# ULAM STABILITY IN C*-ALGEBRAS AND ITS CONNECTIONS TO RIGIDITY CONJECTURES

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# ABSTRACT

## ULAM STABILITY IN C*-ALGEBRAS AND ITS CONNECTIONS TO RIGIDITY CONJECTURES

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Roughly speaking, Ulam stability is the phenomenon of "approximate" morphisms between classes of objects in a category being close to actual morphisms. In this thesis, we study Ulam stability in the context of C*-algebras. More specifically, we survey some results regarding the Ulam stability of finite-dimensional-to-finite-dimensional C*-algebras and finite-dimensional-to-arbitrary C*-algebras, due to Farah and McKenney-Vignati respectively. As a motivating factor, we also briefly analyze the connection between Ulam stability of Borel quotient structures and a conjecture template regarding the rigidity of such structures under additional set-theoretic principles.

Keywords: Ulam Stability, C*-algebras, Borel Quotient Structures, Rigidity Conjectures.

# ÖZ

## C*-CEBİRLERİNDE ULAM KARARLILIĞI VE KATILIK VARSAYIMLARIYLA BAĞLANTILARI

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Genel bir ifadeyle, Ulam kararlılığı bir kategorideki obje sınıfları arasındaki "yaklaşık" morfizmaların gerçek morfizmalara yakın olması fenomenidir. Bu tezde, Ulam kararlılığını C*-cebirleri bağlamında çalışıyoruz. Daha özel olarak, sonlu boyutludan sonlu boyutlu C*-cebirlerine ve sonlu boyutludan keyfi C*-cebirlerine Ulam kararlılığına dair sırasıyla Farah ve McKenney-Vignatiye ait bazı sonuçları inceleyeceğiz. Bunun yanı sıra, motive edici bir etken olarak, Borel bölüm yapılarının Ulam kararlılığı ve böyle yapıların ek küme kuramsal prensipler altındaki katılığına dair bir sanı şablonu arasındaki bağlantıyı da inceleyeceğiz.

Anahtar Kelimeler: Ulam kararlılığı, C*-cebirleri, Borel Bölüm Yapıları, Katılık Varsayımları.

To my parents

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## LIST OF ABBREVIATIONS

| | |
|---|---|
| CH | Continuum Hypothesis |
| $\mathrm{MA}$ | Martin's Axiom |
| $\mathrm{MA}_{\kappa}$ | Martin's Axiom $\kappa$ |
| MAP | Metric Approximation Property |
| $\mathrm{OCA}_{\mathrm{T}}$ | Open Coloring Axiom |
| PFA | Proper Forcing Axiom |
| ZFC | Zermelo–Fraenkel set theory with the axiom of choice |
| $\mathcal{C}^*$ | The class of all C*-algebras. |
| $\mathcal{W}^*$ | The class of all von Neumann algebras. |
| $\mathcal{F}$ | The class of finite dimensional C*-algebras. |
| $\mathcal{AF}$ | The class of approximately finite dimensional C*-algebras. |
| $\mathcal{AB}$ | The class of abeliean C*-algebras. |

# CHAPTER 1

# INTRODUCTION

## 1.1 Ulam Stability

Inspired by the notions of stability as they are seen in mechanics, Ulam asked in his [25] about 'stability' within pure mathematics. He asked about the stability of theorems and the stability of morphisms in different categories, namely, functional equations, metric groups, transformations in Euclidean spaces, and more. Generally, we can say that Ulam Stability results for a category or class indicate that 'approximate' or 'almost' morphisms in a category are close to actual ones

Let's take a look at an example of the Ulam stability definition in the group-theoretic context. First, we shall see examples of what we shall call approximate morphisms. These are maps between objects in a category that are morphisms up to an $\varepsilon$ error. In the group-theoretic context we have the following.

**Definition 1.1.1.** For $\varepsilon > 0$, a function $f : G_1 \to G_2$ between groups where $G_2$ is a metric group with a metric $d$ is called an *$\varepsilon$-homomorphism* if for all $g, h \in G_1$

$$d(f(gh), f(g)f(h)) < \varepsilon.$$

In a general way, we can define Ulam stability for a category. The category $\mathcal{G}$ is said to be *Ulam stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $G_1$ and $G_2$ in $\mathcal{G}$ every $\delta$-homomorphism from $G_1$ to $G_2$ can be uniformly $\varepsilon$-approximated by an actual homomorphism.

Continuing with the framework of our previous example, we can see an instance where Ulam stability does not hold.

**Example 1.1.1.** The category of all finite subgroups of the unit circle, with the metric induced by the topology of complex numbers, is not Ulam Stable.

Note that in this proof we use the additive notation instead of the multiplicative one (from the complex numbers) for better readability.

*Proof.* Let $\varepsilon = 1$ consider $f_n : \mathbb{Z}_{2n+1} \to \mathbb{Z}_{2n}$ such that $f_n : z \mapsto z$. Then

$$|(f_n(k) + f_n(k')) - f_n(k + k')| = 0$$

unless $[k + k'] = [2n + 1]$ in that case we have

$$|(f_n(k) + f_n(k')) - f_n(k + k')| < \frac{2\pi}{2n}.$$

Then each $f_n$ is a $\frac{2\pi}{2n}$-homomorphism. But since the only true homomorphism in this case is trivial, we have that it cannot be 1-approximated by a true homomorphism. □

## 1.2 Ulam Stability of C*-algebras

When it comes to Ulam stability, in many cases, it can be useful to talk about classes of objects in categories. Especially in areas with ongoing exposition, as it can allow us to understand to what extent Ulam stability holds in the given context. This will indeed be the case for Ulam stability in C*-algebras: the primary focus of the thesis.

**Definition 1.2.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be classes of C*-algebras. We say that the pair $(\mathcal{A}, \mathcal{B})$ is Ulam stable if for all $\varepsilon > 0$, there is a $\delta > 0$ such that given a $\delta^*$-homomorphism $\phi : A \to B$ there is a *-homomorphism $\psi : A \to B$ such that for all $x \in A_{\leq 1}$, we have

$$\|\psi(x) - \phi(x)\| < \varepsilon$$

for each $x \in A_{\leq 1}$. If $(\mathcal{A}, \mathcal{A})$ is Ulam stable we can say $\mathcal{A}$ is Ulam stable.

We shall now note some important classes of C*-algebras.

1. $\mathcal{C}^*$: the class of all C*-algebras.
2. $\mathcal{W}^*$: the class of all von Neumann algebras.

3. $\mathcal{F}$: the class of finite dimensional C*-algebras.

4. $\mathcal{AF}$: the class of approximately finite dimensional C*-algebras.

5. $\mathcal{AB}$: the class of abeliean C*-algebras.

The following are some prominent results about Ulam in C*-algebras.

One of the earliest developments in the area was done by Šemrl.

**Theorem 1.2.1.** *[22] The class $\mathcal{AB}$ is Ulam stable.*

The primary focus of this thesis are surveying the following two results.

**Theorem 1.2.2.** *[8] The class $\mathcal{F}$ is Ulam stable.*

**Theorem 1.2.3.** *[15] The pair $(\mathcal{F}, \mathcal{C}^*)$ is Ulam Stable*

Another result from [15] was the following.

**Theorem 1.2.4.** *[15] The pair $(\mathcal{AF}, \mathcal{W}^*)$ is Ulam stable.*

### 1.3 Corona Rigidity

Our primary motivation to delve into the study of Ulam stability came while studying [10] on Corona Rigidity. The same paper provides a good set-theoretic motivation for the study of Ulam stability in C*-algebras; we shall briefly go over the motivation.

We start by looking at some historical results. Note that here $\mathrm{ZFC}$ stands for Zermelo–Fraenkel set theory with the axiom of choice, $\mathrm{CH}$ stands for the continuum hypothesis, and PFA stands for the proper forcing axiom.

**Theorem 1.3.1.** *[20] (ZFC+CH) There are $2^{\mathfrak{c}}$ many automorphisms of the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$.*

**Theorem 1.3.2.** *[23] (ZFC+PFA) All automorphisms of $\mathcal{P}(N)/\mathrm{Fin}$ are induced by almost permutations of $\mathbb{N}$.*

**Theorem 1.3.3.** *[19] (ZFC+CH) The Calkin algebra has $2^{\mathfrak{c}}$ many outer automorphisms.*

**Theorem 1.3.4.** *[8] (ZFC+OCA) All automorphisms of the Calkin algebra are inner.*

If we replace the words 'inner ' and 'almost permutation' with 'trivial', and 'outer' with 'non-trivial' in theorems 1.3.3 and 1.3.4 we notice that the results are similar in nature to theorems 1.3.1 and 1.3.2, respectively. This behaviour of these specific types of quotient structures under additional set-theoretic axioms is a well-studied phenomenon. In fact, these types of questions are leading towards an emergence of a conjecture template.

### 1.3.1 Conjecture Template

A *Borel structure* $\mathcal{M}$ in a signature[1] $\mathcal{L}$ is a structure whose universe $M$ is a Polish space and the interpretation of the function and relation symbols are Borel. A *Borel quotient structure* is obtained from $\mathcal{M}$ by specifying a Borel[2] equivalence relation. This quotient structure is denoted by $\mathcal{M}/E$. The quotient map here is denoted by $\pi_E$. A *lifting* of a morphism $\Phi : \mathcal{M}/E \to \mathcal{N}/F$ between Borel quotient structures is a map $\Phi^* : \mathcal{M} \to \mathcal{N}$ such that the diagram in the following Figure 1.1 commutes.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Phi_*} & \mathcal{N} \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ \mathcal{M}/E & \xrightarrow{\Phi} & \mathcal{N}/F \end{array}$$

Figure 1.1: A lifting $\Phi_*$ of $\Phi$.

We have two types of 'triviality' we care about.

**Definition 1.3.1.** An isomorphism $\Phi$ between quotient structures is *topologically trivial* if it has a Borel measurable lifting.

We also have a more desirable form of triviality, but it is more difficult to define, and the definition depends on the category we are dealing with. Informally speaking, an isomorphism $\Phi$ is *algebraically trivial* if it has a lifting that preserves as much of the algebraic structure as possible.

[1] This is the tuple containing function symbols, relation symbols, constant symbols, arity of function.
[2] The relation defining set is Borel in $M \times M$

Now we can take a look at the conjecture template. The reader should note that the following is not a precise mathematical statement of a conjecture but rather something to be formulated into a conjecture once the mathematical context is established.

**Conjecture Template 1.3.1.** *For Borel quotient structures $\mathcal{M}/E$ and $\mathcal{N}/F$ we have the following statements.*

1. *The Continuum Hypothesis (CH) implies that $\mathcal{M}/E$ has $2^{\mathfrak{c}}$ automorphisms (and therefore $2^{\mathfrak{c}}$ many topologically non-trivial ones). It also implies that $\mathcal{M}/E$ and $\mathcal{N}/F$ are isomorphic, unless for an obvious impediment.*

2. *Forcing axioms imply that all automorphism of $\mathcal{M}/E$ are topologically trivial (Borel liftable). Moreover, every isomorphism between $\mathcal{M}/E$ and $\mathcal{N}/F$ is topologically trivial (Borel liftable).*

The reader might have noticed that the conjecture template does not really talk about algebraic triviality. But algebraic triviality is more desirable than topological triviality, and therefore we have another question to ask.

**Question Template 1.3.1.** *When does topological triviality imply algebraic triviality?*

The key to this question for instances of the conjecture template lies in understanding Ulam Stability. This will be further explored in Chapter 4.

## 1.4 Structure of the Thesis

This thesis primarily aims to prove instances of Ulam stability in C*-algebras while providing the necessary background. The secondary aim is to explore connections of Ulam stability to rigidity conjectures. Chapter 2 is dedicated to providing the necessary background required for the proofs of Ulam stability in Chapter 3. Chapter 3 goes into detailed proofs of the Ulam stability of $(\mathcal{F}, \mathcal{F})$ and $(\mathcal{F}, \mathcal{C}^*)$. In the process, Chapter 3 also explores an important Integral convergence and Ulam stability in group representations, tools foundational to the primary Ulam stability results. Chapter 4

briefly explores the connections between Ulam stability and the conjecture template mentioned above.

# CHAPTER 2

# PRELIMINARIES

## 2.1 Introduction to the chapter

Before we dive deeper into the main contents of the thesis, we need to establish some preliminaries and notation that will be used in the latter parts of the thesis.

## 2.2 Banach Spaces, Hilbert Spaces, and more

We expect the reader to be familiar with elementary Hilbert space theory. However, we shall make some basic definitions and notations clear for readability. Recall that a complete normed space is called a *Banach space*. A complete inner product space is called a *Hilbert space*. Throughout the thesis, we shall usually use $H$ to denote a Hilbert space. By $B(H)$ we shall denote the bounded operators over a Hilbert space. This is an object of great importance in the study of C*-algebras. We will also use $GL(H)$ and $U(H)$ to denote the invertible and unitary elements of $B(H)$, respectively.

## 2.3 Banach Algebras

We start by looking at Banach algebras.

An *algebra* is a vector space $A$ with a bilinear multiplication map that maps $(a, b)$ to $ab$. The multiplication here is associative. A *subalgebra* is a vector subspace that is closed under multiplication. A *normed* algebra is an algebra over a normed space

such that the norm has the submultiplicative property. That is for all $a$, $b$ in a normed algebra $A$, $\|ab\| \leq \|a\|\|b\|$. A complete normed algebra is called a *Banach algebra*. If it has a multiplicative unit it is called a *unital Banach algebra*. Every Banach algebra can be easily unitized. We refer the reader to [17] for further details about unitization of Banach algebras. The following examples were taken from [17].

**Example 2.3.1.** Let $\Omega$ be a topological space. We consider the following examples of Banach algebras.

1. The set of all bounded complex-valued functions on $\Omega$ forms an algebra, denoted by $\ell^\infty(\Omega)$.
2. Important subalgebras of $\ell^\infty(\Omega)$ are $C_b(\Omega)$, the set of all bounded and continuous complex-valued functions on $\Omega$, and $C_0$, the set of functions vanishing at infinity. Note that a continuous function $f : \Omega \to \mathbb{C}$ is said to vanish at infinity if for each positive $\varepsilon$, the set $\{r \in \Omega : |f(\omega)| \geq \varepsilon\}$ is compact.
3. It is clear that $C_0(\Omega) \subseteq C_b(\Omega) \subseteq \ell^\infty(\Omega)$. In fact, $C_0(\Omega)$ is a closed subalgebra of $C_b(\Omega)$, and $C_b(\Omega)$ is a closed subalgebra of $\ell^\infty(\Omega)$.
4. The set of all $n \times n$ matrices over the complex numbers, denoted $M_n(\mathbb{C})$, is a unital Banach algebra, so is the set of upper triangular matrices.

The following is an important fact about Banach algebras, which we will often use in this thesis. This theorem provides us a sufficient condition for invertibility that comes in handy. Note that this condition resembles the one we have for the complex numbers.

**Theorem 2.3.1.** *[17, Theorem 1.2.2] Let $A$ be a unital Banach algebra and $a$ an element of $A$ such that $\|a\| < 1$. Then $1 - a$ is invertible in $A$, moreover,*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

### 2.3.1 Spectral Theory Basics

We shall begin this subsection by defining the spectrum of an element in an algebra. If we view algebras as an analog to matrices, the spectrum can be viewed as the analog

and generalization to the set of eigenvalues. The *spectrum* of an element $a$ of an algebra $A$ is the following set

$$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda \cdot 1 - a \notin \mathrm{Inv}(A)\}.$$

If we are talking about non-unital algebras, we can define the spectrum using their unitization.

The following is an example of the spectrum.

**Example 2.3.2.** We consider the algebra of all continuous $\mathbb{C}$-valued functions over a compact Hausdorff space, or $C(\Omega)$. Then the spectrum of $f \in C(\Omega)$ is exactly its range, that is

$$\sigma(f) = f(\Omega).$$

We shall now give some important facts about spectral theory that we will use throughout this thesis.

**Theorem 2.3.2** (The Polynomial Spectral Mapping Theorem)**.** *Let $\mathbb{C}[z]$ denote the algebra of all polynomials with complex coefficients. Let $a$ be an element of a unital algebra $A$. If $\sigma(a)$ is non-empty and $p \in \mathbb{C}[z]$ then*

$$\sigma(p(a)) = p(\sigma(a)) := \{p(\lambda) : \lambda \in \sigma(a)\}.$$

**Theorem 2.3.3** (Gelfand Theorem)**.** *If $a$ is an element of a unital Banach algebra $A$, then its spectrum $\sigma(a)$ is non-empty.*

Another important aspect of the spectrum is the spectral radius. Let $A$ be a unital Banach algebra, and $a$ be an element. The *spectral radius* of $a$ is

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Continuing with our preceding example, for $f \in C(\Omega)$, $\sigma(f) = \|f\|_\infty$. We have the following fact about the spectral radius.

**Fact 2.3.1.** *Let $A$ be a unital Banach algebra, let $a \in A$, then*

$$r(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}}.$$

### 2.3.2 Characters and their Interplay with the Spectrum

A *character* on an abelian algebra $A$ is a non-zero homomorphism

$$\tau : A \to \mathbb{C}.$$

the set of all characters on $A$, is denoted by $\Omega(A)$. Characters in Banach algebras are analogous to linear functionals in Banach spaces. The following is an important fact about characters that establishes a pertinent relation between the character space and the spectrum.

**Fact 2.3.2.** *Let $A$ be a unital abelian Banach algebra. If $\tau \in \Omega(A)$, then $\|\tau\| = 1$. Moreover, or $a \in A$, if $A$ is unital, then*

$$\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\}.$$

*If $A$ is not unital, then*

$$\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\} \cup \{0\}.$$

One can consider the following set of functions

$$\hat{a} : \Omega(A) \to \mathbb{C},\ \tau \to \tau(a).$$

We consider the weak-*-topology on $\Omega(A)$, by considering this topology we get that $\hat{a} \in C_0(\Omega(A))$., we shall call $\hat{a}$ the *Gelfand transform* of $a$. This brings us to the Gelfand Representation Theorem

**Theorem 2.3.4.** *Let $A$ be an abelian Banach algebra and $\Omega(A)$ be non-empty. Then*

$$A \to C_0(\Omega(A)),\ a \mapsto \hat{a},$$

*is a norm-decreasing homomorphism, and*

$$r(a) = \|\hat{a}\|_\infty\ (a \in A).$$

*Moreover, if $A$ is unital, $\sigma(a) = \hat{a}(\Omega(A))$, and if $A$ is non-unital, for each $a$, $\sigma(a) = \hat{a}(\Omega(A)) \cup \{0\}$.*

**Corollary** *Let $A$ be a unital Banach algebra generated by $1$ and an element $a$ in $A$. Then $A$ is abelian and the map $\hat{a} : \Omega(A) \to \sigma(a)$ mapping $\tau$ to its evaluation at $a$ is a homeomorphism.*

## 2.4 C*-algebras

We shall now talk about *-algebras. Note that an *involution* on an algebra $A$ is a conjugate-linear map $a \mapsto a^*$ on $A$, such that for every $a$ and $b$ in $A$, $a^{**} = a$ and $(ab)^* = b^*a^*$. The pair $(A, *)$ is called a *-algebra. The conjugate transpose of a complex (or real) matrix is an example of an involution.

In general, it is useful to view *-algebras as a sort of generalization of the matrices we are familiar with. Therefore, we will have some familiar terminology.

- An element $a$ in $A$ is *self-adjoint* or *hermitian* if $a^* = a$.
- Note that the elements $a^*a$ and $aa^*$ are always self-adjoint. We say $a$ is *normal* if $a^*a = aa^*$.
- An element $p$ in a *-algebra is a *projection* if $p = p^* = p^2$.
- An element $u$ in a *-algebra is an *isometry* if $u^*u = 1$, a *co-isometry* if $uu^* = 1$, and *unitary* if $u^*u = uu^* = 1$.
- An element $v$ such that $v^*v$ and $vv^*$ are projections is a *partial isometry*.
- An element $a \in A$ is called *skew-hermitian* if $a^* = -a$.

We have further terminology we shall borrow from the terminology used in the matrices. For example, we shall denote the invertible elements of a C*-algebra $A$ by $\mathrm{GL}(A)$. The unitaries shall be denoted by $U(A)$.

Note that it is easy to see that for any invertible $a$, $(a^{-1})^* = (a^*)^{-1}$. It follows that the spectrum of $a$,

$$\sigma(a^*) = \overline{\sigma(a)} = \left\{\overline{\lambda} \in \mathbb{C} : \lambda \in \sigma(a)\right\}.$$

An algebra homomorphism between *-algebras preserving adjoints is called a *-*homomorphism*, other terms can be similarly defined.

A Banach *-algebra is a *-algebra $A$, together with a complete submultiplicative norm such that, for each $a \in A$, $\|a^*\| = \|a\|$. We now have all the tools at our disposal to define C*-algebras

**Definition 2.4.1.** A *C*-algebra* $A$ is a Banach *-algebra such that for every $a \in A$, $\|a^*a\| = \|a\|^2$.

The reader shall note that this is one of the two primary ways to define C*-algebras, but they are equivalent, as we will discuss later. We shall also note that *von Neumann Algebra* is a strongly closed, unital C*-subalgebra of $B(H)$.

**Fact 2.4.1.** *Every *-homomorphism between C*-algebras is norm-decreasing.*

**Example 2.4.1.** The following are some prominent examples of C*-algebras.

1. The scalar field $\mathbb{C}$ is a unital C*-algebra with the usual multiplication, addition, and the conjugation being the involution.

2. Let $\Omega$ be a locally compact Hausdorff space, then $C_0(\Omega)$ is a C*-algebra with the involution $f \mapsto \overline{f}$.

3. Complex valued square matrices, $M_n(\mathbb{C})$, along with the complex transpose, are examples of C*-algebras.

4. If $H$ is a Hilbert space, then the bounded operators over this space, $B(H)$, along with the adjoint operation as the involution, form a C*-algebra. This is arguably one of the most important examples of C*-algebras. We will later see that every C*-algebra is a closed *-subalgebra of some $B(H)$.

There are some important facts one should note about C*-algebras. The norm on a *-algebra that would make it a C*-algebra is unique. Also, *-homomorphisms between C*-algebras are continuous.

The following are some other basic facts about C*-algebras we shall use in the rest of the thesis.

**Fact 2.4.2.** *[17, Chapter 2.1] Every element of a C*-algebra can be written as a sum of two self-adjoint elements. Every self-adjoint element of a C*-algebra can be written as a sum of two unitaries. Therefore, every element of a C*-algebra can be written as a sum of four unitaries.*

The *Hausdorff distance* between compact sets $A$ and $B$ in a metric space is defined as follows

$$d_H(A, B) = \sup_{a \in A, b \in B} d(a, b).$$

**Fact 2.4.3.** *Assume $a$ and $b$ are normal elements of a C*-algebra $A$. Then*

$$d_H(\sigma(a), \sigma(b)) \leq \|a - b\|.$$

### 2.4.1 Gelfand Theorem and the Functional Calculus

In this subsection, we recall some fundamental theorems about C*-algebras.

**Theorem 2.4.1** (Gelfand Theorem)**.** *If $A$ is a non-zero abelian C*-algebra, then the Gelfand representation*

$$\phi : A \to C_0(\Omega(A)),\ a \mapsto \hat{a}$$

*is an isometric *-isomorphism.*

**Theorem 2.4.2** (Spectral Mapping Theorem)**.** *Let $a$ be a normal element of a unital C*-algebra $A$, and let $f \in C(\sigma(a))$. Then*

$$\sigma(f(a)) = f(\sigma(a)).$$

By $C^*(a)$ we shall denote the C*-algebra generated by the element $a$.

**Theorem 2.4.3** (Gelfand Representation Theorem)**.** *If $A$ is a C*-algebra and $a \in A$ is normal then*

$$\phi : C^*(a) \to C_0(\sigma(a) \setminus \{0\})$$

*is an isometric *-isomorphism. If $A$ is unital $C^*(1, a) \cong C(\sigma(a))$.*

The Gelfand representation theorem, also known as the spectral theorem, provides the existence of the functional calculus. This theorem allows us to talk about elements of the form $f(a)$, where $a$ is a normal element in a C*-algebra and $f$ is a continuous function over $\sigma(a)$. Note that $f(a)$ is defined to be $\phi^{-1}(f(\sigma(a))$, in the notation of the preceding result. Because of the functional calculus, it makes sense for us to talk about $a^2$, $\sin(a)$, $e^a$, and more. Note that as $\phi$ is also isometric, we have that we can get information about the norm of $f(a)$ too. For example if $\sigma(a) \subseteq [1, \infty)$, we

have $\|a^2\| \geq \|a\|$. But if $\sigma(a) \in [0,1)$, we have that $\|a^2\| \leq \|a\|$. Note that we use the functional calculus of $a^*a$ to define the element $|a| := (a^*a)^{\frac{1}{2}}$ for any $a$ in a C*-algebra $A$.

An element $a$ of a C*-algebra $A$ is *well-supported* if $0 \notin \sigma(a)$ or $0$ is an isolated point.

**Theorem 2.4.4.** *We have the following two facts about the polar decomposition of elements of C*-algebras.*

1. *For each $a \in B(H)$, for some Hilbert space $H$, there exists a polar decomposition $a = v|a|$, where $v$ is a partial isometry.*
2. *For each $a \in A$, if $a$ is well supported. Then there exists a polar decomposition $a = v|a|$, where $v$ is a partial isometry in $C^*(a)$.*

### 2.4.2 Projections

Recall, as mentioned earlier. An element $p$ in a C*-algebra is a projection if $p^2 = p$. Note also that there is a partial ordering on projections given by

$$p \leq q \iff pq = p.$$

It is straightforward to see that the norm of a projection is always $1$. We also have other formulations for this ordering, that is, for projections $p$ and $q$ in a C*-algebra, $p \leq q \iff qp = p \iff (q-p)^2 = q - p$

We have the following general definition for Murray-von Neumann equivalence of projections. Two projections $p$, and $q$ are considered to be *Murrray-von Neumann equivalent* or (*MvN equivalent*) if there is a partial isometry $v$ such that $v^*v = p$ and $vv^* = q$.

**Fact 2.4.4.** *If $p$ and $q$ are projections in $M_n(\mathbb{C})$ for some $n$, then $p \sim_{MvN} q$ if and only if they are unitarily equivalent and they have the same rank.*

Another important fact to note about projections is the following

**Fact 2.4.5.** *Let $p$ and $q$ be projections in a unital C*-algebra $A$. Then $p$ and $q$ are unitarily equivalent if and only if $1-p$ and $1-q$ are.*

The preceding fact also implies that in $M_n(\mathbb{C})$ the projections are unitarily equivalent if and only if they have the same rank.

### 2.4.3 States and their Properties

Suppose $X$ is any normed space. A *linear functional* is a linear map $\varphi$ from $X$ into the field of scalars (in our case $\mathbb{C}$). It is bounded if $\|\varphi\| := \sup_{x\in X,\|x\|\leq 1} |\varphi(x)|$ is finite.

**Definition 2.4.2.** Note that a linear functional $\varphi$ on a C*-algebra $A$ is positive if for each $a \in A$ $\varphi(a) \geq 0$. Moreover, it is a state if $\|\varphi\| = 1$.

**Definition 2.4.3.** A state $\tau$ on a C*-algebra is *tracial* if for each $a, b \in A$, $\tau(ab) = \tau(ba)$. A state $\varphi$ is *faithful* if $\varphi(a^*a) = 0$ implies $a = 0$. A *trace* is an additive and homogeneous function $\tau : A_+ \to [0, \infty]$ such that $\tau(aa^*) = \tau(a^*a)$.

**Fact 2.4.6.** *The tracial state on matrix algebras,$M_n(\mathbb{C})$, is unique.*

**Fact 2.4.7.** *[9, Corollary 4.1.4] Every tracial state of every simple C*-algebra is faithful.*

### 2.4.4 Gelfand-Naimark-Segal Theorem

We use [9, Chapter 1.7] for this subsection and refer to the same for further details on this. A subalgebra of $B(H)$ is *self-adjoint* if it is closed under adjoint operation. A *concrete C*-algebra* is a norm-closed, self-adjoint, algebra of operators on some Hilbert space.

**Definition 2.4.4.** A *representation* of a C*-algebra is a *-homomorphism

$$\pi : A \to B(H)$$

for some Hilbert space $H$. It is *faithful* if $\ker(\pi) = 0$.

**Theorem 2.4.5** (Gelfand-Naimark-Segal Theorem)**.** *Every abstract C*-algebra $A$ is isomorphic to a concrete C*-algebra*

### 2.4.5 Finite-Dimensional C*-algebras

This subsection is taken from [9, Chapter 2.2]. A finite-dimensional C*-algebra is one where the underlying Hilbert space, in the GNS equivalence, is finite-dimensional. We will be looking at these in detail for this thesis; therefore, it is pertinent to know some of their basic properties.

*Full matrix algebras* are algebras $M_n(\mathbb{C})$ for a natural number $n$. The matrix units in $M_n(\mathbb{C})$ are any $e_{ij}$ for $i, j < n$, where

$$\|e_{ij}\| = 1,\ e_{ij}e_{kl} = \delta_{jk}e_{il} \text{ and } e_{ij}^* = e_{ji}$$

for all $i, j, k,$ and $l$.

The following are facts about finite-dimensional C*-algebras and *-homomorphisms between them that classify them completely. These results also indicate how we can talk about some specific cases instead of all the cases in some statements about finite-dimensional C*-algebras, without loss of generality.

**Fact 2.4.8.** *Every finite-dimensional C*-algebra $A$ is isomorphic to a direct sum of full matrix algebras.*

The following facts work for both the unital and the non-unital cases.

**Fact 2.4.9.** *Every *-homomorphism $\varphi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is uniquely determined up to unitary equivalence by its multiplicity, defined to be the rank of $\varphi(e_{11})$.*

**Fact 2.4.10.** *Every *-homomorphism between finite-dimensional C*-algebras*

$$\varphi : \bigoplus_{j<l'} M_{m(j)}(\mathbb{C}) \to \bigoplus_{j<l} M_{n(j)}(\mathbb{C})$$

*is uniquely determined, up to unitary equivalence, by a matrix $k(i, j)$, for $i < l'$ and $j < l$.*

**Fact 2.4.11.** *The unit ball, any ball around the zero element, in a finite-dimensional C*-algebra, is compact.*

**Theorem 2.4.6.** *[8, Section 5] Assume $\varepsilon < 1$ and $a$ is an element of a finite-dimensional C*-algebra $A$ such that $\|a^*a - 1\| < \varepsilon$. If $a = bu$ is the polar decomposition of $a$, then $u$ is a unitary and $\|a - u\| < \varepsilon$*

**Fact 2.4.12.** *[9, Example 4.1.7] Every tracial state $\tau$ of $\bigoplus_{i<n} M_{n(i)}(\mathbb{C})$ is a convex combination of traces on its direct summands. Let $A = \bigoplus_{i<n} M_{n(i)}$, $B = \bigoplus_{i<k} M_{k(i)}$, $\Psi : A \to B$ be a unital *-homomorphism, and $\tau$ be a tracial state on $B$. Then for $a \in A$ $\sigma(a) := \tau(\Psi(a))$ is a tracial state on $A$.*

## 2.5 Bochner Integral

We will use [4, Appendix E] for most results in this section. Results and concepts from this section are required to prove the technical proposition 3.2.1 in Section 3.2.

For this section, $(X, \Sigma, \mu)$ is a measure space and $E$ is a separable complex Banach space. The space $E$ can also be a real or non-separable Banach space in general. We refer our reader to our source Appendix E in [4] for more information regarding this. We will restrict this section to separable complex Banach spaces for brevity. We will denote the Borel $\sigma$-algebra on $E$ by $\mathcal{B}$. Recall that a map $f : X \to Y$ between measurable spaces is *measurable* if the inverse image of a measurable set $I \subseteq Y$, $f^{-1}(I)$, is also measurable. A map $f : X \to E$ is *Borel measurable* if it is measurable with respect to $\Sigma$ and $\mathcal{B}$. We say $f$ is *simple* if it is finite-valued. If $f : X \to E$ is Borel measurable, then the map $x \mapsto \|f(x)\|_E$ is $\Sigma$-measurable.

**Proposition 2.5.1.** *Let $\{f_n : X \to E\}_{n\in\mathbb{N}}$ be a collection of Borel measurable maps such that for all $x \in X$ $f(x) = \lim_{n\to\infty} f_n(x)$, then $f$ is Borel measurable as well. In other words, the collection of Borel measurable maps from $X$ to $E$ is closed under pointwise limits.*

**Proposition 2.5.2.** *Let $f : X \to E$ be a Borel measurable map; then there is a sequence $\{f_n\}$ of Borel measurable simple functions such that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

*and*

$$\forall x \in X,\ \|f_n(x)\| \leq \|f(x)\|, \text{ for } n = 1, 2, \ldots.$$

**Proposition 2.5.3.** *The set of all Borel measurable maps from $X$ to $E$ forms a vector space.*

We say that a function $f : X \to E$ is *Bochner integrable* (or, if clear by the context, *integrable*) if it is a Borel measurable map and the map $x \mapsto \|f(x)\|$ from $X$ to $\mathbb{R}$ is integrable. We will now define the evaluation of the Bochner integral for simple functions. Then we will extend this definition to 'limits' of simple functions in a standard measure-theoretic fashion.

**Definition 2.5.1.** Assume that $f : X \to E$ is a simple and (Bochner) integrable function, where $f$ can be written as

$$f = \sum_{k=1}^{n} a_k \chi_{A_k}$$

where $A_1, A_2, \ldots, A_n$ are the sets where $f$ takes the non-zero values $a_1, a_2, \ldots, a_n \in E$ respectively. We define the *Bochner integral* of $f$ to be the following

$$\int f(x) d\mu(x) = \sum_{k=1}^{n} a_k \mu(A_k).$$

Since we know that for each $x \in X$ we have $\|a_k \chi_{A_k}(x)\| \leq \|f(x)\|$ we know by the integrability of $f$ that $\mu(A_k)$ must be finite for each $k$, therefore we can say that the previous integral will also be finite and it will make sense. Note that the $x$ in $d\mu(x)$ signifies the variable with respect to which we are integrating. Making this distinction will be necessary for Proposition 3.2.1.

**Proposition 2.5.4.** *Assume that $f : X \to E$ is a simple and (Bochner) integrable function, where $f$ can be written as*

$$f = \sum_{k=1}^{n} a_k \chi_{A_k}.$$

*Then*

$$\left\| \int f(x) d\mu(x) \right\| \leq \int \|f(x)\| d\mu(x).$$

We can also easily obtain the following proposition.

**Proposition 2.5.5.** *If $f$ and $g$ are simple Bochner integrable functions from $X$ to $E$, and $a$ and $b$ are complex numbers, then $af + bg$ is also Bochner integrable.*

Now that we have defined the integral for simple functions, we can expand our definition to all Bochner integrable functions. We know from Proposition 2.5.2 that for

an arbitrary Borel measurable map $f$ there are simple functions $f_n$ approaching $f$ pointwisely and that for each $x \in X$ and $n \in \mathbb{N}$ we have $\|f_n(x)\| \leq \|f(x)\|$. Then, by the continuity of the norm on $E$, and properties of the Lebesgue integral we have that the map $\hat{f}(x) := \sup_{n\in\mathbb{N}} \|f_n(x)\|$ defined from $X$ to $\mathbb{R}$ is integrable. Therefore, we have by the Lebesgue's Dominated Convergence Theorem and Proposition 2.5.4 that

$$\left\{ \int f_n(x) d\mu(x) \right\}$$

is a Cauchy sequence in $E$. Then as $E$ is Banach, $\left\{ \int f_n(x) d\mu(x) \right\}$ must converge to some $I$. We define the Bochner integral of $f$ to be this limit, or in other words

$$\int f_n(x) d\mu(x) = \lim_{n\to\infty} \int f_n(x) d\mu(x).$$

The uniqueness of this integral can be checked easily using standard measure-theoretic methods.

We have the following properties of Bochner Integrals. First, we see that Bochner integrable functions are closed under finite linear combinations and that the integral operates as we would like it to on linear combinations.

**Proposition 2.5.6.** *If $f, g : X \to E$ are Bochner integrable maps and $a, b \in \mathbb{C}$ then $af + bg$ is also integrable, moreover,*

$$\int af(x) + bg(x) d\mu(x) = a \int f(x) d\mu(x) + b \int g(x) d\mu(x).$$

We also have that the analog of Proposition 2.5.4 also holds for all Bochner integrable maps.

**Proposition 2.5.7.** *If $f : X \to E$ is Bochner integrable, then*

$$\left\| \int f(x) d\mu(x) \right\| \leq \|f(x)\| d\mu(x).$$

Finally, we have the analog for the Lebesgue Dominated Convergence Theorem for Bochner integrals.

**Proposition 2.5.8.** *Let $g : X \to [0, +\infty]$ be a Lebesgue integrable map. Suppose that $f$ and $f_1, f_2, \ldots$ are Borel measurable maps from $X$ to $E$ such that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

*and*

$$\|f_n(x)\| \leq g(x)$$

*for* $n = 1, 2, \ldots$ *holds for almost all* $x \in X$*. Then* $f_1, f_2, \ldots$ *are integrable and*

$$\int f(x)d\mu(x) = \lim_{n\to\infty} \int f_n(x)d\mu(x).$$

## 2.6 Groups and their Representations

This section is dedicated to various group theoretic and representation theoretic results we will use in our main results in the thesis.

### 2.6.1 Automatic Continuity in Topological Groups

Automatic continuity is the phenomenon that allows for some 'nice' maps between objects with specific objects to be continuous for free. In this subsection, we shall study an automatic continuity result that we will use in Chapter 3. This section is taken from various parts of [12].

A *topological group* is a group that is also a topological space where the multiplication and inverse functions are continuous.

We recall some fundamental descriptive set theoretic definitions before moving forward. A separable, completely metrizable space is referred to as *Polish*. Note that the bounded closed balls around $0$ in $B(H)$ form a Polish space. A topological space $X$ is called a *Baire space* if every non-empty open set in $X$ is non-meager. There are other equivalent definitions of a Baire space (see [12, Proposition 8.1]). Let $X$ and $Y$ be topological spaces; we say that $f : X \to Y$ is *Baire-measurable* if the inverse image of every open set has the Baire property in $X$. We say that a subset of a Polish space is *meager* if it is a countable union of nowhere dense sets ( i.e., the interior of the closure is empty). A subset of a Polish space has the *Property of Baire* or is *Baire-measurable* if its symmetric difference with some open set is meager.

**Theorem 2.6.1** (Pettis Theorem)**.** *[12, Theorem 9.10] Let* $G_1$ *and* $G_2$ *be topological groups, and let* $f : G_1 \to G_2$ *be a homomorphism. If* $G_1$ *is Baire,* $G_2$ *is separable, and* $f$ *is Baire measurable, then* $f$ *is continuous.*

### 2.6.2 Strongly Continuous One Parameter Group

This subsection is taken from [18, Chapter 5.3].

A *strongly continuous one-parameter unitary group* is a strongly continuous function $U : \mathbb{R} \to U(H)$ ( mapping $t \mapsto U_t$) such that for $s, t \in \mathbb{R}$, $U_{s+t} = U_s U_t$. It is helpful to think of these as an exponential map. Indeed, we have the following result which will further affirm this intuition. We will also use the following theorem in proving Ulam stability in the finite case later.

**Theorem 2.6.2** (Stone's Theorem)**.** *[18, Theorem 5.3.15] If $(U_t)_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group, there is a self-adjoint operator $S$ in $B(H)$ such that $U_t = \exp(itS)$ for all $t$.*

### 2.6.3 Haar Measure

This section is taken from [4, Chapter 9].We have already defined what a topological group is previously; this section will focus on groups equipped with a locally compact topology.

Let $G$ be a locally compact group, and let $\mu$ be a non-zero regular Borel measure on $G$. Then $\mu$ is said to be *left translation invariant* (or *translation invariant*) if for each $A \subseteq G$ and $x \in G$, we have $\mu(xA) = \mu(A)$. We say $\mu$ is a *left Haar measure (or Haar measure)* if it is invariant under left translations (ot translation invariant).

**Example 2.6.1.** The following are some examples relating to the Haar measure.

1. The Lebesgue measure on $\mathbb{R}^n$ for a natural number $n$ is a left and right Haar measure.

2. If $G$ is a group with the discrete topology, then the counting measure on $G$ is a Haar measure.

**Theorem 2.6.3.** *[4, Theorem 9.2.2] Let $G$ be a locally compact group. Then there is a left Haar measure on G.*

**Theorem 2.6.4.** *[4, Theorem 9.2.6] Let $G$ be a locally compact group, and let $\mu$ and*

*$\nu$ be two left Haar measures on $G$. Then one is a positive real constant multiple of the other.*

Note that a corollary of the preceding result is that there is a unique Haar probability measure on a locally compact group.

### 2.6.4 Group Representations and the Peter-Weyl Theorem

This section is taken from Chapter 1 in [13].

Let $G$ be a topological group. A *representation* of $G$ on a complex non-zero Hilbert space is a homomorphism $\pi : G \to GL(H)$, such that the map $\Psi : G \times H \to H$, where $(g, x) \mapsto \pi(g)(x)$ is continuous. An *invariant subspace* for a representation $\pi$ is a vector subspace $H' \subseteq H$ such that for each $g \in G$, $\pi(g)(H') \subseteq H'$. A representation is *irreducible* if it has no other closed invariant subspaces leaving $\{0\}$ and all of $H$ itself. If $\pi(G) \subseteq U(H)$ (the set of unitary operators) then $\pi$ is a *unitary representation*. Two representations of a group $G$ on a Hilbert space $H$, $\pi$ and $\pi'$ are considered *equivalent* if there is an operator $a \in GL(H)$ such that $\pi'(g) = a^*\pi(g)a$. If $a$ is a unitary, they are considered *unitarily equivalent* as representations.

**Proposition 2.6.1** (Schur's Lemma)**.** *Suppose $\pi$ and $\pi'$ are irreducible representations of $G$ on finite-dimensional spaces $V$ and $V'$, respectively. If $L : V \to V'$ is a linear map such that $\pi'(g)L = L\pi(g)$ for all $g \in G$, then $L$ is one-to-one or $L = 0$.*

Let $\{\pi^{(\alpha)}\}$ be a maximal set of mutually inequivalent irreducible unitary representations of $G$. We can choose an orthonormal basis for the underlying vector space and then denote the set of matrices from our representation using $\{\pi^{(\alpha)}_{i,j}\}_{\alpha,i,j}$. The functions $\{\pi^{(\alpha)}_{i,j}\}_{\alpha,i,j}$ form an orthonormal set in $L^2(G)$. The *degree*, $d^{(\alpha)}$, is the dimension of the underlying vector space for the irreducible representation $\pi^{(\alpha)}$.

Let $\pi$ be a unitary representation of a group $G$ on a finite-dimensional Hilbert space. Then the *character* of $\pi$ is the function

$$\chi_\pi(x) = \mathrm{Tr}(\pi(x)) = \sum_i \langle \pi(x)u_i, u_i \rangle.$$

Where $\{u_i\}$ is an orthonormal basis.

**Theorem 2.6.5** (Peter-Weyl Theorem)**.** *[13, Theorem 1.12]*

1. *Let $\pi$ be a unitary representation of $G$ on a Hilbert space $H$. Then $H$ can be given as an orthogonal direct sum $H = \bigoplus H_k$ of finite-dimensional invariant irreducible subspaces.*

2. *Let $\pi$ be a unitary representation of $G$ on a Hilbert space $H$. For each irreducible unitary representation $\pi_k$ of $G$, let $p_k$ be the orthogonal projection on the closure of the sum of all irreducible invariant subspaces of $H$ that are equivalent with $\pi_k$. Then $H_k$ is given by $d_k\overline{\Psi(\chi_k)}$, where $d_k$ is the degree of $\pi_k$ and $\chi_k$ is the character of $\pi_k$. For all $x \in H$ we have*

$$x = \sum_k p_k x,$$

   *where $k$ ranges over the set of representatives of equivalence classes of irreducible unitary representations of $G$.*

Note that in the preceding theorem $\pi$ commutes with the projections $p_k$. This combined with the fact that $1 = \sum p_k$, gives us $\pi_k = \sum p_k \pi p_k$. We shall use this fact in the proof of Theorem 3.5.2.

### 2.6.5 Amenability in Groups

For this section, we rely on [21] and recommend it for further reading. We aim to discuss amenability in this section very briefly as it relates to the thesis. Amenability is a property that helps us avoid paradoxical decompositions. More specifically, one cannot obtain Banach-Tarski-type results using the actions of amenable groups.

Recall that by $L^\infty(G)$ we mean the space of essentially bounded functions over $G$.

**Definition 2.6.1.** Let $G$ be a locally compact group, and let $E$ be a subspace of $L^\infty(G)$ containing the constant functions. A *mean* on $E$ is a functional $m \in E'$ such that $\langle 1, m\rangle = \|m\| = 1$. $E$ is called *left $G$-invariant (or left-invariant* if clear from context) if $\delta_g \cdot \phi \in E$ for all $\phi \in E$ and $g \in G$. If $E$ is left-invariant, then a mean on $E$ is called *left-invariant* if

$$\langle \delta_g \cdot \phi, m\rangle = \langle \phi, m\rangle.$$

**Definition 2.6.2.** A locally compact group $G$ is *amenable* if there is a left-invariant mean on $L^\infty(G)$.

For a compact group $G$, one can define the mean as follows

$$\langle \phi, m \rangle := \int_G \phi(g) dm_G(g),$$

using the Haar measure. Therefore, compact groups are amenable.

**Fact 2.6.1.** *The unitary group of a finite-dimensional C*-algebra with the norm topology is compact, therefore amenable.*

Suppose $G$ is a discrete group. It is *unitary* if its range is a subset of the unitary group of $B(H)$. Note that because of the GNS Theorem (see section 2.4.4) we can have analogous definitions for bounded operators over Hilbert spaces and C*-algebras. We will often use this property of C*-algebras as seen in the next definition.

**Definition 2.6.3.** A representation $\pi : G \to B(H)$ (or $\pi : G \to A$) of a discrete group $G$ into $B(H)$ (or into a C*-algebra $A$) is *unitarizable* if there is $s \in B(H)$ (or if there is $a \in \mathrm{GL}(A)$) such that $s\pi(g)s^{-1}$ (or $g \mapsto a^{-1}\pi(g)a$). for each $g \in G$ is a unitary representation.

We shall use the following theorem in a proof we see later. A sketch of the proof of this theorem is included for a remark.

**Theorem 2.6.6.** *[21, Theorem 1.4.4] Let $G$ be an amenable locally compact group, let $H$ be a Hilbert space, and let $\pi : G \to B(H)$ be a uniformly bounded representation. Then $\pi$ is unitarizable.*

*Proof Sketch.* We first define for $h_1, h_2 \in H$,

$$\eta_{h_1,h_2} : G \to \mathbb{C}$$

as follows

$$\eta_{h_1,h_2}(g) = \langle \pi(g^{-1})h_1, \pi(g^{-1})h_2 \rangle = \langle \pi(g^{-1})^*\pi(g^{-1})h_1, h_2 \rangle.$$

Then it is checked that

$$[h_1, h_2] := \langle \eta_{h_1,h_2}, m \rangle,$$

where $m$ is the left-invariant mean due to amenability, is a positive-definite, sesquilinear form. Then there is an $S \in B(H)$ such that

$$\langle Sh_1, h_2 \rangle = [h_1, h_2].$$

As the sesquilinear form is positive definite we have that $S$ is a positive operator, and thus we define $T := S^{\frac{1}{2}}$. It can then be proved that $T$ has the desired properties. $\square$

**Remark 2.6.1.** We shall use the notation from the preceding proof sketch in this remark. We wish to make a remark about how far away $T$ is from 1. Note that if $\|\pi(g)^*\pi(g) - 1\| < \varepsilon$, then we can observe that

$$|\|\eta_{h,h}(g) - \|h\|^2| < \varepsilon \|h\|^2.$$

As

$$[h, h] = m(\|\pi(g)^{-1}h\|),$$

by the properties of the mean we will get that $|[h, h] - \|h\|^2| \leq \varepsilon \|h\|^2$. This would imply $\|T - 1\| \leq \varepsilon$.

## 2.7 Uniformization

We refer the reader to Subsection 2.6.1 for basic descriptive set theoretic definitions needed for this section. A subset of a Polish space is *analytic* if it is a continuous image of a Borel subset of a Polish space.

We shall now introduce the idea of uniformization. Given two sets $X$ and $Y$ and a subset $P \subseteq X \times Y$, a *uniformization* of $P$ is a subset $P* \subseteq P$ such that for all $x$ in $X$

$$\exists y(x, y) \in P \iff \exists! y(x, y) \in P^*.$$

Notice that this is the same as saying that $P^*$ is a graph of a function with the domain the projection of $P$ onto $X$. The domain is the following set

$$Proj_X(P) := \{x \in X : \exists y \in Y (x, y) \in P\}.$$

We shall refer to such a function as an *uniformizing function* for $P$. The existence of such functions is clear by the axiom of choice; we will typically require these functions to satisfy further properties.

A function $f$ between Polish spaces is *C-measurable* if it is measurable with respect to the smallest $\sigma$-algebra generated by analytic sets.

We will use the following uniformization theorems in the thesis.

**Theorem 2.7.1.** *[12, Theorem 18.1] If $X$ and $Y$ are Polish spaces and $A \subseteq X \times Y$ is analytic, then there is a C-measurable function $f : X \to Y$ such that for all $x \in X$, if $(x, y) \in A$ for some $y$, then $(x, f(x)) \in A$.*

# CHAPTER 3

# EXAMPLES OF ULAM STABILITY

## 3.1 Introduction

This chapter aims to prove the Ulam stability of $(\mathcal{F}, \mathcal{F})$ (as seen in [8]) and the Ulam stability of $(\mathcal{F}, \mathcal{C}^*)$ (as seen in [15]). Section 3.2 will introduce us to a powerful tool that will allow us to perturb our approximate $\varepsilon^*$-homomorphisms to multiplicative $\varepsilon^*$-homomorphisms close by. Section 3.3 will focus on proving an instance of Ulam stability in group representations. This result will be used in the proof of the Ulam stability of $(\mathcal{F}, \mathcal{F})$ case; it is also a good example to observe the techniques used for proving Ulam stability. Section 3.4 and Section 3.5 are dedicated to the proofs of Ulam stability of $(\mathcal{F}, \mathcal{F})$ and the Ulam stability of $(\mathcal{F}, \mathcal{C}^*)$ respectively.

## 3.2 Integral Convergence

The bulk of this chapter is dedicated to proving various instances of Ulam stability. This section is dedicated to a tool used prolifically in all these instances. Most Ulam stability arguments take the original approximate map and find maps with better properties, not too far away from what one started with. Eventually, leading to a morphism within the category. This chapter is dedicated to understanding a technical result, Theorem 3.2.1. This result will allow us to get to a multiplicative map not too far away from our original approximate morphism. The proof of the proposition uses the Bochner integral and Haar measure. We refer the reader to section 2.5 and 2.6.3 in the thesis for details regarding these topics.

**Definition 3.2.1.** Let $G$ be a topological group, and let $\mu$ be the Haar measure on $G$.

Let $A$ be a unital separable Banach algebra. We denote the set of all bounded, Borel measurable maps $f : G \times G \to \mathrm{GL}(A)$ such that

$$\tilde{f} := \int f(u, x) d\mu(x)$$

is also Borel measurable with $\mathcal{D}_{(G,A)}$. When $G$ and $A$ are clear from the context, we just use $\mathcal{D}$.

We first need to prove another lemma that will be needed in the proof of Proposition 3.2.1. We start with the following definition.

**Lemma 3.2.1.** *Assume $G$ is a second-countable group, with the Haar measure $\mu$, and a Banach algebra $A$ is fixed. Then $\mathcal{D}$ contains every bounded, Borel measurable map $f : G \times G \to \mathrm{GL}(A)$.*

*Proof.* First we note that, if $S, T$ are Borel-measurable subsets of $G$, and $a$ is in $\mathrm{GL}(A)$ then $\mathcal{D}$ contains all functions of the form

$$f(u, v) = \begin{cases} a, \text{ if } (u, v) \in S \times T, \\ 0, \text{ otherwise} \end{cases}$$

as

$$\tilde{f}(u) = \int f(u, x) d\mu(x) = \int_{S \times T} a \cdot \chi_S \, d\mu(x).$$

This is also Borel-measurable by simplicity. It is also clear that $\mathcal{D}$ is closed under all finite linear combinations of the functions like $f$. With theorems from section 2.5, we can see that $\mathcal{D}$ contains every bounded Borel measurable map from $G \times G$ to $\mathrm{GL}(A)$. Note that the Borel $\sigma$-algebra on $G \times G$ is generated by Borel rectangles (as $G$ is second countable). □

The following result is an essential piece of machinery needed in the proof of the Ulam stability of $(\mathcal{F}, \mathcal{C}^*)$. One shall note that while the following result refers to the invertible elements of $A$, $\mathrm{GL}(A)$, due to the GNS construction, one can also easily see an analog of the same result over the invertible operators on a Hilbert space.

**Theorem 3.2.1.** *[15, Proposition 2.8] Let $A$ be a unital separable Banach algebra. Let $G$ be a compact, second countable group and $\rho : G \to \mathrm{GL}(A)$ be a Borel measurable map satisfying that for each $u, v \in G$,*

$$\|\rho(u)^{-1}\| \leq \kappa$$

*and*

$$\|\rho(uv) - \rho(u)\rho(v)\| \leq \varepsilon,$$

*where $\kappa$ and $\varepsilon$ are positive constants satisfying $\varepsilon < \kappa^{-2}$. Then there is a Borel-measurable map $\tilde{\rho} : G \to \mathrm{GL}(A)$ such that*

1. *for all $u \in G$,*

$$\|\tilde{\rho}(u) - \rho(u)\| \leq \kappa\varepsilon$$

2. *for all $u \in G$,*

$$\|\tilde{\rho}(u)^{-1}\| \leq \frac{\kappa}{1 - \kappa^2\varepsilon}$$

*and*

3. *for all $u, v \in G$,*

$$\|\tilde{\rho}(uv) - \tilde{\rho}(u)\tilde{\rho}(v)\| \leq 2\kappa^2\varepsilon^2$$

*Proof.* We will define

$$\tilde{\rho}(u) = \int \rho(x)^{-1}\rho(xu)d\mu(x), \tag{3.1}$$

where $\mu$ is the Haar measure on $G$, and the integral is the Bochner integral. Since $\rho(x)^{-1}\rho(xu)$ is Borel measurable. $\tilde{\rho}$ as defined above is Borel measurable.

We will now show the remaining required properties for $\tilde{\rho}$

Note that

$$\begin{aligned}
\|\tilde{\rho}(u) - \rho(u)\| &= \left\| \left( \int \rho(x)^{-1}\rho(xu)d\mu(x) \right) - \rho(u) \right\| \\
&= \left\| \left( \int \rho(x)^{-1}\rho(xu)d\mu(x) \right) - \rho(u)\int d\mu(x) \right\| \\
&= \left\| \left( \int \rho(x)^{-1}\rho(xu)d\mu(x) \right) - \int \rho(u)d\mu(x) \right\| \\
&= \left\| \int \rho(x)^{-1}\rho(xu) - \rho(u)d\mu(x) \right\| \\
&\leq \int \left\| \rho(x)^{-1}\rho(xu) - \rho(u) \right\| d\mu(x) \\
&= \int \left\| \rho(x)^{-1}\left(\rho(xu) - \rho(x)\rho(u)\right) \right\| d\mu(x) \\
&\leq \int \left\| \rho(x)^{-1} \right\| \left\| \left(\rho(xu) - \rho(x)\rho(u)\right) \right\| d\mu(x) \\
&\leq \int \kappa\varepsilon d\mu(x) \\
&= \kappa\varepsilon.
\end{aligned}$$

The second and the last equality use the fact that there is only one Haar measure up to a constant multiple, so we can assume that it is the probability measure (see section 2.6.3). The first inequality uses Proposition 2.5.7. Note also that the invertibility in the range comes from the fact that the range is $\mathrm{GL}(A)$.

We further note that

$$\begin{aligned}
\left\| 1 - \tilde{\rho}(u)\rho(u)^{-1} \right\| &= \left\| (\rho(u) - \tilde{\rho}(u))\rho(u)^{-1} \right\| \\
&\leq \left\| \rho(u) - \tilde{\rho}(u) \right\| \left\| \rho(u)^{-1} \right\| \\
&\leq \kappa^2\varepsilon < \kappa^2\kappa^{-2} = 1.
\end{aligned}$$

Recall $\varepsilon < \kappa^{-2}$ by the given. Then by theorem 2.3.1 we get that $\tilde{\rho}(u)\rho(u)^{-1}$ is invertible. Then there is $a \in \mathrm{GL}(A)$ such that

$$\tilde{\rho}(u)\rho(u)^{-1}a = a\tilde{\rho}(u)\rho(u)^{-1} = 1. \tag{3.2}$$

. Then $\rho(u)^{-1}a$ is the inverse for $\tilde{\rho}(u)$. One side of this is clear by equation 3.2. For the other side,

$$\rho(u)^{-1}a\tilde{\rho}(u) = \rho(u)^{-1}a\tilde{\rho}(u)\rho(u)^{-1}\rho(u) = \rho(u)^{-1}\rho(u) = 1.$$

Therefore $\tilde{\rho}(u)$ is invertible for each $u$. Moreover, again using theorem 2.3.1 we get that

$$a = \sum_{m=0}^{\infty} \left(1 - \tilde{\rho}(u)\rho(u)^{-1}\right)^m.$$

Therefore

$$\tilde{\rho}(u)^{-1} = \rho(u)^{-1} \left( \sum_{m=0}^{\infty} \left(1 - \tilde{\rho}(u)\rho(u)^{-1}\right)^m \right).$$

Recall that $\|1 - \tilde{\rho}(u)\rho(u)^{-1}\| \leq \kappa^2\varepsilon$. By the submultiplicativity of the norm, the triangle inequality, and previously obtained bounds, we get the following

$$\begin{aligned} \left\|\tilde{\rho}(u)^{-1}\right\| &\leq \left\|\rho(u)^{-1}\right\| \left( \sum_{m=0}^{\infty} \left\|1 - \tilde{\rho}(u)\rho(u)^{-1}\right\| \right) \\ &\leq \kappa \left( \sum_{m=0}^{\infty} \kappa^2\varepsilon \right) = \frac{\kappa}{1 - \kappa^2\varepsilon}. \end{aligned}$$

This proves the second condition.

To prove the third condition, we note the following

$$\begin{aligned} \tilde{\rho}(u)\tilde{\rho}(v) &= \left( \int \rho(x)^{-1}\rho(xu)d\mu(x) \right) \left( \int \rho(y)^{-1}\rho(yu)d\mu(y) \right) \quad (3.3) \\ &= \iint \rho(x)^{-1}\rho(xu)\rho(x)^{-1}\rho(xu)d\mu(x)d\mu(y). \quad (3.4) \end{aligned}$$

We can write $\tilde{\rho}(u)\tilde{\rho}(v)$ as we did in equation 3.3 because in the definition for $\tilde{\rho}$ (see equation 3.1) $x$ is a dummy variable. By creating another dummy variable $y$, we can write the product of these integrals as the double integral. Similarly

$$\tilde{\rho}(uv) = \int \rho(x)^{-1}\rho(xuv)d\mu(x) = \iint \rho(x)^{-1}\rho(xuv)d\mu(x)d\mu(y). \quad (3.5)$$

The second equality in the preceding equation holds because the Haar measure on $G$ is a probability measure. Let

$$J := \rho(x)^{-1}\rho(xu)\rho(y)^{-1}\rho(yv) - \rho(x)^{-1}\rho(xuv).$$

Then we have the following,

$$\begin{aligned}
J &= \rho(x)^{-1}\rho(xu)\rho(y)^{-1}\rho(yv) - \rho(x)^{-1}\rho(xuv) + \rho(x)^{-1}\rho(xu)\rho(v) \\
&\quad -\rho(x)^{-1}\rho(xu)\rho(v) + \rho(u)\rho(y)^{-1}\rho(yv) - \rho(u)\rho(y)^{-1}\rho(yv) \\
&\quad -\rho(u)\rho(v) + \rho(u)\rho(v) \\
&= \big(\rho(x)^{-1}\rho(xu)\rho(y)^{-1}\rho(yv) - \rho(x)^{-1}\rho(xu)\rho(v) - \rho(u)\rho(y)^{-1}\rho(yv) + \rho(u)\rho(v)\big) \\
&\quad + \big(\rho(x)^{-1}\rho(xu)\rho(v) + \rho(u)\rho(y)^{-1}\rho(yv) - \rho(u)\rho(v) - \rho(x)^{-1}\rho(xuv)\big) \\
&= \big(\rho(x)^{-1}\rho(xu)\left(\rho(y)^{-1}\rho(yv) - \rho(v)\right) - \rho(u)\left(\rho(y)^{-1}\rho(yv) - \rho(v)\right)\big) \\
&\quad + \big(\rho(x)^{-1}\rho(xu)\rho(v) + \rho(u)\rho(y)^{-1}\rho(yv) - \rho(u)\rho(v) - \rho(x)^{-1}\rho(xuv)\big) \\
&\quad = \big(\rho(x)^{-1}\rho(xu) - \rho(u)\big)\left(\rho(y)^{-1}\rho(yv) - \rho(v)\right) \\
&\quad + \big(\rho(x)^{-1}\rho(xu)\rho(v) + \rho(u)\rho(y)^{-1}\rho(yv) - \rho(u)\rho(v) - \rho(x)^{-1}\rho(xuv)\big)\,.
\end{aligned}$$

Therefore, we let

$$J_1 := \big(\rho(x)^{-1}\rho(xu) - \rho(u)\big)\left(\rho(y)^{-1}\rho(yv) - \rho(v)\right)$$

and

$$J_2 = \big(\rho(x)^{-1}\rho(xu)\rho(v) + \rho(u)\rho(y)^{-1}\rho(yv) - \rho(u)\rho(v) - \rho(x)^{-1}\rho(xuv)\big)\,,$$

and $J = J_1 + J_2$. By equations 3.3 and 3.5 we know that

$$\begin{aligned}
\tilde{\rho}(u)\tilde{\rho}(v) - \tilde{\rho}(uv) &= \iint \rho(x)^{-1}\rho(xu)\rho(x)^{-1}\rho(xu)d\mu(x)d\mu(y) \\
&\quad - \iint \rho(x)^{-1}\rho(xuv)d\mu(x)d\mu(y) \\
&= \iint J d\mu(x)d\mu(y) \\
&= \iint J_1 d\mu(x)d\mu(y) + \iint J_2 d\mu(x)d\mu(y).
\end{aligned}$$

Let

$$I_1 := \iint J_1 d\mu(x)d\mu(y), \text{ and } I_2 := \iint J_2 d\mu(x)d\mu(y).$$

Then we have

$$\begin{aligned}
\|I_1\| &= \left\| \iint \left(\rho(x)^{-1}\rho(xu) - \rho(u)\right)\left(\rho(y)^{-1}\rho(yv) - \rho(v)\right) d\mu(x)d\mu(y) \right\| \\
&= \left\| \iint \left(\rho(x)^{-1}(\rho(xu) - \rho(x)\rho(u))\right)\left(\rho(y)^{-1}(\rho(yv) - \rho(y)\rho(v))\right) d\mu(x)d\mu(y) \right\| \\
&\leq \iint \left\| \rho(x)^{-1}(\rho(xu) - \rho(x)\rho(u))\rho(y)^{-1}(\rho(yv) - \rho(y)\rho(v)) \right\| d\mu(x)d\mu(y) \\
&\leq \iint \|\rho(x)^{-1}\| \, \|\rho(xu) - \rho(x)\rho(u)\| \, \|\rho(y)^{-1}\| \, \|\rho(yv) - \rho(y)\rho(v))\| \, d\mu(x)d\mu(y) \\
&\leq \kappa^2\varepsilon^2.
\end{aligned}$$

Now, for $I_2$ we will have the following,

$$\begin{aligned}
I_2 &= \iint \left(\rho(x)^{-1}\rho(xu)\rho(v) + \rho(u)\rho(y)^{-1}\rho(yv) - \rho(u)\rho(v) - \rho(x)^{-1}\rho(xuv)\right) d\mu(x)d\mu(y) \\
&= \iint \rho(x)^{-1}\left(\rho(xu)\rho(v) - \rho(xuv)\right) d\mu(x)d\mu(y) \\
&- \iint \left(\rho(u)\rho(v) - \rho(u)\rho(y)^{-1}\rho(yv)\right) d\mu(x)d\mu(y) \\
&= \int \rho(x)^{-1}\left(\rho(xu)\rho(v) - \rho(xuv)\right) d\mu(x) \\
&- \int \left(\rho(u)\rho(v) - \rho(u)\rho(y)^{-1}\rho(yv)\right) d\mu(y).
\end{aligned}$$

Recall that by the translation invariance of the Haar measure, we can multiply wherever we have $xu$ with $u^{-1}$ on the right in the first integral. It won't change the value of the integral. Therefore, we have

$$\begin{aligned}
I_2 &= \int \rho(xu^{-1})^{-1}\left(\rho(x)\rho(v) - \rho(xv)\right) d\mu(x) \\
&- \int \left(\rho(u)\rho(v) - \rho(u)\rho(y)^{-1}\rho(yv)\right) d\mu(y).
\end{aligned}$$

Since $x$ and $y$ are dummy variables, we can rewrite this integral as

$$\begin{aligned}
I_2 &= \int \rho(xu^{-1})^{-1}\left(\rho(x)\rho(v) - \rho(xv)\right) d\mu(x) \\
&- \int \left(\rho(u)\rho(v) - \rho(u)\rho(x)^{-1}\rho(xv)\right) d\mu(x) \\
&= \int \rho(xu^{-1})^{-1}\left(\rho(x)\rho(v) - \rho(xv)\right) - \left(\rho(u)\rho(v) - \rho(u)\rho(x)^{-1}\rho(xv)\right) d\mu(x) \\
&= \int \rho(xu^{-1})^{-1}\left(\rho(x)\rho(v) - \rho(xv)\right) - \left(\rho(u)\rho(x)^{-1}(\rho(x)\rho(v) - \rho(xv))\right) d\mu(x) \\
&= \int \left(\rho(xu^{-1})^{-1} - \rho(u)\rho(x)^{-1}\right)\left(\rho(x)\rho(v) - \rho(xv)\right) d\mu(x).
\end{aligned}$$

We further note that,

$$\begin{aligned}
\|\rho(xu^{-1})^{-1} - \rho(u)\rho(x)^{-1}\| &= \|\rho(xu^{-1})\left(1 - \rho(xu^{-1})\rho(u)\rho(x)^{-1}\right)\| \\
&= \|\rho(xu^{-1})^{-1}\left(\rho(x) - \rho(xu^{-1})\rho(u)\right)\rho(x)^{-1}\| \\
&= \|\rho(xu^{-1})^{-1}\left(\rho(xu^{-1}u) - \rho(xu^{-1})\rho(u)\right)\rho(x)^{-1}\| \\
&\leq \|\rho(xu^{-1})^{-1}\|\|\rho(xu^{-1}u) - \rho(xu^{-1})\rho(u)\|\|\rho(x)^{-1}\| \\
&\leq \kappa^2\varepsilon,
\end{aligned}$$

by submultiplicativity and the given bounds. Then finally we have that

$$\begin{aligned}
\|I_2\| &= \left\|\int \left(\rho(xu^{-1})^{-1} - \rho(u)\rho(x)^{-1}\right)\left(\rho(x)\rho(v) - \rho(xv)\right) d\mu(x)\right\| \\
&\leq \int \left\|\left(\rho(xu^{-1})^{-1} - \rho(u)\rho(x)^{-1}\right)\left(\rho(x)\rho(v) - \rho(xv)\right)\right\| d\mu(x) \\
&\leq \int \left\|\left(\rho(xu^{-1})^{-1} - \rho(u)\rho(x)^{-1}\right)\right\| \|\left(\rho(x)\rho(v) - \rho(xv)\right)\| d\mu(x) \\
&\leq \kappa^2\varepsilon.
\end{aligned}$$

Therefore

$$\|\tilde{\rho}(u)\tilde{\rho}(v) - \tilde{\rho}(uv)\| = \|I\| \leq \|I_1\| + \|I_2\| \leq 2\kappa^2\varepsilon.$$

Which is our third condition. □

This theorem provides a powerful tool to get a series of approximate morphisms (we use the term in its generality because of the applicability of this result). If our conditions are met, we can reach a multiplicative map with desirable properties not too far away.

### 3.3 An Example of Ulam Stability in Group Representations

We will need the following definitions and results from [11] to prove the main result, Theorem 3.4.1. In this section, $V$ shall usually denote a Banach space, unless stated otherwise.

We start by defining $\varepsilon$-homomorphsism for the group again. Note that this definition is slightly different from the one seen in Chapter 1.

**Definition 3.3.1.** Let $U$ be a topological group and $d$ be a right-invariant metric on $U$ that induces the topology on $U$. Let $G$ be a topological group, and $\rho : G \to U$ be a continuous map, and $\varepsilon$ be a positive real number. We say that $\rho$ is an $\varepsilon$-homomorphism if $d_U(\rho(gg'), \rho(g)\rho(g')) \leq \frac{\varepsilon}{2}$ for all $g, g' \in G$.

We need a few preliminary definitions for this section.

Note that by $\mathrm{Ad}$ is a group action of the group of unitary elements of $B(H)$ on $B(H)$ by conjugation, defined by $(Ad(u))(a) = uau^{-1}$.

Let $\rho$ be an $\varepsilon$-representation. Let $B_\rho$ be the Banach space of bounded continuous function $f : G \to V$ with norm $\|f\| := \sup_{g\in G} \|f(g)\|$. We shall further introduce $r : G \to U(B_\rho)$ the action of $G$ on $B_\rho$ by right shifts. That is, $g \in G$ is mapped to the element $r(g)$ of $U(B_\rho)$ that, for every $f \in B_\rho$ maps $r(g) : f \mapsto f_g$. If $f : x \mapsto fx$, then $f_g : x \mapsto f(xg)$. We will also define the imbedding map $i : B_\rho \to V$ given by $i(v)(g) = \rho(g)v$ for all $v \in V, g \in G$.

We will also denote the Banach space of continuous bounded functions $c$ from $G^k$ to $V$ by $C^k_\rho$, with the norm

$$|c| = \sup_{g_1,\ldots,g_k\in G} \|c(g_1, g_2, \ldots, g_k)\|.$$

Note that $d^k_\rho : C^k_\rho \to C^{k+1}_\rho$ denotes the linear map given by

$$(d^k_\rho(c))(g_1, \ldots, g_{k+1}) = \rho(g_1)c(g_2, \ldots, g_{k+1}) + \sum_{i=1}(-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) \\ +(-1)^{k+1}c(g_1, \ldots, g_k).$$

For readability we shall add the explicit expressions for $d^1_\rho : C^1_\rho \to C^2_\rho$

$$(d^1_\rho(c))(g_1, g_2) : \rho(g_1)c(g_2) - c(g_1g_2) + c(g_2),$$

and $d^2_\rho : C^2_\rho \to C^3_\rho$

$$(d^2_\rho(c))(g_1, g_2, g_3) = \rho(g_1)c(g_2, g_3) - c(g_1g_2, g_3) + c(g_1, g_2g_3) - c(g_1, g_2).$$

**Definition 3.3.2.** An $\varepsilon$-representation $\rho$ is *$\varepsilon$-acyclic* if for any $c \in C^k_\rho$ there is $b \in C^{k-1}_\rho$ such that $\|b\| \leq \|c\|$, and $\|d^{k-1}_\rho b - c\| \leq \varepsilon\|c\| + \|d^k_\rho c\|$.

**Definition 3.3.3.** A linear map $I : B_\rho \to V$ is an $\varepsilon$-mean if $\|I\| = 1$, $I \circ i = \mathrm{id}$, and $\|I \circ r(g) - \rho(g) \circ I\| \leq \varepsilon$ for every $g \in G$.

We have the following lemmas to be used in the main proposition for this section; we are going to skip the proof of one of these lemmas for brevity.

**Lemma 3.3.1.** *[11, Lemma 1] If there is an $\varepsilon$-mean $I$ on $B_\rho$, then $\rho$ is $\varepsilon$-acyclic.*

**Lemma 3.3.2.** *[11, Lemma 2] Let $G$ be an amenable group, let $\rho = \mathrm{Ad} \circ \tilde{\rho}$ for a $\varepsilon$-representation $\tilde{\rho}$ of $G$ on $H$, then there is an $\varepsilon$-mean on $B_\rho$.*

*Proof.* For any $f \in B_\rho$, we shall define a bilinear form $\tilde{I}(f) : H \times H \to \mathbb{K}$. By the definition of amenability, we know that there is a $G$-invariant linear functional $m \in L^\infty(G)$ such that $m(1) = 1$ and $\|m\| = 1$. We define

$$\tilde{I}(f)(h_1, h_2) := m(a_{h_1,h_2})$$

where

$$a_{h_1,h_2}(g) := \langle f(g)\tilde{\rho}(g)^{-1}h_1, \tilde{\rho}(g)^{-1}h_2\rangle.$$

We have that for any $h_1, h_2 \in H$,

$$\left|\tilde{I}(f)(h_1, h_2)\right| \leq \|f\|\|h_1\|\|h_2\|.$$

Therefore we get that $\tilde{I}(f)$ defines a bounded linear operator $I(f)$ on $H$ by the Riesz Representation Theorem such that

$$\tilde{I}(f)(h_1, h_2) = \langle I(f)h_1, h_2\rangle.$$

Moreover, we have that $\|I(f)\| \leq \|f\|$. It can then be proved that $I$ is indeed an $\varepsilon$-mean. □

We shall need the following theorem for the main result of this section.

**Theorem 3.3.1.** *[11, Theorem 1] Let $G$ be an amenable group and $\tilde{\rho} : G \to U$ be an $\varepsilon$-representation fo $G$ into the group $U$ of unitaries of $B(H)$ for some Hilbert space $H$, for $\varepsilon < \frac{1}{100}$. Then there is a representation $\pi : G \to U$ such that $\|\tilde{\rho}(g) - \pi(g)\| \leq \varepsilon$ for each $g \in G$.*

The proof of Theorem 3.3.1 uses the following proposition, which gives us a better approximate representation $\tilde{\rho}_1$, not too far away from $\tilde{\rho}$.

For the following lemma, for $u \in U$, where $\|u-1\| < 1$, we define, the *natural log* of $u$, denoted $\ln u$ to be

$$\ln u = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(u-1)^i}{i}.$$

The reader should note that for the adjoint of an element $a \in U$ can move inside $\ln$ for elementary reasons, that is

$$a^{-1} \ln(u) a = \ln(aua^{-1}).$$

**Lemma 3.3.3.** *Let $u, u' \in U$ be such that $\|u-1\|$, $\|u'-1\| \leq \frac{\varepsilon}{2}$. Then*

$$\| \ln(uu') - (\ln u + \ln u') \| \leq \frac{\varepsilon^2}{2}$$

*for $\varepsilon < 1/100$.*

The following proposition is the main tool that gives us Theorem 3.3.1.

**Proposition 3.3.1.** *If $\tilde{\rho}$ satisfies the condition of Theorem 3.3.1, then there is an $\varepsilon_1$-representation $\tilde{\rho}_1 : G \to U$ for $\varepsilon_1 = 5\varepsilon^2$ such that $\|\tilde{\rho}(g) - \tilde{\rho}_1(g)\| \leq \varepsilon/2 + \varepsilon^2$.*

*Proof of Proposition 3.3.1.* We again consider $B(H)$ our space of bounded linear operators on $H$. Let

$$\rho = \mathrm{Ad} \circ \tilde{\rho}.$$

This is another $\varepsilon$-representation. Then for any $g, g' \in G$ we further define $W : G \times G \to U(H)$. Here, $W(g, g')$ is the unitary element such that

$$\tilde{\rho}(g)\tilde{\rho}(g') = W(g, g')\tilde{\rho}(gg'),$$

moreover from the same we have

$$\tilde{\rho}(g')^{-1}\tilde{\rho}(g)^{-1} W(g, g') = \tilde{\rho}(gg')^{-1}.$$

Then we have the following

$$\begin{aligned} \|W(g, g') - \mathrm{id}\| &= \|\tilde{\rho}(g)\tilde{\rho}(g')(\tilde{\rho}(gg'))^{-1} - \mathrm{id}\| \\ &\leq \|\tilde{\rho}(gg')\| \|\tilde{\rho}(g)\tilde{\rho}(g') - \tilde{\rho}(gg')\| \leq \frac{\varepsilon}{2}. \end{aligned}$$

Note that the preceding inequality guarantees it makes sense to use $\ln W(g, g')$.

The triple product $\tilde{\rho}(g_1)\tilde{\rho}(g_2)\tilde{\rho}(g_3)$ can be given in the following two ways. Firstly,

$$\begin{aligned}\tilde{\rho}(g_1)\tilde{\rho}(g_2)\tilde{\rho}(g_3) &= W(g_1, g_2)\tilde{\rho}(g_1g_2)\tilde{\rho}(g_3)\\ &= W(g_1, g_2)W(g_1g_2, g_3)\tilde{\rho}(g_1g_2g_3)\end{aligned}$$

and also,

$$\begin{aligned}\tilde{\rho}(g_1)\tilde{\rho}(g_2)\tilde{\rho}(g_3) &= \tilde{\rho}(g_1)W(g_2, g_3)\tilde{\rho}(g_2g_3)\\ &= \tilde{\rho}(g_1)W(g_2, g_3)\tilde{\rho}(g_1)^{-1}\tilde{\rho}(g_1)\tilde{\rho}(g_2g_3)\\ &= \rho(g_1)W(g_2, g_3)W(g_1, g_2g_3)\tilde{\rho}(g_1g_2g_3).\end{aligned}$$

These give us the following equality after cancellation,

$$W(g_1, g_2)W(g_1g_2, g_3) = \rho(g_1)(W(g_2.g_3))W(g_1, g_2g_3).$$

We define an element $\alpha$ in $C^2_\rho$ by $\alpha(g, g') = \ln W(g, g')$.

We are using the $\sup$ metric on $C^2_\rho$. Therefore, we have the following (using the infinite sum with $\|W(g', g) - \mathrm{id}\| \leq \frac{\varepsilon}{2}$),

$$\begin{aligned}|\alpha| &= \sup_{g,g'\in G} \|\ln W(g, g')\|\\ &= \sup_{g,g'\in G} \left\| \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(W(g, g') - \mathrm{id})^i}{i} \right\|\\ &\leq \sup_{g,g'\in G} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\|(W(g, g')) - \mathrm{id}\|^i}{i} \leq \frac{\varepsilon^2}{2} + \frac{\varepsilon}{2}.\end{aligned}$$

Recall that $\rho(g) = \mathrm{Ad} \circ \tilde{\rho}(g)$, and therefore it can move inside the logarithm (using

the Taylor expansion). Now we notice that

$$\begin{aligned}
\|d^2_\rho\alpha\| &= \sup\|(d^2_\rho\alpha)(g_1,g_2,g_3)\| \\
&= \sup\|\rho(g_1)\alpha(g_2,g_3) - \alpha(g_1g_2,g_3) + \alpha(g_1,g_2g_3) - \alpha(g_1,g_2)\| \\
&= \sup\|\rho(g_1)\ln W(g_2,g_3) - \ln W(g_1g_2,g_3) + \ln W(g_1,g_2g_3) - \ln W(g_1,g_2)\| \\
&= \sup\|\ln(\rho(g_1)W(g_2,g_3)) + \ln W(g_1,g_2g_3) - (\ln W(g_1,g_2) + \ln W(g_1g_2,g_3))\| \\
&= \sup \|\ln(\rho(g_1)W(g_2,g_3)) + \ln W(g_1,g_2g_3) - \ln(\rho(g_1)W(g_2,g_3)W(g_1,g_2g_3)) \\
&\qquad + \ln(\rho(g_1)W(g_2,g_3)W(g_1,g_2g_3)) \\
&\qquad + \ln(W(g_1,g_2)W(g_1g_2,g_3)) - \ln W(g_1,g_2) - \ln W(g_1g_2,g_3) \\
&\qquad - \ln(W(g_1,g_2)W(g_1g_2,g_3))\| \\
&\leq \sup\left(\|\ln(\rho(g_1)W(g_2,g_3)) + \ln W(g_1,g_2g_3) - \ln(\rho(g_1)W(g_2,g_3)W(g_1,g_2g_3))\|\right. \\
&\qquad + \|\ln(\rho(g_1)W(g_2,g_3)W(g_1,g_2g_3)) - \ln(W(g_1,g_2)W(g_1g_2,g_3))\| \\
&\qquad \left. + \|\ln(W(g_1,g_2)W(g_1g_2,g_3)) - \ln W(g_1,g_2) - \ln W(g_1g_2,g_3)\|\right) \\
&\leq \frac{\varepsilon^2}{2} + 0 + \frac{\varepsilon^2}{2} = \varepsilon^2.
\end{aligned}$$

Then we know by Lemma 3.3.2 we know that there is $b$ in $C^1_\rho$ such that $\|b\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2}$ and by Lemma 3.3.1 we have $\|d^1_\rho b - \alpha\| \leq 2\varepsilon^2$. Both $\alpha(g,g')$ and $b(g)$ are skew-hermitian operators. We shall define the map $\tilde{\rho}_1 : G \to U$ by $\tilde{\rho}_1 = \exp(b(g))\tilde{\rho}(g)$. This map is exactly what we need. First, we observe the following

$$\begin{aligned}
&\tilde{\rho}_1(gg')^{-1}\tilde{\rho}_1(g)\tilde{\rho}_1(g') \\
&= \tilde{\rho}(gg')^{-1}\exp(-b(gg'))\exp(b(g))\tilde{\rho}(g)\exp(b(g'))\tilde{\rho}(g') \\
&= \tilde{\rho}(g')^{-1}\tilde{\rho}(g)^{-1}W(g,g')\exp(-b(gg'))\exp(b(g))\tilde{\rho}(g)\exp(b(g'))\tilde{\rho}(g') \\
&= \tilde{\rho}(g')^{-1}\tilde{\rho}(g)^{-1}\exp(\alpha(g,g'))\exp(-b(gg'))\exp(b(g))\tilde{\rho}(g)\exp(b(g'))\tilde{\rho}(g') \\
&= \tilde{\rho}(g')^{-1}\tilde{\rho}(g)^{-1}\exp(\alpha(g,g'))\exp(-b(gg'))\exp(b(g)) \\
&\qquad \tilde{\rho}(g)\exp(b(g'))\tilde{\rho}(g)^{-1}\tilde{\rho}(g)\tilde{\rho}(g') \\
&= \tilde{\rho}(g')^{-1}\tilde{\rho}(g)^{-1}\exp(\alpha(g,g'))\exp(-b(gg'))\exp(b(g))\rho(g)(\exp(b(g')))\tilde{\rho}(g)\tilde{\rho}(g') \\
&= \tilde{\rho}(g')^{-1}\tilde{\rho}(g)^{-1}\exp(\alpha(g,g'))\exp(-b(gg'))\exp(b(g))\exp(\rho(g)b(g'))\tilde{\rho}(g)\tilde{\rho}(g').
\end{aligned}$$

Using the preceding equality and the Taylor expansion again, it can be shown that

$$\|\tilde{\rho}_1(gg') - \tilde{\rho}_1(g)\tilde{\rho}_1(g')\| \leq 5\varepsilon^2. \qquad \square$$

The preceding proposition was the main machinery needed for us to prove the main theorem.

*Proof of Theorem 3.3.1.* Let $\tilde{\rho}$ be as in the statement, and $\varepsilon$-representation of $G$ on $U$. We define the sequence $\varepsilon_n$ as follows, $\varepsilon_0 = \varepsilon, \varepsilon_n = 5\varepsilon_{n-1}^2$. We can find $\varepsilon_n$-representations $\tilde{\rho}_n$'s by applying Proposition 3.3.1 repeatedly. The reader shall note that, for each $n$, we have $\varepsilon_n \leq \varepsilon/3^n$ and also that for each $n > 1$ and $g \in G$, $\|\tilde{\rho}_n(g) - \tilde{\rho}_{n-1}(g)\| \leq \varepsilon/3^n$. We define

$$\pi(g) := \lim_{n\to\infty} \tilde{\rho}_n(g).$$

Then $\pi$ is a representation, and for stability, we can see

$$\|\pi(g) - \tilde{\rho}(g)\| \leq \sum_{n=1}^{\infty} \|\tilde{\rho}_n(g) - \tilde{\rho}_{n-1}(g)\| \leq \varepsilon. \qquad \square$$

The reader shall note that this technique of applying a proposition to incrementally obtain a 'more multiplicative' map limiting to a multiplicative map shall be used later with the integral convergence theorem we saw earlier. Indeed, the integral convergence technique is also discussed in [11].

## 3.4 Ulam Stability of ($\mathcal{F}$, $\mathcal{F}$)

This section aims to prove the Ulam stability of finite-dimensional C*-algebras. This section was follows [8], however, at times [9] is also frequently used for reference.

### 3.4.1 Preliminaries

The literature does not agree on the following definition. Therefore, we will update the definitions in different sections and then provide remarks on their equivalence. We start by defining an $\varepsilon$-approximate-*-homomorphism for this section.

**Definition 3.4.1.** Assume $A$ and $B$ are C*-algebras. A map $\phi : A \to B$ is an $\varepsilon$-approximate-$*$-homomorphism if for all $a, b \in A_{\leq 1}$ we have the following:

1. $\|\phi(ab) - \phi(a)\phi(b)\| < \varepsilon$,

2. $\|\phi(a+b) - \phi(a) - \phi(b)\| < \varepsilon$,

3. $\|\phi(a^*) - \phi(a)^*\| < \varepsilon$,

4. $|\|a\| - \|\phi(a)\|| < \varepsilon$.

Note that the definition gives the following property that we will use repeatedly.

**Lemma 3.4.1.** *For each $a \in A_{\leq 1}$ we have $\|\phi(a)\| \leq \varepsilon + 1$*

*Proof.* Observe the following

$$\|\phi(a)\| = \|\phi(a)\| - \|a\| + \|a\| \leq |\|\phi(a)\| - \|a\|| + \|a\| \leq \varepsilon + 1 \qquad \square$$

The main aim of this section is to prove the following, which proves Ulam stability in finite-dimensional C*-algebras.

**Theorem 3.4.1.** *[8, Theorem 5.1] There is a universal constant $K < \infty$ such that the following holds. If $\varepsilon < 1/1000$, $A$ is a finite-dimensional C*-algebra, $m \in \mathbb{N}$, and $\phi : A \to M_m(\mathbb{C})$ is a Borel-measurable unital $\varepsilon$-approximate-*-homomorphism, then $\phi$ can be $K\varepsilon$-approximated by a unital-*-homomorphism.*

### 3.4.2 From Unitary Group Homomorphisms to *-Homomorphisms

We shall also prove the following lemma, which will be a key part of the main proof later on. This specific lemma was taken from [9] instead of [8]. Refer to Section 2.4.3 for definitions about states used in this lemma.

**Lemma 3.4.2.** *[9, Lemma 17.2.4] Let $A$ and $B$ be unital C*-algebras and let $\theta : U(A) \to U(B)$ be a group homomorphism. If $A$ has a faithful tracial state $\tau_A$, and $B$ has a faithful tracial state $\tau_B$, and for each $u \in U(A)$, $\tau_B(\theta(u)) = \tau_A(u)$, then $\theta$ uniquely extends to a *-homomorphism between $A$ and $B$.*

*Proof.* Recall that every element of a C*-algebra can be written as a sum of four unitaries (see fact 2.4.2). Therefore, we shall construct our *-homomorphism defined over a linear combination of unitaries. We claim that

$$\varphi\left(\sum_{j<n} \lambda_j u_j\right) := \sum_{j<n} \lambda_j \theta(u_j)$$

is a well-defined, unital *-homomorphism from $A$ to $B$. The unital part is clear as $\theta$ is a group homomorphism over the unitaries. If well-defined, the map $\varphi$ will be linear and preserve the adjointness by the linear definition of the map and the group homomorphism properties.

We only need to check the well-definedness of the map. It will be enough for us to show that $\sum_{j<n} \lambda_j u_j = 0$ if and only if $\varphi\left(\sum_{j<n} \lambda_j u_j\right) = 0$. We know that $\sum_{j<n} \lambda_j u_j = 0$ if and only if

$$\left(\sum_{j<n} \lambda_j u_j\right)^* \left(\sum_{j<n} \lambda_j u_j\right) = 0.$$

But we also know that

$$\left(\sum_{j<n} \lambda_j u_j\right)^* \left(\sum_{j<n} \lambda_j u_j\right) = \left(\sum_{j<n} \overline{\lambda_j} u_j^*\right) \left(\sum_{j<n} \lambda_j u_j\right) = \sum_{j,k<n} \overline{\lambda_j} \lambda_k u j^* u_k.$$

Moreover, we have by faithfulness of $\tau_A$

$$\begin{aligned} \tau_A\left(\left(\sum_{j<n} \lambda_j u_j\right)^* \left(\sum_{j<n} \lambda_j u_j\right)\right) &= \tau_A\left(\sum_{j,k<n} \overline{\lambda_j} \lambda_k u_j^* u_k.\right) \\ &= \sum_{j,k<n} \overline{\lambda_j} \lambda_k \tau_A\left(u_j^* u_k\right) = 0. \end{aligned}$$

Similarly, as we know that $\tau_B$ is faithful as well, we have $\sum_{j<n} \lambda_j \theta(u_j) = 0$ if and only if

$$\sum_{j,k<n} \overline{\lambda_j} \lambda_k \tau_B(\theta\left(u_j^* u_k\right)) = 0.$$

But by assumption we know that $\tau_A(u_j^* u_k) = \tau_B(\theta(u_j^* u_K)$, therefore we are done. □

### 3.4.3 Connecting the Dots

*Proof of Ulam Stability of $(\mathcal{F}, \mathcal{F})$.* Let $B = M_m(\mathbb{C})$ and consider its representation on the $m$-dimensional Hilbert space $K$. We shall denote $a \approx_\delta b$ if $\|a - b\| < \delta$. We can see that if $a_1 \approx_{\delta_1} a_2 \approx_{\delta_2} \ldots \approx_{\delta_{n-1}} a_n$ then $a_1 \approx_{\sum_{i=1}^n \delta_i} a_n$ as

$$\|a_1 - a_n\| \leq \|a_1 - a_2\| + \|a_2 - a_3\| + \ldots + \|a_{n-1} - a_n\| \leq \sum_{i=1}^{n} \delta_i.$$

We fix a unitary $u$ in $A$. let $a = \phi(u)$. Then

$$\begin{aligned}\|aa^* - \phi(u)\phi(u^*)\| &= \|\phi(u)\phi(u)^* - \phi(u)\phi(u^*)\| \\ &\leq \|\phi(u)\|\|a^* - \phi(u^*)\| \\ &\leq |\|\phi(u)\| - \|u\| + \|u\|\|\|a^* - \phi(u^*)\| \\ &\leq (|\|\phi(u)\| - \|u\|| + \|u\|)\|a^* - \phi(u^*)\| \\ &\leq (1+\varepsilon)\varepsilon.\end{aligned}$$

Therefore we have that $aa^* \approx_{(1+\varepsilon)\varepsilon} \phi(u)\phi(u^*)$, and we know by the properties of $\varepsilon^*$-homomorphism that $\phi(u)\phi(u^*) \approx_\varepsilon \phi(uu^*)$, then $aa^* \approx_{\varepsilon^2+2\varepsilon} \phi(uu^*) = I$ as $\phi$ is unital. Similarly $a^*a \approx_{\varepsilon^2+2\varepsilon} I$. By Theorem 2.4.6, mentioned in the prerequisites chapter, we have that there is a unitary $v \in B$ such that $\|\phi(u) - v\| < 2\varepsilon + \varepsilon^2 = \varepsilon_1$.

Now let $U(A)$ and $U(B)$ be the set of unitaries of $A$ and $B$ respectively. We define $X \subseteq U(A) \times U(B)$ as follows

$$X = \{(u, v) \in U(A) \times U(B) : \|\phi(u) - v\| < \varepsilon_1\}.$$

As $\|\cdot\|$ is continuous and $\phi$ is a Borel map, we can say that $X$ is too. Then by Theorem 2.7.1 we know there is a C-measurable map $\phi' : U(A) \to U(B)$ such that if $(u, v) \in X$ then $(u, \phi'(u)) \in X$ as well. By definition of $X$ we know that $\|\phi(u) - \phi'(u)\| < \varepsilon_1$ for each $u \in U(A)$.

Notice that for all $u, v \in U(A)$ we have

$$\begin{aligned}\|\phi'(u)\phi'(v) - \phi(u)\phi(v)\| &= \|\phi'(u)\phi'(v) - \phi'(u)\phi(v) + \phi'(u)\phi(v) - \phi(u)\phi(v)\| \\ &\leq \|\phi'(u)\|\|\phi'(v) - \phi(v)\| + \|\phi'(u) - \phi(u)\|\|\phi(v)\| \\ &\leq \varepsilon(\|\phi'(u)\| + \|\phi(v)\|) \\ &= \varepsilon(\|\phi'(u) - \phi(u) + \phi(u)\| + \|\phi(v)\|) \\ &\leq \varepsilon(\|\phi'(u) - \phi(u)\| + \|\phi(u)\| + \|\phi(v)\|) \\ &\leq \varepsilon(\varepsilon_1 + 2 + 2\varepsilon)\end{aligned}$$

Moreover we know that $\phi(u)\phi(v) \approx_\varepsilon \phi(uv) \approx_{\varepsilon_1} \phi'(uv)$ by the definition of an $\varepsilon^*$-homomorphism and of $\phi'$. Then we can say that

$$\|\phi'(uv) - \phi'(u)\phi'(v)\| < \varepsilon(\varepsilon_1 + 2 + 2\varepsilon) + \varepsilon + \varepsilon_1 =: \varepsilon_2,$$

and therefore $\phi'$ is a $2\varepsilon_2$ representation of $U(A)$ on $K$. It was proven that if $\delta < 1/100$, then every strongly continuous $2\delta$-representation $\rho$ of a compact group can be $2\delta$-approximated by a (strongly continuous) representation $\rho'$ by Theorem 3.3.1.

Let $\theta : U(A) \to U(B)$ be the continuous homomorphism as described by the previous procedure. We also know that $\|\phi' - \theta\| < 2\varepsilon_2$ on $U(A)$ by our construction, then for all unitary $u$ we have

$$\theta(u) - \phi(u) < 2\varepsilon_2 + \varepsilon_1 =: \varepsilon_3.$$

Let $a \in A$ be self-adjoint; the map $f : \mathbb{R} \to U(K)$ given by $f(r) = e^{ira}$ defines a norm-continuous one-parameter group. By Stone's Theorem (see Theorem 2.6.2) we know that there is a unique $\rho(a) \in U(K)$ such that

$$\theta(\exp(ira)) = \exp(ir\rho(a)).$$

As $B$ is a finite dimensional C*-algebra, we know it is also a von Neumann algebra, therefore, we can say that $\rho(a) \in B$. Note that in this proof we will use $I$ to represent the identity instead of $1$ to differentiate from the complex number $1$.

**Claim 3.4.0.1.** $\rho(\mathrm{id}) = \mathrm{id}$*, hence for each* $r \in \mathbb{R}$ *we have* $\theta(e^{ir}\mathrm{id}) = e^{ir}\mathrm{id}$.

*Proof of Claim* . Assume toward the contradiction that $\rho(\mathrm{I}) = b \neq \mathrm{I}$. As $b$ is self-adjoint, we know that the spectral radius of $b$ is equal to the norm of $b$ and that the spectrum of $b$ is a subset of the reals. If $\sigma(b) = \{1\}$ then $b = \mathrm{id}$, as $b$ is unitary the functional calculus gives us this result. Therefore, there must be $s \neq 1$ in $\sigma(b)$. Then we can choose $r \in \mathbb{R}$ in a way that $r(1-s) = \pi + 2k\pi$ for some $k \in \mathbb{N}$. Note that in this case $e^{ir(1-s)} = -1$. Using Fact 2.4.3 we get

$$\begin{aligned}
\|\exp(ir)I - \exp(irb)\| &\geq d_H(\sigma(\exp(ir)I), \sigma(\exp(irb))) \\
&\geq |\exp(ir) - \exp(irs)| \\
&= |\exp(ir) - \exp(-ir + ir - irs)| \\
&= |\exp(ir) - \exp(-ir)\exp(ir - irs)| \\
&= |\exp(ir) + \exp(-ir)| \\
&= 2.
\end{aligned}$$

However, by the properties of $\varepsilon$-approximate-$^*$-homomorphisms as defined in this section, we also have that

$$\begin{aligned}\|\theta(e^{ir}I) - e^{ir}I\| &= \|\theta(e^{ir}I) - \phi(e^{ir}I) + \phi(e^{ir}I) - e^{ir}I\| \\ &\leq \|\theta(e^{ir}I) - \phi(e^{ir}I)\| + \|\phi(e^{ir}I) - e^{ir}I\| \\ &\leq \varepsilon_3 + \varepsilon < 1,\end{aligned}$$

which gives us the necessary contradiction. □

**Claim 3.4.0.2.** *If $p \in A$ is a projection, then so is $\rho(p)$.*

*Proof.* Akin to the preceding claim, we assume that $p$ is a projection and $\rho(p) = b'$ is not a projection. But then, by the properties of projections, we have that there is $\lambda \in \sigma(b') \setminus \{0, 1\}$. Then we know that $\{\exp(i2k\lambda\pi) : k \in \mathbb{Z}\}$ is a non-trivial subgroup of the unit circle in the complex plane. Moreover, there must be a $k'$ such that

$$|\exp(i2k\lambda\pi) - 1| \geq \sqrt{2},$$

by elementary complex analysis and geometry. We also know that

$$\exp(i2k\pi p) = \exp(i\pi p)^2 = 1.$$

However, we know that $\theta$ is $\varepsilon$-*-homomorphism, therefore

$$\theta(p) = \exp(i2k\pi\rho(p)) \approx_\varepsilon 1.$$

However Fact 2.4.3 about C*-algebras tells us that

$$\|\exp(i2k\pi\rho(p)) - I\| \geq \sqrt{2},$$

which is the required contadiction. □

Let $u$ be a self-adjoint unitary element. Let $p := \frac{1}{2}(I - u)$. Then we can observe that

$$p^2 = \left(\frac{1}{2}(I-u)\right)^2 = \frac{1}{4}(I - 2u + u^2) = \frac{1}{2}(2I - 2u) = p,$$

and

$$p^* = \left(\frac{1}{2}(I-u)\right)^* = \frac{1}{2}(I-u) = p$$

as $u$ is a self-adjoint unitary. Then $u = I - 2p$. Also note that for any given projection $q$ we have that $u_q := \exp(i\pi q) = 1 - 2Q$ will be a self-adjoint unitary. As $\rho(p)$ is going to be a projection, we know that

$$\theta(u) = \theta(\exp(i\pi p)) = \exp(i\pi\rho(p)) = I - 2\rho(p).$$

Therefore $\rho(p) = \frac{1}{2}(I - \theta(u))$.

**Claim 3.4.0.3.** *Let $p$ and $q$ be projections*

1. *If $p$ and $q$ are commuting, then $\rho(p)$ and $\rho(q)$ commute as well.*
2. *If $pq = 0$ then $\rho(p)\rho(q) = 0$.*
3. *If $p \sim_{MvN} q$, then $\rho(p) \sim_{MvN} \rho(q)$.*

*Proof.* This proof has three parts.

1. It is clear that $1 - 2p$ and $1 - 2q$ commute if and only if $p$ and $q$ do. We can see that $\rho(p)$ and $\rho(q)$ commute if and only if $\theta(u_p)$ and $\theta(u_q)$ do as

   $$\rho(P)\rho(Q) = (1 - \theta(u_p))(1 - \theta(u_q)) = 1 - \theta(u_p) - \theta(u_q) + \theta(u_p)\theta(u_q).$$

   But $\theta(u_p)$ and $\theta(u_q)$ commute if and only if $u_p$ and $u_q$ do as $\theta$ is multiplicative.

2. Assume now that $PQ = 0$. Then we note that $(p + q)^* = p^* + q^* = p + q$ and $(p + q)^2 = p^2 + pq + qp + q^2 = p + q$ as $(pq)^* = qp$. Therefore $p + q$ is also a projection. Now we notice that

   $$u_{p+q} = I - 2(p + q) = (I - 2p)(I - 2q) = u_p \cdot u_q,$$

   and therefore

   $$\theta(u_{p+q}) = \theta(u_p \cdot u_q) = \theta(u_p)\theta(u_q)$$

   which is only possible if $\rho(p)\rho(q) = 0$.

3. Since we are in the finite-dimensional case, we know that unitary equivalence coincides with Murray-von Neumann equivalence. Now assume that $v$ is the unitary such that $v^*pv = q$. Now note here that for the unitary $u_p = 1 - 2p$,

   $$v^*u_pv = 1 - 2v^*pv = 1 - 2q = u_q.$$

Then $v$ is the unitary making $u_p$ equivalent to $u_q$. Now note that $\theta$ is a group homomorphism so

$$1 - 2\rho(q) = \theta(u_q) = \theta(v^* u_p v)$$
$$= \theta(v)^* \theta(u_p)\theta(v) = 1 - 2\theta(v)^* \rho(p)\theta(v).$$

Giving us Murray-von Neumann equivalence of $\rho(p)$ and $\rho(q)$. □

Let $u \in A$ be a unitary element. We make the following obervation. Then $\sigma(u)$ will be of the form $e^{i\lambda_j}$ for $0 \leq j \leq n-1$.

We recall again using the Taylor expansion that

$$e^{i\lambda} p = 1 - p - \exp(i\lambda p)$$

if $\lambda \in \mathbb{R}$ and $p$ is a projection. Then we know that for orthogonal projections $p_j$ and real numbers $\lambda_j$, we have

$$\prod_{j<k} \exp(i\lambda_j p_j) = \prod_{j<k} (1 - p_j - e^{i\lambda_j} p_j)$$
$$= 1 - \sum_{j<k} p_j + \sum_{j<k} e^{i\lambda_j} p_j.$$

By applying the Spectral Theorem or the Functional Calculus (see Theorem 2.4.3) for the unitaries, we know that for a unitary $u$ we have

$$u = \sum_{j<k} e^{i\lambda_j} p_j,$$

we also know that $\sum_{j<k} p_j = 1$. To see this one can set $p_j = \chi_{\{e^{i\lambda_j}\}}$ in the Functional calculus. Therefore, by the preceding remarks, we have

$$u = \prod_{j<k} \exp(i\lambda_j p_j).$$

Here $e^{i\lambda_j}$'s are eigenvalues of $u$ and $p_j$'s are rank one projections (note that the eigenvalues can be repeated in the sum). These projections are Murray-von Neumann equivalent, as they are projections of the same rank in finite-dimensional C*-algebras. Therefore $\rho(p_j)$'s are also Murray-von Neumann equivalent by the previous claim. Therefore we have that $\sum_{j<k} \rho(p_j) = 1$. Let $\tau_A$ and $\tau_B$ be the unique tracial states on $A$ and $B$ respectively. We will now try to prove $\tau_A(u) = \tau_B(\theta(u))$.

By the spectral theorem (functional calculus) we know that

$$u = \sum_{j<m} \exp(i\lambda_j) p_j = \prod_{j<m} \exp\left(i\lambda_j p_j\right),$$

where each $p_j$ are rank-1 projections, anf $\lambda_j$ are reals such that $\exp(i\lambda_j)$ are eigenvalues of $u$. Note that these $p_j$'s are Murray-von Neumann equivalent. By the preceding claim we know that $\rho(p_j)$'s are also Murray-von Neumann equivalent. Moreover, $\sum_{j<m} \rho(p_j) = 1$. Then we know that

$$\begin{aligned}\theta(u) = \theta\left(\sum_{j<m} \exp(i\lambda_j) p_j\right) &= \theta\left(\prod_{j<m} \exp\left(i\lambda_j p_j\right)\right) \\ &= \prod_{j<m} \exp\left(i\lambda_j \rho(p_j)\right) = \sum_{j<m} \exp(i\lambda_j)\rho(p_j).\end{aligned}$$

We notice that

$$\tau_B(\theta(u)) = \tau_B\left(\sum_{j<m} \exp(i\lambda_j)\rho(p_j)\right)$$

$$= \sum_{j<m} \exp(i\lambda_j)\tau_B\left(\rho(p_j)\right) = \frac{1}{m}\sum_{j<m} \exp(i\lambda_j).$$

We also notice that

$$\tau_A(u) = \sum_{j<m} \exp(i\lambda_j)\tau_A(p_j) = \frac{1}{m}\sum_{j<m} \exp(i\lambda_j).$$

Therefore we have $\tau_A(u) = \tau_B(\theta(u))$ for any unitary $u$.

Now we can use Theorem 3.4.2 to extend this to a *-homomorphism. Recall that any element of a C*-algebra can be written as a linear combination of four unitary elements (see Fact 2.4.2). □

## 3.5 Ulam Stability of $(\mathcal{F}, \mathcal{C}^*)$

### 3.5.1 Introduction

This part of the thesis aims to prove the Ulam stability for the class of functions from an arbitrary finite-dimensional C*-algebra to some C*-algebra. This part of the thesis is an expansion of [15].

### 3.5.2 Preliminaries

Note that in the preceding section, we had defined what an $\varepsilon$-approximate-*-homomorphism was. In this part of the paper, we shall see a slightly different notation. We shall also see how these two definitions relate before diving deeper into the main result.

**Definition 3.5.1.** Let $A$ and $B$ be C*-algebras. A map $\phi : A \to B$ is called an $\varepsilon$-*-*homomorphism* if for all $x, y \in A_{\leq 1}$ and $\lambda \in \mathbb{C}_{\leq 1}$, we have

$$\begin{aligned} \|\phi(x+y) - \phi(x) - \phi(y)\| &\leq \varepsilon, \\ \|\phi(\lambda x) - \lambda\phi(x)\| &\leq \varepsilon, \\ \|\phi(xy) - \phi(x)\phi(y)\| &\leq \varepsilon, \\ \|\phi(x^*) - (\phi(x)^*)\| &\leq \varepsilon, \\ \|\phi(x)\| &\leq 1 + \varepsilon. \end{aligned}$$

Recall that the operator norm can be defined by taking the supremum over the closed unit ball of the operator in question. We shall use the same definition in our case.

**Definition 3.5.2.** For an arbitrary map $f : A \to B$ between C*-algebras, we define the *norm* of $f$ to be

$$\|f\| := \sup_{x \in A_{\leq 1}} \|f(x)\|$$

Note that it is easy to see that this indeed satisfies the properties of the usual norm.

We have a few other definitions we will look at before moving forward.

**Definition 3.5.3.** A map $f : A \to B$ is called $\varepsilon$-*isometric* if, for all $x \in A$ with $\|x\| = 1$, we have

$$\|f(x)\| \in [1 - \varepsilon, 1 + \varepsilon].$$

It is called $\varepsilon$-surjective if, for all $b \in B_{\leq 1}$, there is $a \in A$ such that

$$\|f(a) - b\| \leq \varepsilon.$$

It is called an $\varepsilon*$-*-*isomorphism* is it is $\varepsilon$-isometric and $\varepsilon$-surjective $\varepsilon^*$-homomorphism. It is called $\varepsilon$-*non-zero* if there is some $a \in A_{\leq 1}$ with $\|a\| = 1$ such that $\|f(a)\| \geq 1 - \varepsilon$.

Our proof for Ulam stability will have additional conditions on $\varepsilon$-*-homomorphisms that can be explained using the following lemmas.

Lemma 3.5.1 will allow us to assume without loss of generality that $\|\phi\| \leq 1$ for a $\varepsilon$-*-homomorphism $\phi$.

**Lemma 3.5.1.** *If $\phi : A \to B$ is an $\varepsilon$-*-homomorphism between C*-algebras where $\|\phi\| \geq 1$, then $\psi := \phi/\|\phi\|$ is a $2\varepsilon$-*-homomorphism. Moreover $\|\phi - \psi\| \leq \varepsilon$.*

*Proof.* We will show here that $\psi$ satisfies the properties for Definition 3.5.1, and then that $\|\phi - \psi\| \leq \varepsilon$. Due to similarity in proofs, we shall only show one of the properties from Definition 3.5.1, the remainder are routine. Let $a, b \in A$. Then,

$$\begin{aligned}
\|\psi(ab) - \psi(a)\psi(b)\| &= \left\| \frac{\phi(ab)}{\|\phi\|} - \frac{\phi(a)}{\|\phi\|} \cdot \frac{\phi(b)}{\|\phi\|} \right\| \\
&= \left\| \frac{\phi(ab)}{\|\phi\|} - \frac{\phi(a)\phi(b)}{\|\phi\|} + \frac{\phi(a)\phi(b)}{\|\phi\|} - \frac{\phi(a)}{\|\phi\|} \cdot \frac{\phi(b)}{\|\phi\|} \right\| \\
&\leq \frac{1}{\|\phi\|} \|\phi(ab) - \phi(a)\phi(b)\| + \frac{\|\phi(a)\phi(b)\|}{\|\phi\|} \left\| 1 - \frac{1}{\|\phi\|} \right\| \\
&\leq \frac{1}{\|\phi\|} \|\phi(ab) - \phi(a)\phi(b)\| + \frac{\|\phi(a)\|\|\phi(b)\|}{\|\phi\|} \left| \frac{\|\phi\| - 1}{\|\phi\|} \right| \\
&\leq \|\phi(ab) - \phi(a)\phi(b)\| + |\|\phi\| - 1| \\
&\leq \varepsilon + \varepsilon.
\end{aligned}$$

For the other condition, we get

$$\|\phi(a) - \psi(a)\| = \left\| \phi(a) - \frac{\phi(a)}{\|\phi\|} \right\| \leq \frac{1}{\|\phi\|} \|\phi(a)\| \|(\|\phi\| - 1)\| \leq (1 + \varepsilon) - 1 = \varepsilon.$$

Note that here we used $\|\phi(a)\| \leq \|\phi\|$. □

The following lemma will further allow us to assume that $\phi(1)$ is a projection.

**Lemma 3.5.2.** *If $\phi : A \to B$ is an $\varepsilon$-*-homomorphism between C*-algebras, where $A$ is unital, then there is $\phi' : A \to B$ such that $\phi'(1)$ is a projection, and $\|\phi - \phi'\| \leq \varepsilon p(\varepsilon)$ for some polynomial $p$.*

*Proof.* We start by considering the functional calculus of $\phi(1)^*\phi(1)$. We know it exists as $\phi(1)^*\phi(1)$ is clearly normal. Then we know that

$$C^*(\phi(1)^*\phi(1)) \cong C_0(\mathrm{sp}(\phi(1)^*\phi(1))).$$

Then we know $p := \mathbb{1}_{\mathrm{sp}(\phi(1)^*\phi(1))}$ or the constant one map on the spectrum of $\phi(1)^*\phi(1)$ is an element of $B$. Moreover, as it is the constant one map, we also know that it is a projection. Let $\phi' : A \to B$ be such that

$$\phi'(x) = \begin{cases} \phi(x) & \text{if } x \neq 1 \\ p & \text{if } x = 1 \end{cases}.$$

Then we have that $\|\phi'(x) - \phi(x)\| = 0$ for all $x \neq 1$. For $x = 1$, we have

$$\begin{aligned} \|\phi'(1) - \phi(1)\| &= \|p - \phi(1)\| \\ &= \|p - \phi(1)^*\phi(1) + \phi(1)^*\phi(1) - \phi(1)\| \\ &\leq \|p - \phi(1)^*\phi(1)\| + \|\phi(1)^*\phi(1) - \phi(1)\| \\ &= \|p - \phi(1)^*\phi(1)\| + \|\phi(1)^*\phi(1) - \phi(1)^2 + \phi(1)^2 - \phi(1)\| \\ &\leq \|p - \phi(1)^*\phi(1)\| + \|\phi(1)^*\phi(1) - \phi(1)^2\| + \|\phi(1)^2 - \phi(1)\| \\ &\leq \|p - \phi(1)^*\phi(1)\| + \varepsilon\|\phi(1)\| + \varepsilon. \end{aligned}$$

Observe that $\phi(1)^*\phi(1)$ is positive. Then as $\mathrm{sp}(\phi(1)^*\phi(1)) \subseteq \mathbb{R}^+$ we have

$$\|p - \phi(1)^*\phi(1)\| \leq r(\phi(1)^*\phi(1)) - 1 = \|\phi(1)^*\phi(1)\| - 1$$

as $\phi(1)^*\phi(1)$ is the identity map on $\mathrm{sp}(\phi(1)^*\phi(1))$ in the functional calculus. Then we have

$$\|\phi'(1) - \phi(1)\| \leq (\|\phi(1)^*\phi(1)\| - 1) + \varepsilon(\|\phi(1)\| + 1).$$

However by the properties of $\varepsilon$-*-homomorphism we have

$$\|\phi'(1) - \phi(1)\| \leq ((1+\varepsilon)^2 - 1) - \varepsilon((1+\varepsilon) + 1) \leq \varepsilon p(\varepsilon),$$

for some polynomial $p$. □

We have the following proposition that shows how that two different definitions, $\varepsilon$-*-homomoprphism and $\varepsilon$-approximate-*-homomorphism relate to each other.

**Proposition 3.5.1.** *[15, Proposition 2.4] Suppose that $\varepsilon < (\sqrt{10} - 3)^2$, $n \in \mathbb{N}$, $B$ is a C*-algebra and $\phi : M_n(\mathbb{C}) \to B$ is a $2\sqrt{\varepsilon}$-non-zero-$\varepsilon$-*-homomorphism with $\|\phi\| \leq 1$. Then $\phi$ is $2\sqrt{\varepsilon}$ -isometric.*

*Proof.* For $a \in M_n(\mathbb{C})$ we have the following using the triangle inequality, properties of the definition of an $\varepsilon$-*-homomorphism and the fact that $\|\phi\| \leq 1$.

$$\begin{aligned}
\left|\|\phi(a^*a)\| - \|\phi(a)\|^2\right| &= \left|\|\phi(a^*a)\| - \|\phi(a^*)\phi(a)\| + \|\phi(a^*)\phi(a)\| - \|\phi(a)\|^2\right| \\
&\leq \left|\|\phi(a^*a)\| - \|\phi(a^*)\phi(a)\|\right| + \left|\|\phi(a^*)\phi(a)\| - \|\phi(a)\|^2\right| \\
&\leq \left|\|\phi(a^*a) - \phi(a^*)\phi(a)\|\right| + \left|\|\phi(a^*)\|\|\phi(a)\| - \|\phi(a)\|^2\right| \\
&= \left|\|\phi(a^*a) - \phi(a^*)\phi(a)\|\right| + \left|\left(\|(\phi(a^*)\| - \|\phi(a)\|\right)\|\phi(a)\|\right| \\
&\leq \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

Therefore we have,

$$\left|\|\phi(a^*a)\| - \|\phi(a)\|^2\right| \leq 2\varepsilon. \tag{3.6}$$

Let $s : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $s(a) = a^*a$. We denote $s^{(n)}$ to mean the composition of $s$ with itself $n$ times. We have the following claim.

**Claim 3.5.1.1.** *There is an $n \in \mathbb{N}$ such that, for each $x \in M_n(\mathbb{C})$ satisfying $\|x\| = 1$ and $\|\phi(x)\| < 1 - 2\sqrt{\varepsilon}$, we have $\|\phi(s^{(n)}(x))\| < 2\sqrt{\varepsilon}$.*

*Proof of Claim .* Let $k$ be in $\mathbb{N}$. Then we can observe the following,

$$(1 - k\sqrt{\varepsilon})^2 + 2\varepsilon \leq 1 - (k+1)\sqrt{\varepsilon} \tag{3.7}$$

if and only if

$$1 - 2k\sqrt{\varepsilon} + k^2\varepsilon + 2\varepsilon - 1 + (k+1)\sqrt{\varepsilon} \leq 0$$

if and only if

$$k^2\varepsilon + -k\sqrt{\varepsilon} + 2\varepsilon + \sqrt{\varepsilon} \leq 0$$

if and only if

$$k^2 - k\frac{1}{\sqrt{\varepsilon}} + \left(2 + \frac{1}{\sqrt{\varepsilon}}\right) \leq 0$$

solving the quadratic inequality further using elementary methods, one obtains that the inequality holds if and only if

$$\frac{1}{2\sqrt{\varepsilon}}(1 - \tau) \leq k \leq \frac{1}{2\sqrt{\varepsilon}}(1 + \tau) \tag{3.8}$$

where

$$\tau = \sqrt{1 - 4(2\varepsilon + \sqrt{\varepsilon})}.$$

Note that since $\varepsilon < (\sqrt{10}-3)^2 < \frac{1}{36}$ we have that $2\varepsilon + \sqrt{\varepsilon} < \frac{2}{9}$. This gives us the following

$$\begin{aligned}\tau^2 &= 1 - 4(2\varepsilon + \sqrt{\varepsilon}) = 1 - 8\varepsilon - 4\sqrt{\varepsilon} \\ &\geq 1 - 8\varepsilon - 4\sqrt{\varepsilon} - 8\varepsilon - 4\sqrt{\varepsilon} \\ &= 1 - 8\sqrt{\varepsilon} - 16\varepsilon = (1 - 4\sqrt{\varepsilon})^2 > \frac{2}{9} > 0,\end{aligned}$$

therefore we can say that $\tau \geq 1 - 4\sqrt{\varepsilon}$. Then we know that

$$1 - \tau \leq 4\sqrt{\varepsilon}$$
$$1 + \tau \geq 2 - 4\sqrt{\varepsilon}.$$

Then we can say by replacing for $\tau$ in the inequality 3.8 that the inequality 3.7 holds for all $k$ such that

$$2 \leq k \leq \frac{1}{\sqrt{\varepsilon}} - 2.$$

Now we assume that $x \in M_n(\mathbb{C})$ such that $\|x\| = 1$, and that $\|\phi(x)\| < 1 - 2\sqrt{\varepsilon}$ as in the conditions for our claim. We shall now prove that for all positive integers $k \leq 1/\sqrt{\varepsilon} - 2$ we have

$$\|\phi(s^{(k-1)}(x))\| \leq 1 - (k+1)\sqrt{\varepsilon}.$$

We will prove this using induction on $k \leq 1/\sqrt{\varepsilon} - 2$.

For the base case , $k = 1$, as $s^{(0)} = \mathrm{id}$ we can see that we need

$$\|\phi(x)\| \leq 1 - 2\sqrt{\varepsilon}.$$

However, this is a given statement for the claim, therefore true.

Assume now that the statement is true for the case $k-1$, then we have the following using inequality 3.6 and inequality 3.7.

$$\begin{aligned}\left\|\phi\left(s^{(k)}(x)\right)\right\| &= \left\|\phi\left(\left(s^{(k-1)}(x)\right)^*\left(s^{(k-1)}(x)\right)\right)\right\| \\ &\leq \left\|\phi\left(s^{(k-1)}(x)\right)\right\|^2 + 2\varepsilon \leq (1 - (k+1)\sqrt{\varepsilon})^2 + 2\varepsilon \\ &\leq 1 - (k+2)\sqrt{\varepsilon}.\end{aligned}$$

Now we choose $n$ to be the maximal integer less than or equal to $1/\sqrt{\varepsilon} - 2$. Then $n + 1 > 1/\sqrt{\varepsilon} - 2$. Therefore we have the following,

$$\|\phi\left(s^{(n-1)}(x)\right)\| \leq 1 - (n+1)\sqrt{\varepsilon} < 1 - \left(\frac{1}{\sqrt{\varepsilon}} - 2\right)\sqrt{\varepsilon} = 2.\sqrt{\varepsilon}$$

Note that the first inequality in the previous expression uses the claim we just proved.

□

Let $p \in M_n(\mathbb{C})$ be a projection. Recall that by definition $p^2 = p$ and $p^* = p$. This implies $s^{(k)}(p) = p$. Note also that since $\varepsilon < (\sqrt{10} - 3)^2$ we have $2\sqrt{\varepsilon} < 1 - 2\sqrt{\varepsilon}$. Then applying claim 3.5.1.1 to $p$ we see that $\|\phi(p)\| < 2\sqrt{\varepsilon}$ or $\|\phi(p)\| \geq 1 - 2\sqrt{\varepsilon}$. We shall call the projections $q \in M_n(\mathbb{C})$ such that $\|\phi(q)\| < 2\sqrt{\varepsilon}$ *small* in this proof.

We shall now assume towards a contradiction that $\phi$ is not $2\sqrt{\varepsilon}$-isometric and then show that all projections in $M_n(\mathbb{C})$ are small. This will lead to a contradiction: $\phi$ not being $2\sqrt{\varepsilon}$-non-zero.

Suppose that $\phi$ is not $2\sqrt{\varepsilon}$-isometric. Then as $\|\phi\| \leq 1$, by definition

$$\sup_{\|x\| \leq 1} \|\phi(x)\| \leq 1.$$

This implies that there must be $x \in M_n(\mathbb{C})$ such that $\|x\| = 1$ and $\|\phi(x)\| < 1 - 2\sqrt{\varepsilon}$. Let $x' := s^{(n)}(x)$ where $n$ is the natural number satisfying thee condition for claim 3.5.1.1. As $\|x\| = 1$, it is clear by elementary properties of C*-algebras, , especially the C*-equality, that $\|x'\| = 1$. We also know that $\|\phi(x')\| < 2\sqrt{\varepsilon}$. Since $x' = \left(s^{(n-1)}(x)\right)^* \left(s^{(n-1)}(x)\right)$ we know that $x'$ is positive too. We know that since $x'$ is positive it's spectral radius is the same as its norm, and then by compactness of the spectrum we can say that $1 \in \sigma(x')$. This implies that there is a projection $p \in M_n(\mathbb{C})$ of rank 1 such that $px'p = p$. Then we have the following using the properties of the $\varepsilon^*$ homomorphism $\phi$, as we did in inequality 3.6.

$$\begin{aligned} \|\phi(p)\| &= \|\phi(pxp)\| \leq \|\phi(px)\phi(p)\| + \varepsilon \\ &\leq \|\phi(p)\phi(x)\phi(p)\| + 2\varepsilon \leq \|\phi(p)\|^2 \|\phi(x)\| + 2\varepsilon \\ &\leq (1)^2(2\sqrt{\varepsilon}) + 2\varepsilon = 2\sqrt{\varepsilon} + 2\varepsilon. \end{aligned}$$

As $\varepsilon < \frac{1}{36}$, we know that we will get

$$2\sqrt{\varepsilon} + 2\varepsilon < 1 - 2\sqrt{\varepsilon},$$

giving us

$$[\|\phi(p)\| < 1 - 2\sqrt{\varepsilon}. \tag{3.9}$$

Therefore $p$ is small. Let $q \in M_n(\mathbb{C})$ be another rank 1 projection. Then we know that there is a unitary $u \in M_n(\mathbb{C})$ such that $p = uqu^*$, see Fact 2.4.5. Then we have the following by the properties of $\varepsilon$-*-homomorphism, inequality 3.9, and $2\sqrt{\varepsilon}$-isometry,

$$\|\phi(q)\| = \|\phi(upu^*)\| \leq \|\phi(u)\|\|\phi(p)\|\|\phi(u^*)\| + 2\varepsilon$$
$$< \|\phi(u)\|\|\phi(u^*)\|(1 - 2\sqrt{\varepsilon}) + 2\varepsilon \leq (1 - 2\sqrt{\varepsilon})^3 + 2\varepsilon \leq 2\sqrt{\varepsilon} + 2\varepsilon.$$

This shows us that $q$ is small as well. Now, if $p_1$ and $p_2$ are two small orthogonal projections, we have the following.

$$\|\phi(p_1 + p_2)\| \leq \|\phi(p_1) + \phi(p_2)\| + \varepsilon$$
$$\leq \|\phi(p_1)\| + \|\phi(p_2)\| + \varepsilon < 4\sqrt{\varepsilon} + \varepsilon \leq 1 - 2\sqrt{\varepsilon}.$$

Note that in the previous inequality we use that

$$4\sqrt{\varepsilon} + \varepsilon \leq 1 - 2\sqrt{\varepsilon}$$

if and only if

$$\varepsilon + 6\sqrt{\varepsilon} - 1 \leq 0$$

if and only if

$$\sqrt{\varepsilon} \leq \sqrt{10} - 3,$$

which we know to be true. Then we can say that every projection in $M_n(\mathbb{C})$ is small. This implies that 1 (the identity matrix) is also small. But then for $a \in M_n(\mathbb{C})$ such that $\|a\| = 1$ we have

$$\|\phi(a)\| = \|\phi(a \cdot 1)\| \leq \|\phi(a)\|\|\phi(1)\| + \varepsilon$$
$$\leq \|\phi(1)\| + \varepsilon \leq 2\sqrt{\varepsilon} + \varepsilon < 1 - 2\sqrt{\varepsilon}.$$

Here the first inequality used the $\varepsilon$-*-homomorphism properties, the second used $\|\phi\| \leq 1$, and the third used the smallness of 1. However, since $a$ was arbitrary we get that $\phi$ is not $2\sqrt{\varepsilon}$-non-zero, a contradiction. □

### 3.5.3 Near-inclusions of C*-algebras

This subsection of the thesis will state a result used in the proof of the main theorem of this section.

We start with the following definition.

**Definition 3.5.4.** For C*-subalgebras $A$ and $B$ of a C*-subalgebra $C$, let $\gamma$ be a positive real number. The relation $A \subseteq^{\gamma} B$ means that for each $a \in A$, there is an operator $b \in B$ such that $\|a - b\| \leq \gamma\|a\|$.

We have the following result that we will use in the main proof.

**Theorem 3.5.1.** *[3, Theorem 5.3] Let $A$ be a finite-dimensional C*-subalgebra and $B$ a C*-subalgebra of a C*-algebra $C$. If $0 \leq \gamma \leq 10^{-4}$, and $A \subseteq^{\gamma} B$, then there is a partial isometry $v$ in $C$ such that $\|\mathrm{id}_A - v\| \leq 120\sqrt{\gamma}$ and $vAv^* \subseteq B$.*

### 3.5.4 Proving Ulam Stability of $(\mathcal{F}, \mathcal{C}^*)$

We have the following definition for Ulam stability for this section.

**Definition 3.5.5.** Let $\mathcal{C}$ and $\mathcal{D}$ be two classes of C*-algebras. We say that the pair $(\mathcal{C}, \mathcal{D})$ is *Ulam stable* if for each $\varepsilon' > 0$ there is a $\delta' > 0$ such that, for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$ and for every $\delta'$-*-homomorphism $\phi : A \to B$ there is a *-homomorphism $\psi : A \to B$ such that $\|\phi - \psi\| \leq \varepsilon'$.

The primary aim of this section is to prove the following statement. Recall that $\mathcal{F}$ represents the class of finite-dimensional C*-algebras, and $\mathcal{C}^*$ represents the class of all C*-algebras.

**Theorem 3.5.2.** *[15, Theorem 2.7] There are $L, \delta > 0$ such that, given $\varepsilon < \delta$, $F \in \mathcal{F}$, $A \in \mathcal{C}^*$ and an $\varepsilon$-*-homomorphism $\phi : F \to A$, there is a $*$-homomorphism $\psi : F \to A$ with*

$$\|\psi - \phi\| < L\varepsilon^{\frac{1}{2}}.$$

The reader should note that Theorem 3.5.2 also proves the Ulam stability of $(\mathcal{F}, \mathcal{C}^*)$. This is because for any $\varepsilon' > 0$ we can choose our $\delta'$ to be the minimum of $(\varepsilon')^2/K^2$ and $\delta - \delta/2026$ (note here that $\delta'$ can be anything smaller than $\delta$). Where $\delta$ and $K$ are as found in the statement of Theorem 3.5.2. These values will satisfy the properties that are needed for Ulam stability.

As mentioned earlier, to prove Theorem 3.5.2, our strategy would be to take a $\varepsilon$-*-homomorphism and improve it consecutively to obtain maps which have the properties we desire. To this aim we start with the following proposition.

**Proposition 3.5.2.** *For $F \in \mathcal{F}$, $A \in \mathcal{C}^*$ and an $\varepsilon$-*-homomorphism $\phi : F \to A$, there is a Borel measurable $4.5\varepsilon$-*-homomorphism $\phi' : F \to A$ with*

$$\|\phi' - \phi\| < \varepsilon.$$

*Proof.* We can assume without loss of generality that $A$ is unital (if not, $A$ can be unitized), $\|\phi\| \leq 1$ and $\phi(1) = 1$ by the preceeding lemmas in this section. We know that since $F$ is finite dimensional, the ball

$$F_{\leq 2} := \{x \in F : \|x\| \leq 2\}$$

is compact. Then we know that this ball is also totally bounded (as $F$ is a metric space with respect to the distance induced by the norm). Then we can say that there is a finite set

$$\{x_0, \ldots, x_k\}$$

such that

$$F_{\leq 2} \subseteq \bigcup_{j=1}^{k} B\left(x_j, \frac{\varepsilon}{2}\right),$$

and $x_0 = 1$. We can assume without loss of generality that the $x_i$'s are ordered in a way that the ones in the unit ball of $F$, denoted $F_{\leq 1}$ have the smaller index. That is to say, for some $0 \leq m \leq k$,

$$\{x_0, \ldots, x_m\} \subseteq F_{\leq 1}.$$

We define $\phi' : F_{\leq 2} \to A$ as follows,

$$\phi'(x) = \phi\left(x_{i_x}\right) \text{ where } i_x = \min\left\{j : \|x_j - x\| \leq \varepsilon\right\}.$$

Then the range of $\phi'$ is a subset of $\{\phi(x_0), \ldots \phi(x_k)\}$. Let $B_i = F_{\leq 2} \cap B(x_i, \varepsilon/2)$. If $\phi'(x) = \phi(x_i)$ it means that $x \in B_i$, and $x \notin B_j$, for any $j < i$. Therefore,

$$\phi'^{-1}(\{x_i\}) = B_i \setminus \bigcup_{j<i} B_j.$$

Since $B_j$'s are open, we know that $\phi'^{-1}(\{x_i\})$'s are Borel measurable (according to the subspace topology). We will now extend the function to the whole space. We define $\phi' : F \to A$, this is the same map for $x \in F_{\leq 2}$, however for $\|x\| > 2$, we have

$$\phi'(x) = \frac{\|x\|}{2}\phi'\left(\frac{2x}{\|x\|}\right).$$

Since we know, that $\|\cdot\| : F_{>2} \to (2, \infty)$, where $F_{>2} := F \setminus F_{\leq 2}$, is continuous and non zero. We know both

$$f_1 : x \mapsto \frac{\|x\|}{2} \text{ and } f_2 : x \mapsto \frac{2x}{\|x\|}$$

are continuous maps. Then we know that $\phi'$ on $F_{>2}$ is $\phi'(x) = f_1(x) \cdot (\phi \circ f_2(x))$, being a composition of a Borel and a continuous map multiplied pointwisely with a continuous map is also Borel.

**Claim 3.5.2.1.** *$\phi'$ as defined above is a $4.5\varepsilon$-*-homomorphism.*

*Proof of Claim .* Let $x, y \in F_{\leq 1}$ and let $\phi'(x) = \phi(x_i)$ and $\phi'(y) = \phi(x_j)$. First we shall show

$$\|\phi(x) - \phi'(x)\| \leq \varepsilon p(\varepsilon), \tag{3.10}$$

for some polynomial $p(\varepsilon)$. This can be seen with the following,

$$\begin{aligned}
\|\phi(x) - \phi'(x)\| &= \|\phi(x_i) - \phi(x)\| = \|\phi(x_i) + \phi(-x) - (\phi(-x) - (-\phi(x)))\| \\
&= \|\phi(x_i) + \phi(-x)\| + \|(\phi(-x) - (-\phi(x)))\| \leq \|\phi(x_i) + \phi(-x)\| + \varepsilon \\
&\leq \|\phi(x_i) + \phi(-x) + \phi(x_i - x) - \phi(x_i - x)\| + \varepsilon \\
&\leq \|\phi(x_i) + \phi(-x) - \phi(x_i - x)\| + \|\phi(x_i - x)\| + \varepsilon \\
&\leq \|\phi(x_i - x)\| + 2\varepsilon = \|x_i - x\| \left\| \frac{\phi(x_i - x)}{\|x_i - x\|} \right\| + 2\varepsilon \\
&\leq \frac{\varepsilon}{2} \left\| \frac{\phi(x_i - x)}{\|x_i - x\|} - \phi\left(\frac{x_i - x}{\|x_i - x\|}\right) + \phi\left(\frac{x_i - x}{\|x_i - x\|}\right) \right\| + 2\varepsilon \\
&\leq \frac{\varepsilon}{2} \left\| \frac{\phi(x_i - x)}{\|x_i - x\|} - \phi\left(\frac{x_i - x}{\|x_i - x\|}\right) \right\| + \frac{\varepsilon}{2} \left\| \phi\left(\frac{x_i - x}{\|x_i - x\|}\right) \right\| + 2\varepsilon \\
&\leq \frac{\varepsilon}{2}\varepsilon + \frac{\varepsilon}{2}(1 + \varepsilon) + 2\varepsilon
\end{aligned}$$

Next we prove that $\phi'$ is an $\varepsilon_1$-*-homomorphism.

$$
\begin{aligned}
&\|\phi'(x+y)-\phi'(x)-\phi'(y)\| \\
&= \|\phi'(x+y)-\phi(x+y)+\phi(x+y)-\phi(x_i+x_j)+\phi(x_i+x_j)-\phi(x_i)-\phi(x_j)\| \\
&\leq \|\phi'(x+y)-\phi(x+y)\|+\|\phi(x+y)-\phi(x_i+x_j)\|+\|\phi(x_i+x_j)-\phi(x_i)-\phi(x_j)\| \\
&\leq \frac{\varepsilon}{2}+\|\phi(x+y)-\phi(x_i+x_j)\|+\varepsilon \\
&\leq \frac{\varepsilon}{2} \\
&+\|\phi(x+y)-\phi(x)-\phi(y)+\phi(x)+\phi(y)-\phi(x_i)-\phi(x_j)+\phi(x_i)+\phi(x_j)-\phi(x_i+x_j)\| \\
&+\varepsilon \\
&\leq \frac{\varepsilon}{2}+\|\phi(x+y)-\phi(x)-\phi(y)\|+\|\phi(x)-\phi(x_i)\| \\
&+\|\phi(y)-\phi(x_j)\|+\|\phi(x_i)+\phi(x_j)-\phi(x_i+x_j)\|+\varepsilon \\
&\leq \frac{\varepsilon}{2}+\varepsilon+\|\phi(x)-\phi(x_i)\|+\|\phi(y)-\phi(x_j)\|+2\varepsilon \leq 4.5\varepsilon.
\end{aligned}
$$

This proves that $\phi'$ has the addition property of a $\varepsilon$-*-homomorphism. For the multiplication property we have the following

$$
\begin{aligned}
&\|\phi'(xy)-\phi'(x)\phi'(y)\| = \|\phi'(xy)-\phi(xy)+\phi(xy)-\phi(x_i)\phi(x_j)\| \\
&\leq \|\phi'(xy)-\phi(xy)\|+\|\phi(xy)-\phi(x_i)\phi(x_j)\| \leq \frac{\varepsilon}{2}+\|\phi(xy)-\phi(x_i)\phi(x_j)\| \\
&\leq \frac{\varepsilon}{2}+\|\phi(xy)-\phi(x_ix_j)+\phi(x_ix_j)-\phi(x_i)\phi(x_j)\| \\
&\leq \frac{\varepsilon}{2}+\|\phi(xy)-\phi(x_ix_j)\|+\|\phi(x_ix_j)-\phi(x_i)\phi(x_j)\| \\
&\leq 1.5\varepsilon+\|\phi(xy)-\phi(x_ix_j)\| \\
&\leq 1.5\varepsilon+\|\phi(xy)-\phi(x)\phi(y)+\phi(x)\phi(y)-\phi(x_i)\phi(x_j)+\phi(x_i)\phi(x_j)-\phi(x_ix_j)\| \\
&\leq 1.5\varepsilon+\|\phi(xy)-\phi(x)\phi(y)\|+\|\phi(x)\phi(y)-\phi(x_i)\phi(x_j)\|+\|\phi(x_i)\phi(x_j)-\phi(x_ix_j)\| \\
&\leq 3.5\varepsilon+\|\phi(x)\phi(y)-\phi(x_i)\phi(x_j)\| \\
&\leq 3.5\varepsilon+\|\phi(x)\phi(y)-\phi(x_i)\phi(x_j)+\phi(x_i)\phi(x_j)-\phi(x_i)\phi(x_j)\| \\
&\leq 3.5\varepsilon+\|\phi(x)\phi(y)-\phi(x_i)\phi(y)+\phi(x_i)\phi(y)-\phi(x_i)\phi(x_j)\| \\
&\leq 3.5\varepsilon+\|(\phi(x)-\phi(x_i))\phi(y)+\phi(x_i)(\phi(y)-\phi(x_j))\| \\
&\leq 3.5\varepsilon+\|\phi(x)-\phi(x_i)\|\|\phi(y)\|+\|\phi(x_i)\|\phi(y)-\phi(x_j)\| \leq 4.5\varepsilon.
\end{aligned}
$$

Let $\lambda \in \mathbb{C}$. For the scalar multiplication property, we have the following,

$$\begin{aligned}
\|\phi'(\lambda x) - \lambda\phi'(x)\| &= \|\phi'(\lambda x) - \lambda\phi(x_i)\| \\
&= \|\phi'(\lambda x) - \phi(\lambda x_i) + \phi(\lambda x_i) - \lambda\phi(x_i)\| \\
&\leq \|\phi'(\lambda x) - \phi(\lambda x_i)\| + \|\phi(\lambda x_i) - \lambda\phi(x_i)\| \\
&\leq \|\phi'(\lambda x) - \phi(\lambda x_i)\| + \varepsilon \\
&= \|\phi'(\lambda x) - \phi(\lambda x) + \phi(\lambda x) - \phi(\lambda x_i)\| + \varepsilon \\
&\leq \|\phi'(\lambda x) - \phi(\lambda x)\| + \|\phi(\lambda x) - \phi(\lambda x_i)\| + \varepsilon \\
&\leq 1.5\varepsilon + \|\phi(\lambda x) - \phi(\lambda x_i)\| \\
&= 1.5\varepsilon + \|\phi(\lambda x) - \lambda\phi(x) + \lambda\phi(x) - \phi(\lambda x_i)\| \\
&\leq 1.5\varepsilon + \|\phi(\lambda x) - \lambda\phi(x)\| + \|\lambda\phi(x) - \phi(\lambda x_i)\| \\
&\leq 2.5\varepsilon + \|\lambda\phi(x) - \phi(\lambda x_i)\| \\
&= 2.5\varepsilon + \|\lambda\phi(x) - \lambda\phi'(x) + \lambda\phi(x_i) - \phi(\lambda x_i)\| \\
&\leq 2.5\varepsilon + \|\lambda\phi(x) - \lambda\phi'(x)\| + \|\lambda\phi(x_i) - \phi(\lambda x_i)\| \\
&\leq 2.5\varepsilon + |\lambda|0.5\varepsilon + \varepsilon \leq 4\varepsilon.
\end{aligned}$$

The other properties do not require much effort since for those cases we retain the bounds from $\phi$ directly. We have

$$\begin{aligned}
\|\phi'(x) - \phi'(x)^*\| &= \|\phi(x_i) - \phi(x_i)^*\| \leq \varepsilon \\
\|\phi'(x)\| &= \|\phi(x_i)\| \leq 1 + \varepsilon. \quad \square
\end{aligned}$$

We also have $\|\phi'\| \leq 1$, as $\phi'$ is bounded by $\phi$. $\square$

*Proof of Theorem 3.5.2.* We start by assuming that we have positive $\gamma$ and $K$ that satisfy the conditions for Theorem 3.4.1. We can assume using proposition 3.5.2 and lemma 3.5.2 that $\phi$ is Borel measurable and $\phi(1) = 1$. As we have that $\varepsilon < 1$ and $\phi$ is unital we can say that for each $u \in U(F)$ (Unitary elements of $F$), we have

$$\|\phi(u^*)\phi(u) - 1\| = \|\phi(u^*)\phi(u) - \phi(u^*u)\| \leq \varepsilon < 1.$$

Therefore $\phi(u^*)\phi(u)$ is invertible by Theorem 2.3.1. We know by the same theorem that the inverse of $\phi(u^{-1})\phi(u)$ is

$$c = \sum_{k=0}^{\infty} (1 - \phi(u^{-1})\phi(u))^k,$$

moreover,

$$\|c\| \leq \sum_{k=0}^{\infty} \|1 - \phi(u^{-1})\phi(u)\|^k$$
$$= \frac{1}{1 - \|1 - \phi(u^{-1})\phi(u)\|} \leq \frac{1}{1-\varepsilon} \leq 2,$$

by our choice of $\varepsilon$. Now note that we know

$$c\phi(u^{-1})\phi(u) = \phi(u^{-1})\phi(u)c = 1.$$

We wish to show that $\phi(u)c\phi(u^{-1}) = 1$. However note that $c$ is the limit of partial series of the form

$$c_K = \sum_{k=0}^{k=K} (1 - \phi(u^{-1})\phi(u))^k.$$

Similarly we know $c_K\phi(u^{-1})\phi(u) \to 1$. Each term in $c_K$ however is of the form

$$c_{K,i} = \alpha_i(\phi(u^{-1})\phi(u))^{r_i}$$

but then we have

$$c_{K,i}\phi(u^{-1})\phi(u) = \alpha_i(\phi(u^{-1})\phi(u))^{r_i+1}$$
$$= \phi(u^{-1})\alpha_i(\phi(u^{-1})\phi(u))^{r_i}\phi(u) = \phi(u^{-1})c_{K,i}\phi(u).$$

Therefore $c_K\phi(u^{-1})\phi(u) = \phi(u^{-1})c_K\phi(u)$ and $\phi(u^{-1})c_K\phi(u) \to 1$. Hence, $\phi(u)$ is invertible and $\phi(u^{-1})c$ is its inverse.

$$\|\phi(u)^{-1}\| \leq \|\phi(u^{-1})c\| \leq \frac{1+\varepsilon}{1-\varepsilon} \leq 2 \tag{3.11}$$

as $\varepsilon < \frac{1}{3}$.

Now let $\rho_0$ be the restriction of $\phi$ to $U(F)$. By the preceding paragraph it is clear how $\rho_0 : U(F) \to \mathrm{GL}(A)$. As $F$ is finite, we know that $U(F)$ is a compact second-countable group, and $\rho_0$ is also a Borel map the conditions for Proposition 3.2.1 are satisfied. The $\kappa$ in this case is 2 by Equation 3.11. Then, using Theorem 3.2.1 we can inductively define the sequence of Borel measurable maps $\rho_k : U(F) \to \mathrm{GL}(A)$, where $\rho_0$ is as previously defined and $\rho_{k+1} = \rho'_k$ for each natural number $n \geq 1$. We also have $\delta_k$'s and $\kappa_k$'s defined as follows, $\delta_0 = \varepsilon$, $\kappa_0 = 2$, and

$$\delta_{k+1} = 2\kappa_k^2\delta_k^2, \text{ and } \kappa_{k+1} = \frac{\kappa_k}{1 - \kappa_k^2\delta_k}.$$

Note that these bounds occur naturally from proposition 3.2.1. We have the following claim

**Claim 3.5.2.2.** *For each $k$, $\kappa_{k+1} - \kappa_k < 2^{-n}$ and $\delta_k \leq 2^{5(1-2^k)}\varepsilon$. Consequently, $\kappa_k < 4$ for all $k$, and*

$$\sum_{n=0}^{\infty} \kappa_k \delta_k < 8\varepsilon.$$

*Proof of Claim .* We shall prove this claim by induction over $k$. For $k = 0$, $\delta_0 = \varepsilon < 2^{-10}$ and $\kappa_0 = 2$. Then

$$\kappa_1 - \kappa_0 = \frac{\kappa_0}{1 - \kappa_0^2\varepsilon} - \kappa_0 = \frac{\kappa_0^2\varepsilon}{1 - \kappa_0^2\varepsilon} < 1.$$

Assume now that $\kappa_0, \ldots, \kappa_n$ and $\delta_0, \ldots, \delta_n$ satify the inductive hypothesis. Then as $\kappa_{k+1} - \kappa_k < 2^{-k}$ for each $k = 0, \ldots, k-1$ we have that

$$\begin{aligned}
&\kappa_0 = 2\\
&\kappa_1 < \kappa_0 + 1 = 2 + 1\\
&\kappa_2 < \kappa_1 + 2^{-1} < 2 + 1 + 2^{-1}\\
&\ldots\\
&\kappa_n < 2 + 1 + 2^{-1} + \ldots + 2^{-n} < 2\left(\frac{1}{1 - \frac{1}{2}}\right) = 4.
\end{aligned}$$

We also know that $\delta_n \leq 2^{5(1-2^n)}\varepsilon$. Therefore

$$\begin{aligned}
\delta_{n+1} = 2\kappa_n^2\delta_n^2 < 2(4^2)\left(2^{10(1-2^n)}\right)\varepsilon^2 &< 2(4^2)(2^{-10})\left(2^{10(1-2^n)}\right)\varepsilon\\
&= 2^{-5}\left(2^{10(1-2^n)}\right)\varepsilon = 2^{5(1-2^{n+1})}\varepsilon.
\end{aligned}$$

as $\varepsilon < 2^{-10}$. As $2 - 5 \cdot 2^n \leq -n$ for every $n \geq 0$, we have that $\delta_{n+1}$ we also have the following

$$\begin{aligned}
\kappa_{n+1} - \kappa_n = \frac{\kappa_n}{1 - \kappa_n^2\delta_n} - \kappa_n &= \frac{\kappa_n^3\delta_n}{1 - \kappa_n^2\delta_n}\\
&\leq \frac{4^3 \cdot 2^{5(1-2^n)}\varepsilon}{1 - 4^2 \cdot 2^{5(1-2^n)}\varepsilon} < \frac{4^3 \cdot 2^{5(1-2^n)}2^{-10}}{1 - 4^2 \cdot 2^{5(1-2^n)}2^{-10}}\\
&= \frac{2^{6+5-5\cdot 2^n-10}}{1 - 2^{4+5-5\cdot 2^n-10}} = \frac{2^{1-5\cdot 2^n}}{1 - 2^{-1-5\cdot 2^n}} = \frac{2^{2-5\cdot 2^n}}{2 - 2^{-5\cdot 2^n}} < 2^{2-5\cdot 2^n}.
\end{aligned}$$

as $0 < \varepsilon < 2^{-10}$ we know $\varepsilon^{-1} > 2^{-10}$. Then as $2 - 5 \cdot 2^n \leq -n$ for all $n \geq 0$. As $\kappa_n < 4$. We also have

$$\sum_{n=0}^{\infty} \kappa_n\delta_n < 4\varepsilon_1 \sum_{n=0}^{\infty} 2^{5(1-2^n)} < 4\varepsilon_1 \sum_{n=0}^{\infty} 2^{-n} = 8\varepsilon_1. \qquad \square$$

Then we can say that $\rho : U(F) \to \mathrm{GL}(A)$ given by $\rho(u) = \lim_{n\to\infty} \rho_n(u)$ is a Borel measurable multiplicative map such that $\|\rho - \phi\| \leq 8\varepsilon =: \varepsilon_2$.

Let $\sigma : A \to B(H)$ be faithful representation of $A$ on a separable Hilbert space $H$ by the GNS construction. Since $\sigma$ is continuous and $\rho$ is Borel measurable, we can define, $\tau : U(F) \to GL(B(H))$, where $\tau = \sigma \circ \rho$. $\tau$ will be a Borel measurable map with respect to the strong operator topology on $B(H)$ and the norm topology on $U(F)$, it will also be a group homomorphism. We also have that for each $u \in U(F)$,

$$\|\tau(u)\| = \|\sigma(\rho(u)\| \leq \|\rho(u)\| \leq \|\rho(u) - \phi(u)\| + \|\phi(u)\| \leq 1 + \varepsilon_2 + \varepsilon$$

and

$$\begin{aligned}
\|\tau(u)^*\tau(u) - 1\| &= \|\sigma(\rho(u))^*\sigma(\rho(u)) - \sigma(1)\| = \|\sigma(\rho(u)^*\rho(u) - 1)\| \\
&\leq \|\rho(u)^*\rho(u) - 1\| = \|\rho(u)^*\rho(u) - \rho(u)^{-1}\rho(u)\| \\
&\leq \|\rho(u)^* - \rho(u)^{-1}\|\|\rho(u)\| = \|\rho(u)^* - \rho(u^*) + \rho(u^*) - \rho(u)^{-1}\|\|\rho(u)\| \\
&\leq \left(\|\rho(u)^* - \rho(u^*)\| + \|\rho(u^*) - \rho(u)^{-1}\|\right)\|\rho(u)\| \\
&\leq 2(\varepsilon_2 + \varepsilon)(1 + \varepsilon_2 + \varepsilon) =: \varepsilon_3.
\end{aligned}$$

By the compactness of $U(F)$, we know that $\tau$ is unitarizable (see Theorem 2.6.6), i.e. there exist some $T \in \mathrm{GL}(B(H))$ such that $\pi(u) := T\tau(u)T^{-1}$ is a unitary representation. That is, $\pi : U(F) \to U(H)$. Moreover, by Remark 2.6.1 we have $\|T - 1\| \leq \varepsilon_3$. Moreover, using this with Theorem 2.3.1 we get that $\|T^{-1}\| \leq \frac{1}{1+\varepsilon_3}$. Therefore we have the following,

$$\begin{aligned}
\|\pi(u) - \tau(u)\| &= \left\|T\tau(u)T^{-1} - \tau(u)\right\| \\
&= \left\|T\tau(u)T^{-1} - \tau(u)T^{-1} + \tau(u)T^{-1} - \tau(u)\right\| \\
&= \left\|(T-1)\tau(u)T^{-1} + \tau(u)(T^{-1} - 1)\right\| \\
&\leq \|T - 1\|\left\|\tau(u)T^{-1}\right\| + \|\tau(u)\|\left\|T^{-1} - 1\right\| \\
&= \|T - 1\|\|\tau(u)T^{-1}\| + \|\tau(u)\|\|T^{-1}(1 - T)\| \\
&\leq \|T - 1\|\|\tau(u)\|\|T^{-1}\| + \|\tau(u)\|\|T^{-1}\|\|1 - T\| \\
&= 2\|T - 1\|\|\tau(u)\|\|T^{-1}\| = 2\varepsilon_3(1 + \varepsilon_2 + \varepsilon)\|\|T^{-1}\| \\
&\leq \frac{2\varepsilon_3(1 + \varepsilon_2 + \varepsilon)}{1 + \varepsilon_3} =: \varepsilon_4.
\end{aligned}$$

As $U(F)$ and $U(H)$, with the norm topology and the strong operator topology respectively, are Polish groups. As $\pi$ is a composition of a Borel map with a continuous map,

it is also Borel; moreover, by Pettis' Theorem (theorem 2.6.1)(which uses separability in the range), we obtain automatic continuity for $\pi$. Moreover, using the Peter-Weyl Theorem (theorem 2.6.5) we have that $H = \bigoplus H_k$, where $H_k$ are finite dimensional, and $\pi \restriction_{H_k}$ is irreducible for each $H_k$. Moreover, by the Peter-Weyl Theorem there are also projections $p_k : H \to H_k$ such that $[p_k, \pi(u)] = 0$, and $\pi(u) = \sum p_k \pi(u) p_k$. We already have that $\|\phi(u) - \rho(u)\| \leq \varepsilon_2$ for each $u \in U(F)$; therefore we have the following

$$\begin{aligned}
\|\sigma(\phi(u)) - \pi(u)\| &= \|\sigma(\phi(u)) - \tau(u) + \tau(u) - \pi(u)\| \\
&\leq \|\sigma(\phi(u)) - \tau(u)\| + \|\tau(u) - \pi(u)\| \\
&= \|\sigma(\phi(u)) - \sigma(\rho(u))\| + \|\tau(u) - \pi(u)\| \\
&= \|\sigma(\phi(u) - \rho(u))\| + \|\tau(u) - \pi(u)\| \\
&= \|\phi(u) - \rho(u)\| + \|\tau(u) - \pi(u)\| \\
&\leq \varepsilon_2 + \varepsilon_4.
\end{aligned}$$

Then for each $u \in U(F)$ and $k \in \mathbb{N}$ we know that

$$\begin{aligned}
\|[\sigma(\phi(u)), p_k]\| &= \|\sigma(\phi(u))p_k - p_k\sigma(\phi(u))\| \\
&= \|\sigma(\phi(u))p_k - \pi(u)p_k + p_k\pi(u) - p_k\sigma(\phi(u))\| \\
&\leq \|\sigma(\phi(u)) - \pi(u)\|\|p_k\| + \|p_k\|\|\pi(u) - \sigma(\phi(u))\| \\
&\leq 2(\varepsilon_2 + \varepsilon_4).
\end{aligned}$$

By fact 2.4.2 we know that every element $a$ of a unital C*-algebra is a linear combination of four unitaries whose coefficients are at most $\|a\|$. That is to say for each $a$, there exist unitaries $a_1$, $a_2$, $a_3$, $a_4$, and complex numbers $c_1, c_2, c_3$, and $c_4$ with $\|c_i\| \leq \|a\|$ such that $a = \sum c_i a_i$. Then as we know that $\|[\sigma(\phi(a_i)), p_k]\| \leq 2(\varepsilon_2 + \varepsilon_4)$ for each $i$, we can say that if $\|a\| \leq 1$

$$\begin{aligned}
\left\| \left[ \sum c_i \sigma(\phi(a_i)), p_k \right] \right\| &\leq \sum \|c_i\| \, \|[\sigma(\phi(a_i)), p_k]\| \\
&\leq 4(2(\varepsilon_2 + \varepsilon_4)) = 8(\varepsilon_2 + \varepsilon_4)
\end{aligned}$$

It is also clear that

$$\begin{aligned}\|[\sigma(\phi(c_i a_i)), p_k]\| &= \|[\sigma(\phi(c_i a_i)), p_k]\| \\ &= \|[\sigma(c_i\phi(a_i)) - \sigma(c_i\phi(a_i)) + \sigma(c_i\phi(a_i)), p_k]\| \\ &\leq \|[\sigma(c_i\phi(a_i)) - \sigma(\phi(c_i a_i)), p_k]\| + \|[\sigma(c_i\phi(a_i)), p_k]\| \\ &\leq 2\varepsilon + \|[c_i\sigma(\phi(a_i)), p_k]\|.\end{aligned}$$

Similarly, using the properties of an $\varepsilon^*$-homomorphism one can show that

$$\left\|\left[\sigma\left(\phi\left(\sum c_i a_i\right)\right), p_k\right]\right\| \leq \left\|\left[\sum \sigma\left(\phi\left(c_i a_i\right)\right), p_k\right]\right\| + 4\varepsilon.$$

Therefore, by combining all the previous results, we get

$$\begin{aligned}\|[\sigma(\phi(a)), p_k]\| &= \left\|\left[\sigma\left(\phi\left(\sum c_i a_i\right)\right), p_k\right]\right\| \\ &\leq \left\|\left[\sum \sigma\left(\phi\left(c_i a_i\right)\right), p_k\right]\right\| + 4\varepsilon \\ &= \left\|\sum \left[\sigma\left(\phi\left(c_i a_i\right)\right), p_k\right]\right\| + 4\varepsilon \\ &\leq \left\|\sum \left[c_i\sigma\left(\phi\left(a_i\right)\right), p_k\right]\right\| + 12\varepsilon \\ &\leq 8(\varepsilon_2 + \varepsilon_4) + 12\varepsilon,\end{aligned}$$

for all $a \in F$, and $\|a\| \leq 1$. Now, we define

$$\phi_k(a) = p_k((\sigma \circ \phi)(a))p_k.$$

It is clear that $\phi_k[F] \subseteq B(H_k)$, then we can say $\phi_k$ is a map from $F$ to $B(H)$. Moreover, one can show $\phi_k$ is an $\varepsilon_5$ homomorphism akin to the preceding results that are similar, where $\varepsilon_5 := 8(\varepsilon_2 + \varepsilon_4) + 12\varepsilon$.

Then as our constants $\gamma$, $K$, $\varepsilon$, are chosen in a way that they satisfy Theorem 3.4.1 we know there are $*$-homomorphisms $\psi_k : F \to B(H_k)$ such that $\|\phi_k - \psi_k\| \leq K\varepsilon_5$. Now as the range of each $\psi_k$ has a trivial intersection we can define $\psi' := \bigoplus_{k\in\mathbb{N}} \psi_k$. Now we consider $C = \psi'[F]$ and $B = \sigma[A]$, both of which are *-subalgebras of $B(H)$. We now try to prove the following claim.

**Claim 3.5.2.3.** *$C \subseteq^{\varepsilon_6} B$, where $\varepsilon_6 = K\varepsilon_5 + 2\varepsilon_4 + 2\varepsilon_2$.*

*Proof of Claim* . First we notice that for each $u \in U(F)$, we have

$$\begin{aligned}\|\psi'(u) - \pi(u)\| &= \sup_k \|\psi_k(u) - p_k\pi(u)p_k\| \\ &= \sup_k \|\psi_k(u) - \phi_k(u) + \phi_k(u) - p_k\pi(u)p_k\| \\ &\leq \sup_k \|\psi_k(u) - \phi_k(u)\| + \|\phi_k(u) - p_k\pi(u)p_k\| \\ &\leq \sup_k K\varepsilon_5 + \varepsilon_4 + \varepsilon_2.\end{aligned}$$

We need to prove that for each $\psi'(b) \in C$ there is a $\sigma(a) \in B$ such that $\|\psi'(b) - \sigma(a)\| \leq \varepsilon_6\|\psi'(b)\|$. We can choose $a = \phi(b)$, then

$$\begin{aligned}\|\psi'(b) - \sigma(a)\| &= \|\psi'(b) - \sigma(\phi(b))\| \\ &= \|\psi'(b) - \pi(b) + \pi(b) - \sigma(\phi(b))\| \\ &= \|\psi'(b) - \pi(b)\| + \|\pi(b) - \sigma(\phi(b))\| \\ &\leq K\varepsilon_5 + 2\varepsilon_4 + 2\varepsilon_2\end{aligned}$$ □

Then by theorem 3.5.1 there is a partial isometry $v \in B(H)$ such that $\|v - 1_H\| < 120\varepsilon_6^{1/2}$, and $vCv^* \subseteq B$. Let $\eta : F \to B(H)$ be such that $\eta(a) = v\psi'(a)v^*$. Then $\eta$ is a $*$-homomorphism (easy to see as it has just been conjugated by a partial isometry), moreover

$$\begin{aligned}\|\eta(a) - \psi'(a)\| &= \|v\psi'(a)v^* - \psi'(a)\| \\ &= \|v\psi'(a)v^* - v\psi'(a) + v\psi'(a) - \psi'(a)\| \\ &\leq \|v\psi'(a)v^* - v\psi'(a)\| + \|v\psi'(a) - \psi'(a)\| \\ &\leq \|v\psi'(a)\|\|v^* - 1\| + \|v - 1\|\|\psi'(a)\| \\ &\leq \|v\|\|\psi'(a)\|\|v^* - 1\| + \|v - 1\|\|\psi'(a)\| \\ &\leq \|v^* - 1\| + \|v - 1\| < 240\varepsilon_6^{1/2}.\end{aligned}$$

Since $\sigma$ was a faithful representation of $A$ on $H$, we can say that $\psi : F \to A$ such

that $\psi(x) = \sigma^{-1}(\eta(x))$ is well defined. Moreover, we also have the following

$$\begin{aligned}
\|\psi - \phi\| &= \|\sigma(\psi - \phi)\| \\
&= \|\sigma(\psi) - \sigma(\phi)\| \\
&= \|\eta - \sigma \circ \phi\| \\
&\leq \|\eta - \psi\| + \|\psi - \sigma \circ \phi\| \\
&< 240\varepsilon_6^{1/2} + K\varepsilon_5
\end{aligned}$$

The polynomial bound found can be seen to be $O(\varepsilon^{1/2})$, therefore there is some $L$ such that

$$\|\psi - \phi\| < L\varepsilon^{1/2}.$$

which doesn't depend on $F$, $A$, or $\phi$, therefore, the proof is complete. □

# CHAPTER 4

# CONNECTIONS TO QUOTIENT STRUCTURES IN C*-ALGEBRAS

## 4.1 Introduction to the Connection

As mentioned in the introductory chapter, one of the reasons to study Ulam stability is its connection to the conjecture template in Borel quotient structures mentioned in Chapter 1. We shall explore this connection briefly in this chapter. Section 4.2 will contain some more preliminary concepts only needed for this chapter. Section 4.3 and 4.4 are about understanding the connection of Ulam stability to the conjecture template in more detail. Section 4.3 focuses more on the Boolean algebra case, and Section 4.4 focuses on the C*-algebra case. For the brevity of this thesis, we will only provide a bird's-eye view of this very rich area of mathematical research and not get into the details of the proofs.

## 4.2 Preliminaries to this Chapter

### 4.2.1 Boolean Algebras

We will follow both [10] and [6] for this section. We equip $\mathcal{P}(\mathbb{N})$ with the Cantor space topology, identifying $\mathcal{P}(\mathbb{N})$ with the product space $2^{\mathbb{N}}$. This is done by taking the characteristic sequence of each subset of $\mathbb{N}$ and mapping it to an element of the Cantor space $2^{\mathbb{N}}$. Alternatively, one can view this as the topology on the binary tree, where the distance between two subsets of $\mathbb{N}$ is determined at the first element they differentiate at as a sequence. We explore further the connection between Ulam Stability and our conjecture template.

*Boolean algebras* are structures with operations $\wedge$, $\vee$, and $\neg$, and constants $0$ and $1$ that function like $\cap$, $\cup$, complement, $\emptyset$ and $X$ in $\mathcal{P}(X)$. One should note that we can view these as algebras by thinking of $\cup$ or $\vee$ as addition, and $\cap$ or $\wedge$ as multiplication. A *Boolean algebra homomorphism* will be a map preserving these properties. An *ideal* $\mathcal{I}$ of a Boolean algebra $\mathcal{X}$ is a non-empty subset of $\mathcal{X}$ such that for each $x, y \in \mathcal{I}$, $x \vee y \in \mathcal{I}$, and for each $a \in X$, $a \wedge x \in \mathcal{I}$. In the subset setting, we can interpret the $a \in \mathcal{X}$, $a \wedge x \in \mathcal{I}$ to mean the ideal is closed downward. We now look at the dual definition. An *filter* $\mathcal{I}$ of a Boolean algebra $\mathcal{X}$ is a non-empty subset of $\mathcal{X}$ such that for each $x, y \in \mathcal{I}$, $x \wedge y \in \mathcal{I}$, and for each $a \in \mathcal{X}$, $a \vee x \in \mathcal{I}$. In the subset setting, we can interpret the $a \in \mathcal{X}$, $a \vee x \in \mathcal{I}$ to mean the filter is closed upward.

### 4.2.2 The Forcing Idea and Forcing Axioms

The following is a very brief description of mathematical forcing. We suggest [2] for a beginner-friendly approach to forcing, and [14] for a detailed text on the topic.

Forcing was introduced by Cohen in order to prove the independence of the Continuum Hypothesis from the ZFC axioms. In essence, forcing takes a model $M$ of ZFC and extends it by adding a special type of filter $G$, obtaining a larger model $M[G]$. This filter contains information about how the extended model behaves when it is added. For example, a filter may contain 'conditions' which enforces the existence of an injective map from $\aleph_2$ to $\mathcal{P}(\mathbb{N})$ in the extended model. Such a model would prove that there is a model of ZFC in which the continuum hypothesis does not hold.

The forcing technique can be quite tedious to apply, therefore certain forcing axioms were developed to bypass the forcing process and use it as a sort of 'black box.' Since this thesis does not aim to go into detail regarding these axioms nor their applications, we shall only briefly mention these forcing axioms very briefly in this section as they are merely conditions used in Theorem 4.4.3.

#### 4.2.2.1 Martin's Axiom

This section is taken from [14, Chapter II.2]. A *partial order* is a pair $(\mathbb{P}, \leq)$ that is a nonempty set with a transitive, reflexive relation $\leq$. A *chain* in $\mathbb{P}$ is a subset $C$ where all elements are comparable, that is $\forall p, q \in C((p \leq q) \vee (q \leq p)$. Two elements in a partial order are *compatible* if

$$\exists r \in \mathbb{P}\ (r \leq p) \vee (r \leq q),$$

otherwise they are *incompatible.* An antichain is a subset of $\mathbb{P}$ where all elements are incompatible.

A partial order $(\mathbb{P}, \leq)$ has the *countable chain condition* or is c.c.c. if every antichain in $\mathbb{P}$ is countable.

**Definition 4.2.1.** Let $(\mathbb{P}, \leq)$ be a partial order.

1. A subset $D \subseteq \mathbb{P}$ is considered dense in $\mathbb{P}$ if and only if $\forall p \in \mathbb{P}\ \exists q \leq p\ (q \in D)$.
2. A subset $G \subseteq \mathbb{P}$ is a filter in $\mathbb{P}$ if and only if
   (a) $\forall p, q \in G\ \exists r \in G\ (r \leq p \wedge r \leq q)$, and
   (b) $\forall p \in G\ \forall q \in \mathbb{P}\ (q \leq p \rightarrow q \in G)$.

We shall now introduce one of the earliest forcing axioms, namely, Martin's axiom.

**Axiom 1.** *Let $\kappa < 2^{\aleph_0}$ be a cardinal. The axiom $\mathrm{MA}_\kappa$ is the following statement: Whenever $(\mathbb{P}, \leq)$ is a c.c.c partial order and $\mathcal{D}$ is a family of $\leq_\kappa$ dense subsets of $\mathbb{P}$, then there is a filter $G$ in $\mathbb{P}$ such that $\forall D \in \mathcal{D}\ (G \cap D \neq 0)$.*

**Axiom 2.** *Martin's axiom (*MA*) states the following: $\mathrm{MA}_\kappa$ holds for every $\kappa < 2^{\aleph_0}$.*

#### 4.2.2.2 Open Coloring Axiom

This section is taken from [9, Chapter 8.6].

For a set $X$, let $[X]^2$ denote its two element subsets. For $x, y \in X$, let $\{x, y\}_<$ denote an element of $[X]^2$ where $x < y$. If $L \subseteq [X]^2$, $Y \subseteq X$, such that $[Y]^2 \subseteq L$ we say

that $Y$ is $L$-homogeneous. One can see this definition as $Y$ being a monochromatic set in a coloring of edges between points of $X$. $Y$ is $L$-homogeneous if all edges between points of $Y$ are of the same color as edges in $L$. Recall Ramsey's theorem.

**Theorem 4.2.1** (Ramsey's Theorem)**.** *Let $[\mathbb{N}]^2 = L_0 \sqcup \ldots \sqcup L_k$, where $k$ is a natural, be a coloring. Then there is an infinite $L_i$-homogeneous set for $0 \leq i \leq k$.*

One can ask a similar question for uncountable spaces, whether every edge coloring will provide a monochromatic set of a color. However, this is false in general, but for a specific type of coloring, 'open' colorings we get a statement independent of ZFC.

Let $X$ be a topological space. We can see $[X]^2 = (X^2 \setminus \Delta_X)/\sim$ where $\sim$ is the equivalence relation such that $(x, y) \sim (y, x)$, and $\Delta_X$ is the diagonal on $X \times X$. Then $[X]^2$ can be equipped with the topology induced from the product topology $X \times X$. We say a coloring of edges of a topological space $[X]^2 = L_0 \sqcup L_1$ in two colors is an *open coloring* if $L_0$ is open in the aforementioned topology.

**Axiom 3.** $\mathbf{OCA_T}$ *Let $X$ be a separable metrizable space and $[X]^2 = L_0 \sqcup L_1$ be an open coloring. Then either*

1. *There is an uncountable $L_0$-homogeneous subset of $X$. Or,*
2. *There are $L_1$-homogeneous sets $X_n$, for $n \in \mathbb{N}$ such that*

$$\bigcup_{n\in\mathbb{N}} X_n = X.$$

$\mathrm{OCA_T}$ is relatively consistent with ZFC and is a consequence of the Proper Forcing Axiom.

**Theorem 4.2.2.** $\mathrm{OCA_T}$ *is incompatible with the continuum hypothesis.*

There are other equivalent formulations of $\mathrm{OCA_T}$, such as $\mathrm{OCA}_\infty$ and $\mathrm{OCA}^\#$, however we shall not cover them in this section.

### 4.2.3 Direct Sums and Products of C*-algebra

The material in this section is following [9, Chapter 2.1].

A *direct sum of two C*-algebras* $A$ and $B$ is the algebra of all pairs in the cartesian product with pointwise defined operations and for $(a,b) \in A \oplus B$, $\|(a,b)\| = \max\{\|a\|, \|b\|\}$. Direct sum of any number of finitely many C*-algebras can be defined similarly.

Lt $J$ be an index set. We can also define the direct sum of C*-algebras $A_j$ for $j \in J$ as follows

$$\bigoplus_{i\in J} A_i := \{(a_i : i \in J) : a_i \in A_i \text{ for all } i, \text{ and } \{i : \|a_i\| > 1/n\} \text{ is finite for all } n\}.$$

We shall now define the direct product of C*-algebras. If we have finitely many C*-algebras then their direct product and direct sum are the same. Let $J$ be an index set such that. $A_i$ for $i \in J$ are C*-algebras. We define their *direct product* as

$$\prod_{i\in J} A_i = \left\{(a_i : i \in J) : \sup_{i\in J} \|a_i\| < \infty\right\}.$$

The operations here are again pointwisely defined. If the index set is infinite then $\prod_{i\in J} A_i$ contains $\bigoplus_{i\in J} A_i$ as a proper ideal.

## 4.3 Connection to Liftings: Part I

This section is taken from [10, Section 5] for the most part. Other sources are reffered if used for a specific subsection.

**Convention 4.3.1.** For this section, $\mathcal{G}$ shall denote a category of finite algebraic structures equipped with a metric (with uniformly bounded diameter) in a fixed finite signature.

We shall begin by defining an $\varepsilon$-homomorphism between structures in $\mathcal{G}$. These will be analogs of all the other such definitions we have seen before but we shall mention them in two relevant cases: Boolean algebras and groups.

### 4.3.1 Ulam Stability in Boolean Algebras

We use [5] for this subsection.

A map $\mu : \mathcal{P}(X) \to [0, \infty]$ is a *submeasure supported by* $X$ if $\mu(\emptyset) = 0$, and $\mu$ is monotonic and subadditive. Note that we can use this measure as a sort of "metric" in order to define and discuss Ulam stability.

We denote subsets of $m$ and $n$ by $\{0,1\}^{[m]}$ and $\{0,1\}^{[n]}$ respectively. These are finite structures, and they agree with the established convention for this section.

**Definition 4.3.1.** Assume $n$ (as a set) supports a submeasure $\mu$ and let $\varepsilon, \delta > 0$. A map $H : \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ is a *$\varepsilon$-approximate homomorphism* (with respect to $\mu$) if for all $s$ and $t$ subsets of $m$ we have

$$\mu((H(s) \cup H(t))\Delta H(s \cup t)) < \varepsilon \text{ and} \tag{4.1}$$

$$\mu(H(s^c)\Delta H(s)) < \varepsilon. \tag{4.2}$$

A boolean algebra homorphism $\Phi : \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ is a $\delta$-approximation for $H$ if

$$\mu(\Phi(s)\Delta H(s)) < \delta. \tag{4.3}$$

Recall that in the case of finite metric groups, Definition 1.1.1 can be used.

**Theorem 4.3.1.** *[5, Theorem 4] For every map $H : \{0,1\}^{[m]} \to \{0,1\}^{[n]}$ there is a homomorphism $\Phi : \{0,1\}^{[m]} \to \{0,1\}^{[n]}$. Such that for every measure $\mu$ supported by $n$ and for every $\varepsilon$, if $H$ is an $\varepsilon/K_M$-approximate homomorphism with respect to $\mu$, then $\Phi$ is an $\varepsilon$-approximation to $H$. Also, $K_M \leq 521$ univerally.*

The preceding theorem states Ulam stability in finite Boolean algebras. There are results for Ulam stability of finite groups in [7].

### 4.3.2 Ideals and Reduced Products

We now look at some essential definitions to be used later in this section.

**Definition 4.3.2.** An ideal $\mathcal{I}$ is a *$P$-ideal* if for any countable collection $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{I}$, for $n \in \mathbb{N}$, there is a $A \in \mathcal{I}$ such that $A_n \setminus A \in \mathrm{Fin}$ for all $n$.

**Definition 4.3.3.** $\mathcal{G}$ is the same as before. If $(A_n, d_n)$, for $n \in \mathbb{N}$ is a sequence of metric structures in $\mathcal{G}$ and $\mathcal{I}$ is an ideal on $\mathbb{N}$, the *reduced product*, denoted

$$\prod_n A_n / \bigoplus_{\mathcal{I}} A_n$$

is the quotient of the product $\prod_n A_n$ modulo the congruence identifying $(x_n)_{n\in\mathbb{N}}$ with $(y_n)_{n\in\mathbb{N}}$ if and only if

$$\lim_{n\to\mathcal{I}} d_n(x_n, y_n) = \inf_{X\in\mathcal{I}} \sup_{n\notin X} d_n(x_n, y_n) = 0.$$

Note that in the preceeding definition

$$\lim_{n\to\mathcal{I}} d_n(x_n, y_n) = \inf_{X\in\mathcal{I}} \sup_{n\notin X} d_n(x_n, y_n) = 0$$

can also be seen as

$$\lim_{n\to\mathcal{I}} d_n(x_n, y_n) = \inf_{Y\in\mathcal{F}} \sup_{n\in Y} d_n(x_n, y_n) = 0$$

where $\mathcal{F}$ is the filter obtained by taking complements of the elements of the ideal $\mathcal{I}$,

$$\mathcal{F} = \{X^c : X \in \mathcal{I}\}$$

**Definition 4.3.4.** Let $\mathcal{I}$ and $\mathcal{J}$ be Borel ideals in $\mathcal{P}(\mathbb{N})$. A homomorphism

$$\Phi : \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$$

is said to be *algebraically trivial* if it has a lifting $\Phi_* : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ which is a Boolean algebra homomorphism.

### 4.3.3 Connections to Triviality

This subsection aims to show how Ulam stability connects topological triviality to algebraic triviality in the case of finite algebraic structures. This section is mostly taken from [10, Section 5].

For this section, let $\mathcal{G}$ denote a category of finite algebraic structures equipped with a metric (with uniformly bounded diameter) in a fixed finite signature. Finite Boolean algebras and finite groups are examples of this. Let $\mathcal{G}$'s signature contain a constant $0$ and binary operation $+$. Here $0$ is interpreted as the neutral element for $+$ for every

element of $\mathcal{G}$. We have the analogs of $+$ and $0$ in the Boolean algebra case. We can even consider $\cdot$ if needed.

We will use the following notation for the next few sections. For $I \subset \mathbb{N}$, $A_I = \prod_{n\in I} A_n$, identifying $A_I$ with

$$\{a \in A_{\mathbb{N}} : a_n = 0 \text{ for all } n \notin I\}.$$

This allows us to see $A_I$ as a quotient of $A_{\mathbb{N}}$ We use $\pi_I : A_{\mathbb{N}} \to A_I$ to mean the natural projection.

**Definition 4.3.5.** A function $\Phi_* : \prod_n A_n \to \prod_n B_n$ is *asymptotically additive* if there are partitions $\mathbb{N} = \bigsqcup_n I(n) = \bigsqcup_n J(n)$ into finite intervals and maps $f_n : A_{I(n)} \to B_{J(n)}$ such that $\Phi_*((a_m)) = (f_n(a_m))_{m\in I(n)}$, for each $m \in I(n)$.

We identify $\mathcal{P}(\mathbb{N})$ with $\prod_{n\in\mathbb{N}}\{0,1\}$

**Theorem 4.3.2.** *[6, Theorem 1.5.2] If $\mathcal{I}$ is an analytic $P$-ideal then every topologically trivial homomorphism $\Phi : \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ has an asymptotically additive lifting.*

In terms of our convention we have the following result

**Theorem 4.3.3.** *[10, Lemma 5.8] Suppose that $\mathcal{G}$ is a category of Boolean algebras or groups as in convention 4.3.1 and $A_n$ and $B_n$ belong to $\mathcal{G}$. Let $\mathcal{I} \supset \mathrm{Fin}$ and $\mathcal{J} \supset \mathrm{Fin}$ be ideals in $\mathcal{P}(\mathbb{N})$. If*

$$\Phi : \prod_n A_n / \bigoplus_{\mathcal{I}} A_n \to \prod_n B_n / \bigoplus_{\mathcal{J}} B_n$$

*is a topologically trivial homomorphism, then $\Phi$ has an asymptotically additive lifting.*

One shall note here that the viability of using Ulam stability to connect topological triviality to algebraic triviality often depends on 'asymptotic additivity' or similar properties (we will also see this in the C*-algebra case in this chapter). Therefore, the following problem was posed in [10].

**Problem 4.3.1.** *[10, Problem 5.9] Finding other cases where the analog of the preceding lemma holds.*

Progress made on Problem 4.3.1 connects Ulam Stability to topologically trivial homomorphisms being algebraically trivial.

The following proposition and the sketch of the proof are taken from [10]. We now assume that $\mathcal{G}$ is closed under taking finite products.

**Proposition 4.3.1.** *[10, Proposition 5.10] The following are equivalent*

1. *The class $\mathcal{G}$ is Ulam-stable.*
2. *If a homomorphism*

$$\Phi : \prod_n A_n / \bigoplus_{\mathcal{I}} A_n \to \prod_n B_n / \bigoplus_{\mathcal{J}} B_n$$

*between reduced products of structures in $\mathcal{G}$ over ideals that include* $\mathrm{Fin}$ *has an asymptotically additive lifting then it is algebraically trivial.*

*Proof Sketch.* $(2 \Rightarrow 1)$ Assume toward the contrapositive that $\mathcal{G}$ is not Ulam stable. Then there is some $\varepsilon$ for each $n \geq 1$ such that there is some $1/n$-homomorphism $f_n : A_n \to B_n$ that cannot be uniformly $\varepsilon$-approximated by a homomorphism. Then these $f_n$'s define an asymptotically additive lifting of a homomorphism as $n \to \infty$ the $f_n \to$ actual homomorphisms (and use the definition of the reduced product). However, the lifting these $f_n$'s will define will not be algebraically trivial.

$(1 \Rightarrow 2)$ Let $\Phi^*$ be an asymptotically additive lifting of $\Phi$. $\mathcal{G}$ is closed under finite products, we can asssume $I(n) = J(n) = \{n\}$ (see definition 4.3.5) for each $n$. Now we can define $f_n : A_n \to B_n$ using the preceding fact and the asymptotically additive lifting. By the finiteness of the signature and the structure, we can say that for each $m \geq 1$,

$$X_m = \{n : f_n \text{ is not a } 1/m - \text{homomorphism}\}$$

is in $\mathcal{I}$ as it contains $\mathrm{Fin}$. Choose $\varepsilon_m$ such that every $1/m$-homomorphism can be $\varepsilon_m$-approximated by a true homomorphism and $\varepsilon_m \to 0$. Let $X_0 = \mathbb{N}$ and for each $j \in X_n \setminus X_{n+1}$ we replace $f_n$ with a true homomorphism $g_n$ that $\varepsilon_n$-approximates it. Any $f_n$ that is not in

$$\bigcup_n X_n \setminus X_{n+1}$$

is already a homomorphism. The lifting is now a true homomorphism, and hence algebraically trivial. □

## 4.4 Connections with Liftings: Part II

### 4.4.1 The Metric Approximation Property

We shall note again that most of the content in this section is from [16].

A unital *-closed vector subspace of $B(H)$ is called an *operator system*.

**Definition 4.4.1.** A C*-algebra $A$ has the *metric approximation property* or *MAP* if for all finite $F \subseteq A$ and $\varepsilon > 0$ there is a finite-dimensional operator system $E$ and unital linear contractions $\varphi : A \to E$ and $\psi : E \to A$ with $\|\psi \circ \varphi(a) - a\| < \varepsilon$ for all $a \in F$. That is to say, $A$ has the MAP if the identity map can be uniformly approximated on finite sets by contractive linear maps of finite rank.

We have the following as examples regarding the MAP. The reader shall note that nuclearity in C*-algebras is the analog of amenability in groups.

**Theorem 4.4.1.** *[1] All nuclear C*-algebras have the MAP.*

If the contractions condition in definition 4.4.1 is dropped, we get the *approximation property*.

**Theorem 4.4.2.** *[24] The C*-algebra $B(H)$ does not have the approximation property. Therefore, $B(H)$ does not have the MAP.*

### 4.4.2 Connections to Triviality

This subsection is taken from [16, Section 3]. The role Ulam stability plays in bridging the gap between topologically trivial isomorphisms and algebraically trivial ones is quite similar to the Boolean algebra case. First, a link between topological triviality and asymptotically algebraic isomorphisms (analog of asymptotically additive) is established. Then Ulam stability of a class implies that the asymptotic additivity and

algebraic triviality of isomorphisms between quotients of the reduced products from the same class coincide.

We begin by looking at two results connecting liftings of isomorphisms between reduced products to a sequence of $\varepsilon$-*-homomorphisms. The following proposition shows how $\varepsilon$-*-homomorphims in C*-algebras induce a *-homomorphism in the reduced product.

**Proposition 4.4.1.** *Suppose $A_n$ and $B_n$ are sequences of C*-algebras, and define $A = \prod A_n / \bigoplus A_n$ and $B = \prod B_n / \bigoplus B_n$. Then any sequence $\varphi_n : A_n \to B_n$ od $\varepsilon_n$-*-homomorphisms, where $\varepsilon_n \to 0$, induces a *-homomorphism $\Phi : A \to B$.*

The following lemma connects liftings of maps between reduced products to sequences of $\varepsilon_n$-*-isomorphisms.

**Lemma 4.4.1.** *[16, Lemma 3.12] Let $A_n$, $B_n$ be C*-algbras, let $g$ be an almost permutation of $\mathbb{N}$ and $\varphi_n : A_n \to B_{g(n)}$ be maps such that $\prod \varphi_n$ is the lift of an isomorphism*

$$\Lambda : \prod A_n / \bigoplus A_n \to \prod B_n / \bigoplus B_n$$

*on a dense ideal $\mathcal{I}$. Then there is a sequence $\varepsilon_n \to 0$ such that each of the $\varphi_n$'s are $\varepsilon_n$-*-isomorphism.*

**Definition 4.4.2.** Let $A_n$ and $B_n$ be C*-algebras with no nontrivial central projections. An isomorphism

$$\Lambda : \prod A_n / \bigoplus A_n \to \prod B_n / \bigoplus B_n$$

is *asymptotically algebraic* if there are finite sets $F_1, F_2$, a bijection $g : \mathbb{N} \backslash F_1 \to \mathbb{N} \backslash F_2$ and maps $\varphi_n : A_n \to B_{g(n)}$ such that the map $\Phi : \prod A_n \to \prod B_n$ defined by

$$\Phi(x)_m = \begin{cases} \varphi_n(x_n) \text{ if } m = g(n) \\ 0 \text{ if } m \in F_2 \end{cases}$$

is a lift of $\Lambda$. The maps $g$ and $(\varphi_n)_n$ are said to *witness the well-behavedness* of $\Lambda$. If there is $n_0$ such that for all $n \geq n_0$ the map $\varphi_n$ can be chosen to be an isomorphism, we say that. $\Lambda$ is *algebraically trivial*.

The curious reader may ask why we require the C*-algebras $A_n$'s to not have non-trivial central projections. We can consider the following example for our answer.

**Example 4.4.1.** [16, Remark 3.14] Without loss of generality, we shall assume that the C*-algebras are unital. Otherwise, we can work with projections. Let each $A_n = C_n \oplus D_n$, a direct sum of two unital C*-algebras. Let $B_i$ be $C_i$ for even $i$'s and $D_i$ for odd $i$'s, respectively. Notice that $A_n$ has a central projection of the form $(1_{C_n}, 0)_{n\in\mathbb{N}}$, where $1_{C_n}$ is the identity in $A_n$. Such projections are also non-trivial as they commute with $(0, d_n)_{n\in\mathbb{N}}$, where $d_n \in D_n$. Over here, we shall note that $\prod_{n\in\mathbb{N}} A_n/\bigoplus_{n\in\mathbb{N}} A_n \cong \prod_{n\in\mathbb{N}} B_n/\bigoplus_{n\in\mathbb{N}} B_n$, as in terms of product and direct sums there are the same entities. We consider the identity map

$$\phi : \prod_{n\in\mathbb{N}} A_n/\bigoplus_{n\in\mathbb{N}} A_n \to \prod_{n\in\mathbb{N}} B_n/\bigoplus_{n\in\mathbb{N}} B_n.$$

This is the map that sends

$$[((c_0, d_0), (c_1, d_1), \ldots)] \mapsto [(c_0, d_0, c_1, d_1, \ldots)]$$

Assume towards a contradiction that this identity map is asymptotically algebraic. Then, by definition, there are finite sets $F_1, F_2$, a bijection $g : \mathbb{N} \setminus F_1 \to \mathbb{N} \setminus F_2$ and maps $\varphi_n : A_n \to B_{g(n)}$ such that the map $\Phi : \prod A_n \to \prod B_n$ defined by

$$\Phi(x)_m = \begin{cases} \varphi_n(x_n) \text{ if } m = g(n) \\ 0 \text{ if } m \in F_2 \end{cases}$$

is a lift of $\Phi$. By Lemma 4.4.1 we know that there are $\varepsilon_n$-*-isomorphisms between $\varphi_n : A_n \to B_{g(n)}$ where $\varepsilon_n \to 0$. But notice that $A_n = C_n \oplus D_n$ but $B_{g(n)}$ is either $C_{g(n)}$ or $D_{g(n)}$. But there is no way for us to have $\varepsilon$-*-isomorphism between $A_n$ and $B_{g(n)}$.

The following is one of the main results of [16]. It connects topologically trivial isomorphisms to asymptotically algebraic ones. The proof of this theorem is outside the scope of this thesis.

**Theorem 4.4.3.** *[16, Theorem E]* $(\mathrm{OCA} + \mathrm{MA}_{\aleph_1})$ *Let $A_n$ and $B_n$ be seperable unital C*-algebras with no non-trivial central projections. Suppose that each $A_n$ has the metric approximation property. Then all topologically trivial isomorphisms*

$$\Lambda : \prod A_n/\bigoplus A_n \to \prod B_n/\bigoplus B_n$$

*are asympototically algebraic.*

The following theorem, analogous to Proposition 4.3.1, shows an instance of how the gap between asymptotically algebraic isomorphisms and algebraically trivial ones is abridged using Ulam stability.

**Theorem 4.4.4.** *[16, Theorem 3.16] Let $\mathcal{C}$ be a class of unital separable C*-algebras with no nontrivial central projections. Then the following are equivalent.*

1. *The class $\mathcal{C}$ is Ulam stable.*
2. *For $A_n$ and $B_n$, elements of $\mathcal{C}$, all asymptotically algebraic isomorphisms between $\prod A_n/\bigoplus A_n$ and $\prod B_n/\bigoplus B_n$ are algebraically trivial.*

*Proof.* $(1 \Rightarrow 2)$ Let $\mathcal{C}$ be Ulam stable. Fix sequences of C*-algebras $A_n$ and $B_n$ in $\mathcal{C}$ and an asymptotically algebraic isomorphism

$$\Lambda : \prod A_n/\bigoplus A_n \to \prod B_n/\bigoplus B_n.$$

Then by definition, there exist finite subsets $F_1, F_2 \subseteq \mathbb{N}$, and a bijection $g : \mathbb{N} \setminus F_1 \to N \setminus F_2$ such that

$$\Phi(x)_m = \begin{cases} \varphi_n(x_n) \text{ if } m = g(n) \\ 0 \text{ if } m \in F_2 \end{cases}$$

is a lifting of $\Lambda$. By Lemma 4.4.1, there is a sequence $\varepsilon_n$ going to $0$, such that each $\varphi_n$ is an $\varepsilon_n$-*-isomorphism. Without loss of generality, by re-indexing the $B_n$'s and by ignoring finitely many of them, we can assume that $g$ is the identity map and $\varepsilon_n$'s are decreasing. We can do this as for every $\varepsilon' > \varepsilon$, a $\varepsilon$-*-homomorphism is also a $\varepsilon'$-*-homomorphism. We can ignore the set $F_2$ as it is finite and therefore won't affect algebraic triviality. Similarly, by the finiteness of the lower bound $n_0$ after which for all $n > n_0$ all $\varphi_n$ have to be *-isomorphisms for Ulam stability, we can assume $g : m \mapsto m$ for each $m \in \mathbb{N}$.

We shall now define a sequence $k_n$ inductively. We let $k_0 = 0$. Assume now that $k_i$ has been defined for $i < n$. We set $k_n$ to be the least natural number such that $k_n > k_{n-1}$, and all $\varepsilon_{k_n}$-*-homomorphisms can be $2^{-n}$-approximated by a *-homomorphism. Such a $k_n$ exists as $\mathcal{C}$ is Ulam stable. Then for all $j \in [k_n, k_{n+1})$, we

done the *-homomorphism close to $\varphi_n$ by $\psi_n$, and we have that $\|\psi_j - \varphi_j\| < 2^{-n}$. But then we have that $\prod \varphi_n$ and $\prod \psi_n$ define the same lifting; however, the lifting $\Psi := \prod \psi_n$ is mostly (leaving finitely many indices) is made from *-homomorphisms. Therefore, we obtain topological triviality.

$(2 \Rightarrow 1)$ Assume now that $\mathcal{C}$ is not Ulam stable. Then there exist an $\varepsilon > 0$, such that for all $\delta > 0$, there are $A_\delta$ and $B_\delta$ in $\mathcal{C}$, there is a $\delta$-*-homomorphism $\varphi_\delta$ such that for any *-homomorphism $\psi_\delta : A_\delta \to B_\delta$, $\|\varphi_\delta - \psi_\delta\| \geq \varepsilon$. As we can do this for any $\delta > 0$, we let $A_n := A_{\frac{1}{n}}$ and $B_n := B_{\frac{1}{n}}$ and $\varphi_n := \varphi_{\frac{1}{n}} : A_{\frac{1}{n}} \to B_{\frac{1}{n}}$. Then, by Proposition 4.4.1, these produce an asymptotically algebraic lifting. This is not an algebraically trivial map. If that were the case, there would be some $n_0$ such that for all $m > n_0$, the $\psi_m$'s that constitute the lifting would be *-isomorphisms. However, if that were the case then $\|\psi_m - \varphi_m\| \to 0$, which would contradict $\mathcal{C}$ not being Ulam stable. $\square$

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