Fig. 11):

Episode 5

73 T: If we look at the scores of both players, who scored the highest? Arda or Barış? Who has the highest score?
74 S3: Barış. He had 19.
75 T: Barış had a score of 19. Did this score not influence your decision? Can it affect your opinion? (She waits for a few seconds.) We mostly preferred Arda because we can trust him; he is consistent, and, as S4 said, we can easily predict his next score. That's why we can choose Arda, I agree. I think our choices are consistent here. (Another student again presented her decision of Arda by giving similar reasons.)
76 S5: If we take a risk, we can choose Barış.
77 T: Hımm. If we want to take a risk…Why?
78 S5: Because Barış's scores are getting up and down. We cannot know his next move.
79 T: Yes, we cannot know his next score. Let's make a comment thinking about some single games. For example, the fifth game. Who is more successful, Arda or Barış? Class: Barış.
80 T: Then, if the coach had chosen Barış in the fifth game, then he might have been very happy that he preferred him. However, as you said, if he had chosen him in the first game, Barış had scored 4 points in that game. Then, the coach might have been disappointed.

In Lines 73 and 75, leading her students to consider Barış's highest score within the given data set, she funneled them to a specific path. Especially in Line 75, Alicia attended to students' ideas to clarify the reasons for choosing Arda and to make students contemplate their ideas deeply and to reflect on whether the highest score affect their decision. Thus, her attending to students' ideas and the questions indicated that she used the ESM to encourage the class to also consider the player Barış for the final game. Upon S5's response in line 76, Alicia asked her to articulate her response about taking a risk (Line 77). With this move, she oriented S5 to explain her decision (EDM), which illustrates another example of an ESM and EDM cycle. As mentioned in the method section, the teachers were expected to notice such a student strategy—to decide based on the highest data values—in the pre-test. Moreover, they discussed their opinions about this strategy in the online PDP. Although our

teacher believed this strategy was completely incorrect in the pre-test and at the beginning of the online asynchronous discussion (see Çatman Aksoy et al., 2022), about the development of Alicia's noticing skills), it was interesting to observe that she guided her students to this strategy during her implementation.

Moreover, during the asynchronous discussion, a teacher presented the idea of more and less risky decisions. This idea was further elaborated during the synchronous discussion led by the facilitator, and the importance of the rationale behind the decisions was discussed with the teachers. Alicia emphasized this idea throughout her implementation, starting with the dialogue above. She also commented on this in the post-lesson interview. She said, "My aim in guiding my students to player Barış was to make them realize this: What is the reason behind our decisions? We can also choose the player Barış. I wanted my students to feel that if an appropriate rationale was provided correct rationale is presented, selecting Barış could also be considered valid."

Towards the middle of the lesson, many students seemed to agree on two opinions. While the first one was about the importance of the rationale, as explained in the previous paragraph, the other one was about examining the mean and range of the given data to make a better decision. At that time, the teacher asked the class, "Can we use any other method with or without using the mean and range?" (Suggesting alternative solutions-ESM). Upon a student suggesting that the data could be displayed in a bar graph, the teacher showed the bar graph of each player's scores on the board and asked the students to interpret it. During the interview, Alicia stated that "When numerical data are given, the students tend to reach a conclusion by calculating the average, whereas when the data is on the graph, they may prefer to focus on specific data points, such as the extremes." She explained that she had noticed this tendency during the instruction and adjusted her moves accordingly by asking the students to explain the reason behind the change in their reasoning. After the bar graph, the teacher preferred shifting the students' focus from the mean to the range. To achieve this, she tried to focus students' attention on the balance point interpretation of the mean. In other words, she supported students in finding alternate solutions within the context of particular mathematical concepts, asking whether they could use the other meaning of the mean (Focusing-ESM). Following a few students' ideas about displaying the data on a number line, the teacher showed the dot plots in Fig. 9 and asked the class if their opinions had

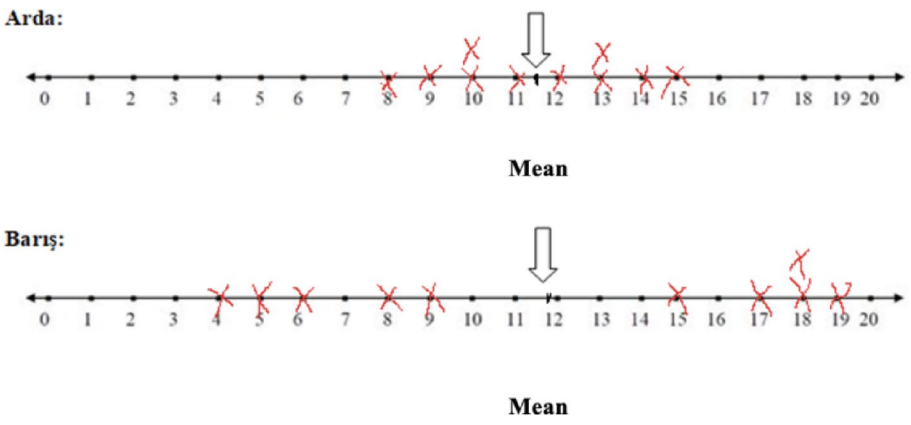


Fig. 9 The dot plots of the two players' scores

changed based on this representation (Focusing-ESM). The following excerpt displays the dialogue between the teacher and S6 (27:55–28:30 in Fig. 11).

Episode 6

85 S6: Arda has many values around a number without being too scattered around that number.
86 T: Is it the mean, right? There are many data around the balance point.
87 S6: Yes. On the other hand, Barış is more scattered.
88 T: What do you consider? What concept comes to mind when you think of the data being scattered or together?
89 S6: It is more about the comparison… (She had difficulty in answering the question.)
90 T: It is related to which data analysis tool?
91 S6: The range.
92 T: Range, right? When we look at these graphs, which one do you think has a greater range?
93 The class: Barış.

In line 85, Alicia noticed that S6's response was about center and spread using informal language and interpreted this as an emerging understanding of the mean as a balance point (Line 86), and variability/spread (Line 88). Then, her responses in Lines 88 and 90 indicated bridging the student's thinking to mathematical concepts move—CM— to connect S6's thinking with the range concept. In the post-lesson interview, she stated, "It was an unexpected answer for me. It also caught my attention in class. I thought that, even if the students did not say it, they felt the range was low when the data clustered around the mean. It was a moment of awareness I experienced during the lesson". This explanation showed how she embedded her noticing in the classroom to her responsive teaching moves. She further explained her reason for guiding the class to use the balance point meaning and the dot plots as follows: "If I did not implement the balance point activity first, I probably would not use it [the dot plot]. Therefore, I wanted to test whether they understood that meaning or not. It surprised me that some students changed their views when the representation changed."

Moreover, during the online PDP, a similar discussion took place between the facilitator and the teachers regarding the dot plots in the synchronous discussion that followed the online discussion of the basketball problem. To emphasize the reasoning about distribution, the facilitator formed the dot plots in Fig. 10 and asked the teachers to compare the data based on them. Alicia held a similar discussion in her class after the synchronous discussion and used various teaching moves during her implementation.

The codeline in Fig. 11 summarizes Alicia's teaching moves in the basketball problem. Similar to the balance point activity, she used all responsive teaching moves. However, there were some differences in the particular moves she employed. Firstly, as opposed to the balance point activity, the making sense move—MSM—was used only at the beginning of the class. The most frequent moves observed in the basketball problem were the EDM and ESM (see the green—EDM—and orange—ESM—boxes in Fig. 11). Indeed, the teacher applied a cycle of the ESM and EDM throughout her lesson. That is to say, in this problem, the teacher usually encouraged the students to consider different strategies—the ESM. After exploring the details of a student's strategy—the EDM—she again motivated them to find alternative

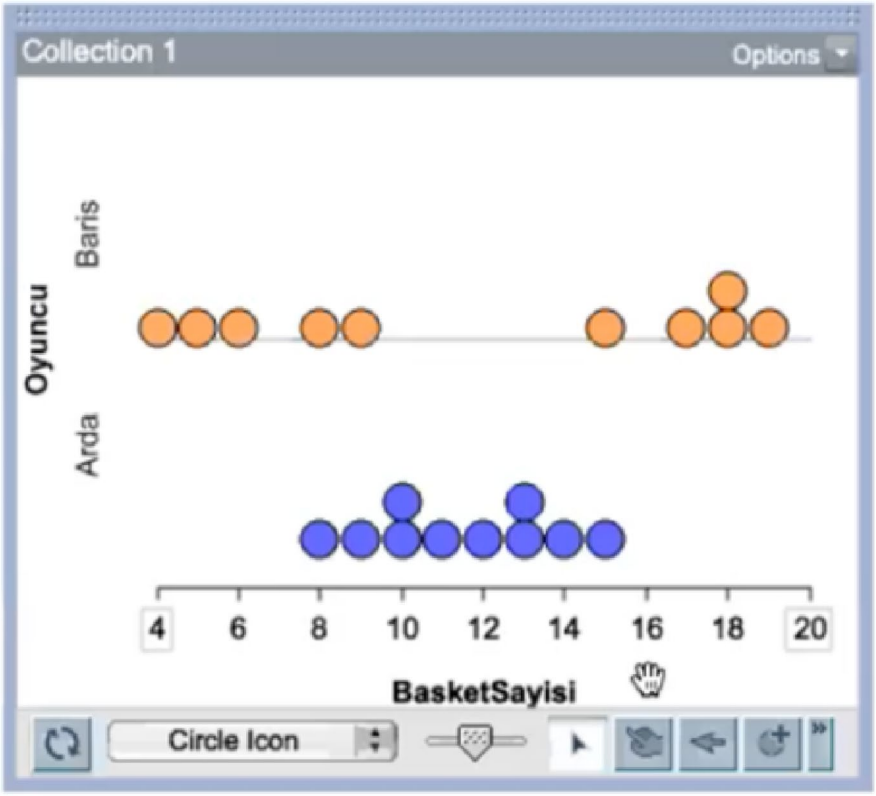


Fig. 10 The dot plots were formed by the facilitator during the synchronous discussion

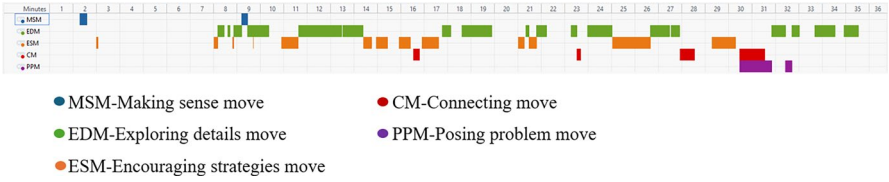


Fig. 11 Teaching moves in the basketball problem

methods (e.g., Episodes 4&5). The connecting move—CM—was observed after half of the lesson on the basketball problem (see the red boxes), showing a lower frequency compared to the balance point activity. Lastly, she preferred giving less priority to the last teaching move, which is about posing a more comprehensive problem.

Discussion and conclusion

This study examined the nature of the teaching moves employed by a teacher (Alicia) when teaching the concept of mean after participating in an online PDP. To this end, the basketball problem, which involved applying the concepts of mean and variability to compare two data sets, and the balance point activity, which aimed to develop an understanding of the balance point interpretation of the mean, were first discussed on the online platform as part of the PDP. Then, Alicia, whose noticing skills had been developed through the online PDP, volunteered to implement these tasks in two sixth-grade classrooms. The teaching moves she enacted during these lessons were analyzed in the present study. Overall, the analysis revealed that the teacher employed the anticipated responsive teaching moves, albeit in varying orders and with differing frequencies and durations.

The data analysis indicated that the teacher employed a range of responsive teaching moves when teaching the concept of mean. This finding suggested that she structured her lessons in ways that were responsive to students' statistical thinking, engaging with their ideas without imposing her own. Given that responsive teaching entails attention to evidence of students' thinking and its development (Empson, 2014), the findings indicate that the teacher, whose noticing skills were supported through the online PDP (Çatman Aksoy et al., 2022; Tekin-Sitrava et al., 2025) , appeared to begin transforming these skills into teaching moves in a classroom setting. More specifically, Alicia conceptualized students' thinking during instruction and made instructional decisions immediately in response to it. For instance, in the balance point activity, she noticed that S3 confused the numbers on the dots representing different toys and the numbers on the number line showing their prices. This was a student's thinking, which was not discussed in the online PDP. Alicia noted in the post-lesson interview that this confusion was unexpected for her, too. Nevertheless, rather than proceeding with her planned implementation, she chose to focus on S3's thinking first. After attending to S3's thinking, she first asked questions to help the student recognize her misunderstanding, then posed additional questions to ensure the class understood the problem (see Episode 1). This finding suggested that her noticing was not confined to familiar instructional contexts but could be flexibly enacted in novel classroom situations. Furthermore, during the post-interview, she stated that "after the project, especially when I saw an incorrect solution in the classroom, I started to notice more clearly why this problem happens". As mentioned earlier, in addition to the two scenarios involving the concepts of mean and variability, 15 further scenarios were discussed in the online PDP over two years, focusing on student strategies related to mathematics content, such as integers and fractions. Previous findings suggested that most of the participating teachers, including Alicia, showed growth in their noticing skills across all content areas (see Çatman Aksoy et al., 2022; Tekin-Sitrava et al., 2024). This extensive professional development might have supported the transfer of her noticing skills into the classroom, as demonstrated by her awareness of an unfamiliar student's thinking during instruction.

At the same time, her classroom practice differs from her contributions in the online PDP, where responsive teaching moves were discussed hypothetically. Whether such moves are responsive and productive depends on the emerging students' thinking in a specific classroom situation. In this sense, the teaching moves cannot be considered responsive independently of context (Jacobs & Empson, 2016); rather, they become responsive when they are meaningfully connected to students' thinking. Alicia's decision to deviate from her plan

to respond to S3's thinking in the above example illustrated that although her instruction did not unfold as originally intended, she could adjust her moves to the needs of the context based on her noticing, which indicates her responsiveness in relation to the context.

Another important finding of this study is the sequencing of teaching moves. In prior work, Jacobs and Empson (2016) introduced the teaching moves in one-on-one interactions around fraction story problems in both teacher-student interviews and classroom settings. They observed that the teacher applied the teaching moves in the same order proposed by the framework in both settings (Jacobs & Empson, 2016; Jacobs et al., 2015). The current study showed that Alicia did implement the teaching moves in various orders and sometimes simultaneously during her teaching in real classroom settings. This finding may arise from either the students' understanding that emerged during the whole-class discussions or from the nature of the teaching moves. For instance, during the balance point activity, the teacher frequently revisited the making sense and exploring details moves because students experienced difficulty understanding the task context and its representations. Hence, after responding to students through the exploring details move, Alicia often returned to the making sense move to ensure that students understood the problem context and the representations involved. The nature of the teaching moves also warrants attention. As Jacobs and Empson (2016) stated, the making sense move is at the heart of solving problems correctly and is essential. Thus, after Alicia noticed that the students did not make sense of the problem, she aimed to make the problem more understandable to them. Additionally, it is essential to examine the details of students' existing strategies, enabling teachers to determine how to implement the other three teaching moves effectively. Hence, Alicia might need to ensure that the students understood the problem and explore the details of their strategy, and turn to these moves, even though she employed the encouraging strategies and connecting moves.

In addition to using teaching moves in a different order Alicia implemented them with varying frequencies and durations (see Figs. 7, 8, and 11). For example, the encouraging strategies and exploring details moves were the most commonly used moves in the basketball problem. Alicia generally encouraged students to consider different strategies for this problem first. After exploring the details of a student's strategy, she again motivated them to find alternative methods. On the other hand, since the aim of the balance point activity was to teach one of the conceptual meanings of the mean, she might have used the move of connecting the child's thinking to symbolic notation/mathematical concepts much more than in the basketball problem. These differences imply that the use of teaching moves—specifically, their order and frequency—may depend on the workload of mathematics required in the tasks.

The hierarchical extension of the teaching moves also made it possible to distinguish how different sub-moves within the same teaching move functioned at varying levels of responsiveness. For instance, within the exploring details move, Alicia's enactments ranged from asking students to describe their strategy in general terms—clarifying—to more demanding sub-moves, such as re-voicing the student's reasoning for the whole class and inviting others to interpret or critique it—connection with invitation. For example, the former was exemplified by the question "How did you find 26?" (Line 20 in Episode 2) in the dialogue between the teacher and S5, whereas the latter was illustrated by the question "Why do you think that she might have said the multiples of 6?" (Line 26 in the same dialogue). Although these actions all fell under exploring details move, the hierarchical organization foregrounded

meaningful differences in how the teacher shaped the depth and direction of classroom practice. By differentiating among sub-moves, the hierarchy made qualitative variation visible in teachers' enactment that would remain obscured if sub-moves were treated as analytically equivalent. This finding aligned with research on whole-class discourse showing that teachers' in-the-moment decisions vary in the extent to which they engage student reasoning and shape the trajectory of discussion (Kooloos et al., 2022). Methodologically, this hierarchical organization of Jacobs and Empson's framework (2016) enhanced its explanatory power by enabling analysis not only of whether a teaching move occurred, but also of how specific enactments of that move shaped the direction and depth of classroom discussion across activities.

The other issue raised by prior research was the pre-service and in-service mathematics teachers' difficulty in considering different students' strategies for their instructional decisions and determining appropriate instructional strategies based on the students' misunderstanding of teaching statistics (Callingham & Watson, 2011; Gökce & Kazak, 2017; Shin, 2020; Watson et al., 2008). The above-mentioned findings also suggested that the online PDP designed to support teachers' noticing skills and responsive teaching might have provided resources that helped Alicia overcome these obstacles. The online PDP appeared to offer opportunities for developing competence in considering student thinking and responding to actual student answers in the classroom. For example, the basketball problem can be approached using different strategies, some of which were discussed in the online PDP. As described earlier, many teachers, including Alicia, initially believed that S2's strategy—deciding based on the higher data values—was completely incorrect in the pre-test and at the beginning of the asynchronous discussion. After this strategy was discussed online, along with alternative approaches such as more and less risky decisions, Alicia guided her students to consider these strategies and the underlying rationale during her implementation (see Episode 5). Moreover, during the interview, she stated the following: "If I had not been involved in the project, I do not think I would have focused so much on risky decisions in these problems (She is referring to the data comparison tasks.) Therefore, I do not think my students could have come up with this strategy in the basketball problem." In other words, the discussions in the online PDP might play a role in Alicia's developing this repertoire of strategies, and she could enrich classroom discussions by using a combination of encouraging strategies and exploring detailed moves.

The balance point activity was also introduced during the online PDP to promote students' conceptual understanding of the mean, as suggested in the GAISE II report (Bargagliotti et al., 2020). Alicia stated that this activity was the first time she had focused on the balance point interpretation of the mean, marking a notable development in her approach to teaching statistics. This shift was particularly significant because one challenge teachers encounter in teaching statistics is that students tend to understand the mean only procedurally (Koparan, 2015).

In conclusion, this research contributes to mathematics education by illustrating how a long-term professional development program focused on noticing students' thinking appeared to support a teacher's in-the-moment response to students' ideas, particularly in relation to the concept of mean. The study also offers a practical contribution by outlining a possible teaching strategy to promote students' conceptual understanding of the mean, which is often regarded as a challenging concept to teach and learn. However, in line with Stockero (2021), further research is needed to explore teachers' in-the-moment responses

during their classroom teaching following participation in professional development programs focused on other mathematics topics. This need is particularly essential given that our analyses of the teacher's noticing during instruction were based primarily on inferences drawn from her in-the-moment instructional actions and questioning. Although post-lesson interviews provided additional support for these interpretations, this analytic approach necessarily involves substantial interpretive work (Jacobs & Spangler, 2017) and thus constitutes a limitation of the present study. Hence, future research may benefit from greater reliance on video-augmented interviews to more directly access teachers' perspectives on what they noticed during instruction.

Also, given the single case design of this study, our conclusions are limited to examining how a teacher whose noticing development had been previously documented enacted her noticing through responsive teaching moves. While the study offers a fine-grained account of this transformation within one case, it does not aim to establish causal relationships between participation in professional development and in-the-moment instructional practices. Hence, future research involving multiple participants would be needed to more directly examine potential relationships between professional development and in-the-moment instructional practices. Lastly, future researchers could investigate refined responsive teaching framework with sub-moves to explore the implementation of responsive teaching moves and their interactions with tasks, students, and the teacher.

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Declarations

Ethical approval The necessary permissions were taken from both the ethics committees of Middle East Technical University (METU) and the Ministry of National Education (MoNE). The teachers, along with the students and their caregivers, provided their informed consent to participate in this study.

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References

Alsina, Á., Pincheira, N., Barquero, B., Sellas, I., & Marimon-Martí, M. (2025). Responsive mathematics teaching and mathematics teacher noticing: A systematic review in early childhood and primary education. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-025-00519-2

Çatman Aksoy, E., Ucuzoğlu, A., Işıksal Bostan, M., Kazak, S., Tekin Sitrava, R., & Yemen Karpuzcu, S. (2022). How does online professional development nurture teachers' noticing skills of students' thinking?: The case of data analysis. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti, F. (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education* (CERME12) (pp. 853-860). Free University of Bozen-Bolzano and ERME. *)* https://hal.science/hal-03751818v2

Bakker, A. (2018). *Design research in education*. Routledge.

Bargagliotti, A., Franklin, C., Arnold, P., Gould, R., Johnson, S., Perez, L., Spangler, D. A. (2020). *The Pre-K–12 guidelines for assessment and instruction in statistics education II (GAISE II): A framework for statistics and data science education*. American Statistical Association.

Blömeke, S., Gustafsson, J. E., & Shavelson, R. J. (2015). Beyond dichotomies: Competence viewed as a continuum. *Zeitschrift Für Psychologie, 223*, 3–13. https://doi.org/10.1027/2151-2604/a000194

Cai, J. (1998). Exploring students' conceptual understanding of the averaging algorithm. *School Science and Mathematics, 98*(2), 93–98. https://doi.org/10.1111/j.1949-8594.1998.tb17398.x

Callingham, R., & Watson, J. (2011). Measuring levels of statistical pedagogical content knowledge. In C. Batanero, G. Burrill, & C. Reading (Eds.), *Teaching statistics in school mathematics-Challenges for teaching and teacher education* (pp. 283–293). Springer.

Coskun, S. D., Sitrava, R. T., & Bostan, M. I. (2023). Pre-service elementary teachers' noticing expertise of students' mathematical thinking: The case of fractions. *International Journal of Mathematical Education in Science and Technology, 54*(6), 982–999. https://doi.org/10.1080/0020739X.2021.1979260

Ellis, A., Özgür, Z., & Reiten, L. (2019). Teacher moves to support student reasoning. *Mathematics Education Research Journal, 31*(2), 107–132. https://doi.org/10.1007/s13394-018-0246-6

Empson, S. B. (2014). Responsive teaching from the inside out: Teaching base ten to young children. *Investigations in Mathematics Learning, 7*(1), 23–53. https://doi.org/10.1080/24727466.2014.11790337

Enisoglu, D., & Isiksal-Bostan, M. (2017). Identifying students' possible solution strategies while solving questions regarding the concept of mean. *The Eurasia Proceedings of Educational and Social Sciences, 6*, 24–30.

Gal, I. (1995). Statistical tools and statistical literacy: The case of the average. *Teaching Statistics, 17*(3), 97–99. https://doi.org/10.1111/j.1467-9639.1995.tb00720.x

Garfield, J., & Ben-Zvi, D. (2007). How students learn statistics revisited: A current review of research on teaching and learning statistics. *International Statistical Review, 75*(3), 372–396. https://doi.org/10.1111/j.1751-5823.2007.00029.x

Girit Yildiz, D., Osmanoglu, A., & Gundogdu Alayli, F. (2023). Providing a video-case-based professional development environment for prospective mathematics teachers to notice students' misconceptions in measurement. *Journal of Mathematics Teacher Education, 26*(2), 179–209. https://doi.org/10.1007/s10857-021-09525-0

Gökce, R. (2019). *Examining Middle School Mathematics Teachers' Content and Pedagogical Content Knowledge of Statistical Reasoning* [Doctoral dissertation, Pamukkale University]. Council of Higher Education Thesis Center.

Gökce, R., & Kazak, S. (2017). Middle school mathematics teachers' pedagogical content knowledge in relation to statistical reasoning. In *CERME 10*.

Gökce, R., & Kazak, S. (2017). Middle school mathematics teachers' pedagogical content knowledge in relation to statistical reasoning. In Dooley, T. & Gueudet, G. (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (CERME 10) (pp. 772–779). DCU Institute of Education and ERME . https://hal.science/hal-01927849v1

Haverly, C., Calabrese Barton, A., Schwarz, C. V., & Braaten, M. (2020). "Making space": How novice teachers create opportunities for equitable sense-making in elementary science. *Journal of Teacher Education, 71*(1), 63–79.

Işıksal-Bostan, M., Tekin-Sitrava, R., Özel, Z.,& Yemen-Karpuzcu, S. (2023). Teachers' in-the-moment response to student thinking: the case of pattern generalization. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, E. Herendiné-Kónya (Eds.), *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education* (CERME13) (pp. 3586-3593). Alfréd Rényi Institute of Mathematics, and ERME. https://hal.science/hal-04398420v1

Jacobbe, T., & Carvalho, C. (2011). Teachers' understanding of averages. In C. Batanero, G. Burrill, & C. Reading (Eds.), *Teaching statistics in school mathematics-Challenges for teaching and teacher education* (pp. 199–209). Springer.

Jacobs, J. K., & Morita, E. (2002). Japanese and American teachers' evaluations of videotaped mathematics lessons. *Journal for Research in Mathematics Education, 33*(3), 154–175.

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM - Mathematics Education, 48*, 185–197. https://doi.org/10.1007/s11858-015-0717-0

Jacobs, V. R., Empson, S. B., Jessup, N. A., Dunning, A., Pynes, D. A., Krause, G., & Franke, T. M. (2024). Profiles of teachers' expertise in professional noticing of children's mathematical thinking. *Journal of Mathematics Teacher Education, 27*(3), 295–324.

Jacobs, V. R., Empson, S. B., Krause, G. H., & Pynes, D. (2015). *Responsive teaching with fractions* [Paper Presentation]. National Council of Teachers of Mathematics Annual meeting & Exposition, Boston, MA, United States.

Jacobs, V. R., Lamb, L. C., Philipp, R. A., & Schappelle, B. P. (2011). Deciding how to respond on the basis of children's understandings. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 97–116). Routledge.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169–202. https://doi.org/10.5951/jresematheduc.41.2.0169

Jacobs, V. R., & Spangler, D. A. (2017). Research on core practices in K–12 mathematics teaching. In J. Cai (Ed.), *The compendium for research in mathematics education* (pp. 766–792). National Council of Teachers of Mathematics.

Kaiser, G., Blömeke, S., König, J., Busse, A., Döhrmann, M., & Hoth, J. (2017). Professional competencies of (prospective) mathematics teachers—Cognitive versus situated approaches. *Educational Studies in Mathematics, 94*(2), 161–182. https://doi.org/10.1007/s10649-016-9713-8

Karaca, ŞA., & Ay, Z. S. (2024). Investigation of eighth grade students' performance on tasks involving statistical thinking about measures of central tendency. *Participatory Educational Research, 12*(1), 18–42. https://doi.org/10.17275/per.25.2.12.1

Kooloos, C., Oolbekkink-Marchand, H., van Boven, S., Kaenders, R., & Heckman, G. (2022). Making sense of student mathematical thinking: The role of teacher mathematical thinking. *Educational Studies in Mathematics, 110*(3), 503–524.

Koparan, T. (2015). Difficulties in learning and teaching statistics: Teacher views. *International Journal of Mathematical Education in Science and Technology, 46*(1), 94–104. https://doi.org/10.1080/0020739X.2014.941425

Lee, M. Y., & Francis, D. C. (2018). Investigating the relationships among elementary teachers' perceptions of the use of students' thinking, their professional noticing skills, and their teaching practices. *The Journal of Mathematical Behavior, 51*, 118–128. https://doi.org/10.1016/j.jmathb.2017.11.007

Levin, D., Hammer, D., Elby, A., & Coffey, J. (2013). *Becoming a responsive science teacher: Focusing on student thinking in secondary science*. National Science Teachers Association.

McGatha, M., Cobb, P., & McClain, K. (2002). An analysis of students' initial statistical understanding: Developing a conjectured learning trajectory. *The Journal of Mathematical Behavior, 21*, 339–355. https://doi.org/10.1016/S0732-3123(02)00133-5

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematics success for all*. National Council of Teachers of Mathematics.

Pfannkuch, M., & Ben-Zvi, D. (2011). Developing teachers' statistical thinking. *Teaching Statistics in School Mathematics-Challenges for Teaching and Teacher Education: A Joint ICMI/IASE Study: The 18th ICMI Study*, 323–333.

Scheiner, T. (2021). Towards a more comprehensive model of teacher noticing. *Zdm–mathematics Education, 53*(1), 85–94.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011). *Mathematics teacher noticing: Seeing through teachers' eyes*. Taylor & Francis.

Shin, D. (2020). Prospective mathematics teachers' professional noticing of students' reasoning about mean and variability. *Canadian Journal of Science, Mathematics and Technology Education, 20*(3), 423–440. https://doi.org/10.1007/s42330-020-00091-w

Stockero, S. L. (2021). Transferability of teacher noticing. *Zdm–mathematics Education, 53*(1), 73–84. https://doi.org/10.1007/s11858-020-01198-y

Tekin-Sitrava, R., Özel, Z., Işıksal-Bostan, M., & Yemen-Karpuzcu, S. (2025). How does online professional development program enriched with collaborative discussion develop teachers' noticing skills? I*nternational Journal of Science and Mathematics Education, 23*(1), 71-96 https://doi.org/10.1007/s10763-024-10461-7

van De Walle, J. A., Karp, K. S., & Bay-Williams J. M. (2019). *Elementary and middle school mathematics: Teaching developmentally (10th ed*.). Pearson Education, Inc.

van Es, E. A., Hand, V., Agarwal, P., & Sandoval, C. (2022). Multidimensional noticing for equity: Theorizing mathematics teachers' systems of noticing to disrupt inequities. *Journal for Research in Mathematics Education, 53*(2), 114–132.

van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education, 10*(4), 571–596.

van Es, E. A., & Sherin, M. G. (2021). Expanding on prior conceptualizations of teacher noticing. *Zdm–mathematics Education, 53*(1), 17–27.

van Es, E. A., & Sherin, M. G. (2025). Exploring teacher noticing in practice: The role of shaping. *ZDM—Mathematics Education*. https://doi.org/10.1007/s11858-025-01740-w

VERBI Software. (2021). MAXQDA 2022 [computer software]. Berlin, Germany: VERBI Software. Available from maxqda.com

Warshauer, H. K., Starkey, C., Herrera, C. A., & Smith, S. (2021). Developing prospective teachers' noticing and notions of productive struggle with video analysis in a mathematics content course. *Journal of Mathematics Teacher Education, 24*, 89–121.

Watson, J., Callingham, R., & Donne, J. (2008). Establishing PCK for teaching statistics. In C. Batanero, G. Burrill, C. Reading, & A. Rossman (Eds*.), Joint ICME/IASE study: Teaching statistics in school mathematics. Challenges for teaching and teacher education. Proceedings of the ICMI Study 18 and the 2008 IASE Round Table Conference.* International Association for Statistical Education.

Watson, J. M., & Moritz, J. B. (2000). The longitudinal development of understanding of average. *Mathematical Thinking and Learning, 2*(1–2), 11–50. https://doi.org/10.1207/S15327833MTL0202_2

Weyers, J., Koenig, J., Scheiner, T., Santagata, R., & Kaiser, G. (2024). Teacher noticing in mathematics education: a review of recent developments. *Videobasierte Kompetenzmessung in der universitären Lehrkräfteausbildung*, 103–133.

Zawojewski, J. S., Chamberlin, M., Hjalmarson, M. A., & Lewis, C. (2008). Designing design studies for professional development in mathematics education. In A. E. Kelly, R. Lesh, & J. Baek (Eds.), *Handbook of design research methods in education* (pp. 219–245). Routledge.

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