SUPPORTING THE DEVELOPMENT OF PROSPECTIVE MATHEMATICS TEACHERS' MATHEMATICAL KNOWLEDGE AND PROFESSIONAL NOTICING SKILLS ON THE PERIMETER CONCEPT THROUGH THE MODEL GROUNDED ON THE CYCLES OF ENACTMENT AND INVESTIGATION

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ABSTRACT

SUPPORTING THE DEVELOPMENT OF PROSPECTIVE MATHEMATICS TEACHERS' MATHEMATICAL KNOWLEDGE AND PROFESSIONAL NOTICING SKILLS ON THE PERIMETER CONCEPT THROUGH THE MODEL GROUNDED ON THE CYCLES OF ENACTMENT AND INVESTIGATION

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This study aims to support the improvement of prospective middle school mathematics teachers' mathematical knowledge and professional noticing skills on the perimeter concept through the development of the model based on the cycles of enactment and investigation within the framework of design-based research. Data were collected from senior prospective mathematics teachers who are enrolled in one of the sections of the Teaching Practice 2 course in the spring semester of 2020-2021 at one of the state universities in Istanbul. The data obtained from these prospective teachers were analyzed as a group, while the results obtained from one prospective teacher who had perimeter teaching experience in different classroom settings as part of the current study were presented. Multiple data sources were used in the data collection process, such as pre- and post-tests, post-interviews, initial and revised lesson plans, individual responses to classroom activities, videotapes of the model implementations, and field notes. Qualitative methods were used to analyze the data. Findings from the pretest indicated that the prospective teacher lacked mathematical

knowledge and professional noticing skills on the perimeter concept prior to participating in the current study. In contrast, the findings obtained from the post-test and post-interview results showed that there were significant improvements in her mathematical knowledge and professional noticing skills on the perimeter concept throughout the study. Based on the findings of this study, it can be argued that the model based on the cycle of enactment and investigation can be used as an effective practical tool to develop preservice teachers' mathematical knowledge and professional noticing skills on the perimeter concept.

Keywords: The Cycles of Enactment and Investigation, Mathematical Knowledge, Professional Noticing Skills, Prospective Teacher, The Perimeter Concept

ÖZ

MATEMATİK ÖĞRETMEN ADAYLARININ ÇEVRE KAVRAMINA İLİŞKİN MATEMATİKSEL BİLGİ VE MESLEKİ FARK ETME BECERİLERİNİN GELİŞİMİNİN UYGULAMA VE İNCELEME DÖNGÜLERİNE DAYANAN MODEL YOLUYLA DESTEKLENMESİ

Dışbudak Kuru, Özge
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Bu çalışmanın amacı, ortaokul matematik öğretmen adaylarının çevre kavramına ilişkin matematiksel bilgi ve mesleki fark etme becerilerini, tasarım tabanlı araştırma çerçevesinde uygulama ve inceleme döngülerine dayalı bir model geliştirerek desteklemektir. Veriler, İstanbul'daki devlet üniversitelerinden birinde 2020-2021 bahar döneminde Öğretmenlik Uygulaması 2 dersinin bir şubesine kayıtlı son sınıf matematik öğretmeni adaylarından toplanmıştır. Öğretmen adaylarından elde edilen veriler grup olarak analiz edilirken, mevcut çalışma kapsamında farklı sınıf ortamlarında çevre öğretimi deneyimi olan bir öğretmen adayından elde edilen bulgular sunulmuştur. Veri toplama sürecinde ön ve son testler, son görüşmeler, ilk ve revize edilmiş ders planları, sınıf içi etkinliklere verilen bireysel yanıtlar, modelin uygulanmasına ilişkin video kayıtları ve saha notları gibi çoklu veri kaynakları kullanılmıştır. Verileri analiz etmek için nitel yöntemler kullanılmıştır. Ön testten elde edilen bulgular öğretmen adayının çevre kavramına ilişkin matematiksel bilgi ve mesleki fark etme becerilerinde eksiklikler olduğunu göstermiştir. Buna karşın, son test ve son mülakat sonuçlarından elde edilen bulgular, çalışma boyunca öğretmen adayının çevre kavramına ilişkin matematiksel bilgi ve mesleki fark etme

becerilerinde önemli gelişmeler olduğunu göstermiştir. Bu çalışmanın bulgularına dayanarak, uygulama ve inceleme döngüsüne dayanan modelin öğretmen adaylarının çevre kavramına yönelik matematiksel bilgilerini ve mesleki fark etme becerilerini geliştirmede etkili bir uygulama aracı olarak kullanılabileceği söylenebilir.

Anahtar Kelimeler: Uygulama ve İnceleme Döngüsü, Matematiksel Bilgi, Mesleki Fark Etme Becerisi, Öğretmen Adayı, Çevre Kavramı

To my darling daughter and To all children around the world

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LIST OF ABBREVIATIONS

ABBREVIATIONS

CCK	Common Content Knowledge
CEIs	Cycles of Enactment and Investigation
EME	Elementary Mathematics Education
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
KFLM	Knowledge of Features of Learning Mathematics
KMLS	Knowledge of Mathematics Learning Standards
KMT	Knowledge of Mathematics Teaching
KoT	Knowledge of Topics
KPM	Knowledge of Practices in Mathematics
KPSS	Public Personnel Selection Examination
KSM	Knowledge of the Structure of Mathematics
KQ	Knowledge Quartet
MK	Mathematical Knowledge
MoNE	Ministry of National Education
MTSK	The Mathematics Teacher's Specialised Knowledge
NCTM	National Council of Teachers of Mathematics
PCK	Pedagogical Content Knowledge
SCK	Specialized Content Knowledge

CHAPTER 1

INTRODUCTION

> "Effective teaching requires attending to students' mathematical thinking and reasoning during instruction. Well-prepared beginners commit themselves to noticing, eliciting, and using student thinking to assess student progress in understanding mathematics and to adjust instruction in ways that further support and advance learning toward the intended learning goals" (Association of Mathematics Teacher Educators, 2017, p. 16).

Focusing on students' mathematical ideas and adapting the instructions based on these ideas are key to effective teaching. However, effective teaching is a complex endeavor because responding to students' mathematical understanding is not an easy task for teachers (Jacobs et al., 2010; 2011; Sherin et al., 2011; Star & Strickland, 2008; van Es & Sherin, 2021). To recognize and respond to students' mathematical understanding, effective teachers need to have well-developed mathematical knowledge (Copur-Gencturk & Tolar, 2022; Jarry-Shore & Borko, 2023).

Considering the role of teachers' knowledge in effective teaching, researchers in mathematics education have paid attention to teachers' knowledge throughout the years (e.g., Ball & Bass, 2000; 2003; Ball et al., 2008; Carpenter et al., 1989; Hill et al., 2007; 2008; Grossman, 1990; Rowland et al., 2005; Shulman, 1986; Tirosh, 2000) and categorized it differently (e.g., Ball et al., 2008; Grossman, 1990; Rowland et al., 2003; 2005; 2009; Shulman, 1986; 1987). In the most general sense, teachers' knowledge was first defined and put into categorization by Shulman (1986). Using Shulman's categorization by reference, many researchers in mathematics education have developed different conceptualizations of teacher knowledge with different focuses and aims (e.g., Ball et al., 2008; Carrillo-Yañez et al., 2018; Rowland et al., 2005). Indeed, some of these researchers categorized teacher knowledge into

pedagogical knowledge and subject matter knowledge (e.g., Ball et al., 2008), while some of them advocated that the concepts of pedagogical content knowledge and subject matter knowledge should be examined together (e.g., Huckstep et al., 2006; Rowland et al., 2003; 2005; Rowland & Turner, 2007; Turner, 2007). For example, Rowland et al. (2003; 2005) developed the Knowledge Quartet Model (KQ) as mathematical knowledge in teaching, which is a combination of pedagogical content knowledge and subject matter knowledge with four dimensions, which are Foundation Dimension, Transformation Dimension, Connection Dimension, and Contingency Dimension, and several components. In the most general sense, they believe that teachers should know and understand mathematics, including operations, rules, relations, and the underlying reasons behind them conceptually, and use that knowledge to adapt instruction considering students' understanding. Indeed, as Ball (1990) states,

> They (teachers) need to understand the subject flexibly enough so that they can interpret and appraise students' ideas, helping them to extend and formalize intuitive understandings and challenging incorrect notions (p. 458).

Among these dimensions of the KQ model, the Foundation, which includes theoretical knowledge about mathematics and its teaching in the most general sense (Rowland et al., 2005), addresses preservice teachers' knowledge through predetermined components. Among these components, for example, *overt subject knowledge* concerns how preservice teachers define mathematical concepts, and *using terminology* is related to how they use mathematical language and terminology. *Identifying errors* is related to how they identify students' difficulties regarding mathematics, and the *theoretical underpinning of pedagogy* is related to what kinds of teaching strategies and materials they use or plan to use in their teaching. In this sense, mathematics and the teaching of mathematics were simultaneously in this dimension.

As stated before, although there is still no consensus on how teachers' knowledge should be categorized, it is accepted that it plays a crucial role in developing students' understanding and responding to their mathematical thinking (Ball et al., 2008; Borko, 2004; Hill et al., 2007; 2008; Rowland et al., 2003; 2005; Shulman, 1986). Unfortunately, most studies on teachers' knowledge have shown that teachers and preservice teachers lack mathematical knowledge of various mathematical concepts such as fractions (e.g., Işıksal & Çakıroğlu, 2011) and the perimeter concept (e.g., Doğan-Coşkun & Işıksal-Bostan, 2019; Reinke, 1997; Yenilmez & Koza-Çiftçi, 2014) and have difficulty of responding students' mathematical understanding due to a lack of knowledge (e.g., Doerr & Lesh, 2003; Zeytun et al., 2010).

Researchers have emphasized that understanding is not a static state but a dynamic process that varies from person to person (e.g., Carpenter et al., 1996). Therefore, focusing on students' mathematical understanding is difficult for teachers and preservice teachers, who lack experience communicating with students and teaching in a complex and dynamic classroom environment where many things are happening simultaneously (Jacobs et al., 2010). Indeed, preservice teachers generally struggle to decide which and how students' thinking should be focused and "what mathematics to pursue and how to pursue it" during instruction (Sherin, 2002, p. 122) since they are mainly interested in managing classrooms to establish their authority rather than focusing on students' mathematical understanding during their instruction (Star et al., 2011; Star & Strickland, 2008). Hence, they tend to focus on the results rather than the process of solving problems by overlooking the essence of students' ways of thinking. Thus, in addition to teachers' knowledge, teachers need other competencies to recognize and respond to students' mathematical understanding since this process needs to be able to hear and interpret students' way of thinking. Thus, teacher's noticing skill is one that effective teacher should have (e.g., Jacobs & Empson, 2015; Jacobs et al., 2010; Kaiser et al., 2015; 2017; Sherin et al., 2011; van Es & Sherin, 2021).

In the most general sense, noticing is "a part of everyday life, indicating the act of observing or recognizing something" (Liu, 2014, p. 1). If the ability to notice is considered a skill that teachers have, then noticing deals with "a blooming, buzzing confusion of classroom events" (Sherin & Star, 2011, p. 73) as teachers meet. State differently, teacher noticing is a skill that helps teachers make sense of what they observe in classrooms (Star & Strickland, 2008; van Es & Sherin, 2002; Jacobs et al., 2010). van Es and Sherin (2002) emphasized three essential features of teacher noticing as follows:

> (a) identifying what is important or noteworthy in a classroom situation,
>
> (b) making connections between the specific events and the broader principles of teaching and learning, and
>
> (c) using what one knows in the context to reason about classroom interactions" (p. 573).

As van Es and Sherin (2002) explained, teacher noticing is an active process that includes being aware of noteworthy events in the classroom environment and making sense of them. While van Es and Sherin (2002) focus on classroom events in describing noticing, Jacobs et al. (2010) narrow their focus and restrict teachers' noticing to students' mathematical thinking. Regarding this idea, they are interested in how and to what extent teachers notice students' mathematical thinking; therefore, they call their framework Professional Noticing of Children's Mathematical Thinking, which consists of a set of three interrelated subskills: "(a) attending to children's strategies, (b) interpreting children's understanding, and (c) deciding how to respond based on children's understandings" (Jacobs et al., 2010, p.169).

In the most general sense, the first component, *attending*, includes identifying mathematically important details in students' problem-solving strategies regarding particular mathematical content. The second component, *interpreting*, involves making inferences about students' mathematical understandings of concepts based

on their strategies. Finally, the third component, *deciding*, includes determining the following steps to extend the students' mathematical understanding of concepts. Overall, focusing on the details of students' strategies, making sense of their understanding based on their strategies, and deciding how to respond to these students' mathematical thinking is called a particular type of noticing skill, namely, teachers' *professional noticing skills* (Jacobs et al., 2010; van Es, 2011; van Es & Sherin, 2008; Star & Strickland, 2008). Since this study was interested in preservice teachers' professional noticing skills, decision-making processes, and in-the-moment decisions in response to students' mathematical thinking, Jacobs et al.'s (2010) definition was chosen as the study's theoretical framework.

Considering its vital role in responding to students' mathematical thinking, researchers in mathematics education have been increasingly interested in professional noticing skills in recent years (e.g., Mason, 2002; van Es & Sherin, 2002; Star & Strickland, 2008; Jacobs et al., 2010). However, considering the domain specificity of professional noticing skills, most research studies conducted in various mathematical domains have concluded that preservice teachers and even in-service teachers' professional noticing skills are low (e.g., Amador et al., 2021; Dışbudak-Kuru et al., 2022; Dogan-Coskun et al., 2021a; Güner & Akyüz, 2020; Fisher et al., 2018; 2019; Ivars et al., 2020). The perimeter is one of these mathematical domains that preservice teachers have difficulty in professionally noticing students' mathematical thinking (e.g., Caylan-Ergene & Isiksal-Bostan, 2021; Girit-Yıldız et al., 2022; Güner & Akyüz, 2020).

The concept of perimeter is crucial in mathematics learning and teaching as it establishes a base for constructing knowledge of measurement and other concepts in geometry (Clements & Battista, 2001). For example, in terms of geometry, knowing the perimeter of a shape could enable learners to find unknown sides of the shape (Güner, 2017). Simultaneously, the perimeter is a particular form of length measurement. Actually, the length measurement is concerned with measuring the distance between two points, while the perimeter is related to the distance around the

two-dimensional shape. However, since it is necessary to consider the main concepts related to measurement when measuring length and perimeter, a conceptual understanding of these concepts can be established at the end of a long and challenging process for learners (Barret et al. 2006; Martin 2009).

Most research studies have shown that learners, including students and preservice teachers, lack a mathematical understanding of the concept (e.g., Bragg & Outhred, 2000; Lehrer, 2003; Yenilmez & Koza-Çiftçi, 2014; Reinke, 1997; Simon & Blume, 1994; Steele, 2006). However, a conceptual understanding of perimeter could be a basis for learning or engagement with other concepts of measurement, such as area and volume. Therefore, learners' lack of understanding of length measurement and perimeter could affect the learning area and volume because an understanding of length (a one-dimensional measurement) is essential for understanding area (a two-dimensional measurement) and volume (a three-dimensional measurement) (Martin, 2007; Outhred et al., 2003). At that point, it is essential to question the quality of perimeter instruction in schools because, as Kamii and Clark (1997) say, " something is clearly wrong with the instruction given in measurement of length" (p. 116). Similarly, research studies have shown that teachers tend to provide rote and formula-based education about perimeter because of a lack of knowledge about perimeter and perimeter teaching (e.g., Doğan, 2017; Doğan-Coskun & Işıksal-Bostan, 2019; Kellogg, 2010). Hence, today's preservice teachers will be tomorrow's in-service teachers, so strengthening preservice teacher education could be the first step in supporting the perimeter concept.

When examining traditional teacher education programs, it is observed that they mainly focus on courses on how to teach mathematics theoretically by providing limited field-based experience. In other words, traditional teacher education programs treat theory and practice separately and then expect preservice teachers to apply what they have learned in the field when they become teachers. Unfortunately, considering the complex nature of teaching and learning, responding to students' mathematical needs and adapting instruction according to those needs are different

from what preservice teachers learn during their undergraduate. Put another way, given that they need to shape their teaching to take into account students' ways of thinking, enacting teaching in the classroom may not happen as preservice teachers learn theoretically in teacher education programs (Grossman et al., 2009a; 2009b). At the same time, teaching could differ between lesson plans prepared in advance and the moment of teaching practice carried out by that plan since preservice teachers might encounter unexpected responses, misconceptions, difficulties, errors, or high levels of students' thinking during instructions (Jacobs et al., 2010). Therefore, it could be inferred that, unfortunately, traditional teacher education programs fail to prepare preservice teachers for interactive teaching work (Darling-Hammond, 2006; Ghousseini, 2009; Hiebert et al., 2003), and studies have shown that preservice teachers and teachers need to help them adapt instruction based on students' thinking and responses to students' mathematical ideas (Amador & Lamberg, 2013; Jacobs & Empson, 2016; Lampert, 2001). Van Es and Sherin (2002) highlight this problem regarding teacher education programs as the following:

> Current teacher education programs often do not focus on helping teachers learn to interpret their classroom interactions. Instead, they focus on helping teachers learn to act, often providing them with instruction concerning new pedagogical techniques and new activities that they can use (p. 572).

Considering these shortcomings of teacher education programs, research studies have recently indicated that preservice teachers need more space to reflect on the theoretical knowledge and skills they gain in undergraduate education in their practice. (e.g., Grosman et al., 2009b; Goodlad, 1990; Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013). In other words, researchers have claimed that teacher education programs should allow preservice teachers to have authentic experiences of how students learn mathematical concepts, imagine how they would respond to them, and enact teaching practices. Indeed, they need to learn to look at the teaching process through the eyes of students with authentic experiences during their undergraduate education. Thus, building a stronger connection between theory

and practice through courses in teacher education programs is essential to enable preservice teachers to gain more experience before going to the field.

In the last decade, researchers have sought to answer the following question: How can the structure of undergraduate courses enable preservice teachers to become more effective teachers in real classrooms by recognizing and responding to students' mathematical understanding? In light of research suggesting that providing authentic and collective experiences can be helpful in preparing preservice teachers to be effective teachers, researchers have proposed making greater use of pedagogies in teacher education practice to answer this question (e.g., Kazemi et al., 2009; Lampert et al., 2010; 2013; Lampert & Graziani, 2009; McDonald et al., 2013). Some of these pedagogies focus on approximating teaching practices in simulated classrooms (e.g., Grossman et al., 2009a), enacting teaching practices in the classroom (e.g., Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013), and investigating and reflecting on teaching practices (e.g., Grossman et al., 2009b). Subsequently, bringing together various pedagogies, researchers have proposed a learning cycle called cycles of enactment and investigation for use in teacher education to support the development of preservice teachers' mathematical knowledge, skills, and practices through authentic and collective experiences (e.g., Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013). In the broadest sense, the cycles of enactment and investigation involve a cyclical process of investigating and enacting teaching practices using a variety of pedagogies, such as observation, approaching teaching practice, enacting teaching, and reflecting on these practices to provide preservice teachers with a chance to co-review their knowledge, skills, and practices (Note: More details on these topics are provided in the Literature Review section). Consequently, researchers have argued that this could be one way to prepare preservice teachers, the teachers of tomorrow, to be effective teachers (e.g., Grossman et al., 2009b; Hiebert et al., 2007). Therefore, the current study examines the ways to support the development of preservice teachers' mathematical knowledge and professional noticing skills on the perimeter concept through the development

of the model based on the cycles of enactment and investigation within the framework of design-based research.

1.1 The Purpose of Study and Research Questions

This study aims to support the improvement of prospective middle school mathematics teachers' mathematical knowledge and professional noticing skills on the concept of perimeter through the development of the model based on the cycles of enactment and investigation within the framework of design-based research. In light of the purpose of the study, this study addresses the following research questions:

1. How does a prospective middle school mathematics teacher develop her mathematical knowledge (overt subject knowledge, using terminology, identifying the errors and theoretical underpinning of the pedagogy) regarding the perimeter concept through the model based on the cycles of enactment and investigation?

2. How does a prospective middle school mathematics teacher develop her professional noticing skills (attending, interpreting, and deciding) on the perimeter concept through the model based on the cycles of enactment and investigation?

1.2 Significance of the Study

The contribution to both the literature and educational practice is the significance of this dissertation study. This study's primary contribution to the literature and educational practice is to support the development of prospective teachers' mathematical knowledge and professional noticing skills on the perimeter concept during undergraduate education. In the most general sense, it might provide evidence of a contemporary understanding of pedagogies of and in teacher education to

support such developments. In the literature, it is observed that researchers have mainly integrated several pedagogies in teacher education into their courses through the cycle of enactment and investigation, aiming to develop preservice teachers' knowledge, skills, and practices (e.g., Arbaugh et al., 2019; Grossman et al., 2009a; 2009b; Grossman & McDonald, 2008; Kazemi et al., 2009; Lampert et al., 2013). their main aim is to collectively investigate and enact teaching practices through instructional activity in which mathematics, pedagogy, and practice are integrated in a variety of settings, such as simulated and real classrooms (Kazemi et al., 2009; Lampert et al., 2013). Consequently, they do not address different activities in which mathematics, pedagogy, and practice are handled separately throughout the study. At the same time, most of the models in the literature grounded on the cycle of enactment and investigation do not include the individual process before the collective analysis of practices (e.g., Lampert et al., 2013; McDonald et al., 2013). Considering these shortcomings and the aim of this study, the current study offers a new model, designed in the light of a designed-based research perspective, by adding individual analysis before collective analysis and handling the mathematics, pedagogy, and skills separately in similar types of mathematical activities. From this point of view, the present study could make a significant contribution to the existing literature by providing evidence of the development of prospective teachers' mathematical knowledge and professional noticing skills by using similar types of activities and adding individual analysis throughout the model grounded on the cycle of enactment and investigation.

This study's other contribution to the literature is to unpack the development of prospective teachers' mathematical knowledge of the perimeter concept since much of the research has been descriptive in preservice teachers' knowledge of the perimeter concept, such as how they define the concept and the perimeter measurement (e.g., Reinke, 1997; Yenilmez & Çiftçi Koza, 2014). State it differently; the tendency of these studies is to determine the mathematical understanding, knowledge, and misconceptions of teachers or preservice teachers about the perimeter concept. Even if the participants of the studies are at different

grade levels or ages, their findings often share many similarities. More specifically, most of these studies have concluded that students, preservice teachers, and even in-service teachers have difficulty learning and teaching the concept (e.g., Clement & Stephan, 2004; D'Amore & Fandino Pinilla, 2006; Danielson, 2005; Lehrer, 2003; Reinke, 1997; Steele, 2006). In this regard, it is essential to examine the ways to support their knowledge of the perimeter concept. It is also essential to know which stages they go through during their development in order to reshape the courses regarding perimeter teaching in teacher education. In addition, although previous research studies have established that having authentic experiences and collective analysis of teaching practices could promote preservice teachers' knowledge and skills (e.g., Amador et al., 2021; 2023; Arbaugh et al., 2019; Güner, 2017; Lampert et al., 2013; Uygun, 2016), there is a little known how prospective teachers develop their mathematical knowledge on the particular domain through authentic and collective experiences. In this regard, the findings of the present study may contribute to the literature on unpacking the development of prospective teachers' mathematical knowledge of the perimeter concept through the model grounded on the cycle of enactment and investigation based on authentic and collaborative experiences.

The other significant contribution of the present study to the literature is to unpack the development of the prospective teachers' professional noticing skills. It might provide evidence of contemporary understanding of the development of professional noticing skills through the model grounded on the cycle of enactment and investigation. To address the effective ways to support the development of professional noticing skills, researchers in mathematics education have recently examined ways to enhance noticing skills through appropriate interventions such as video club discussions (e.g., Girit-Yildiz et al., 2022; Llinares & Valls, 2010; Osmanoğlu et al., 2012), animations (e.g., Amador et al., 2021; Wieman & Webel, 2019), and professional development programs (e.g., Schack et al., 2013). Even though most of these studies have concluded that noticing is a developable skill, most were grounded on long-term interventions. In other words, they lasted one or more

semesters. But, when considering professional noticing that is skill grounded on particular mathematical knowledge (Jacobs & Empson, 2016) and the diversity of the concepts in middle school mathematics, adequate and practical ways to do that within shorter periods than one or more semesters should be investigated. Furthermore, in these interventions, researchers have primarily promoted the improvement of preservice teachers' noticing abilities by students' written artifacts or using videos rather than allowing them to participate in real situations. In these situations, preservice teachers could display their intended or hypothetical actions. However, it is essential to examine, analyze, and support preservice teachers' in-the-moment decisions since they could struggle to decide "what to pursue and how to pursue it are continually adjusted during instruction in response to children's content-specific thinking instead of being determined in advance" (Jacobs & Emspon, 2016, p. 185). From this point of view, the current study can make a significant contribution to the literature on identifying prospective teachers' in-the-moment actions to respond to students' mathematical understanding. Additionally, research studies have clarified that taken-as-shared understanding of students' mathematical strategies, their mathematical understanding, and ways to respond to such understanding could contribute to the development of noticing skills collectively (Amador et al., 2023; van Es & Sherin, 2021). Although there is still work to be done in this field, the current study can make a significant contribution to the literature on supporting the development of prospective teachers' professional noticing skills during less than one semester by providing them with authentic and collective analysis experiences through the model grounded on the cycle of enactment and investigation.

The other contribution of the present study to the literature is supporting the development of mathematical knowledge and professional noticing skills. Indeed, although the main focus of the current study was not related to focusing on the relationship between mathematical knowledge and professional noticing skills, studies in the literature have shown that teachers' knowledge and professional noticing skills are conjoined structures that play a crucial role in recognizing and responding to children's mathematical thinking (e.g., Ball et al., 2008; Empson &

Levi; 2011; Jacobs & Empson, 2016) and also professional noticing is a particular domain-specific skill (Jacobs & Empson, 2016). However, little is known about how preservice teachers' mathematical knowledge and professional noticing skills develop simultaneously in a particular domain and how the development of mathematical knowledge nurtures the development of subject-specific professional noticing skills or vice versa—for this reason, supporting the development of mathematical knowledge and professional noticing skills separately but simultaneously could nurture prospective teachers' abilities related to recognizing and responding to children's mathematical thinking.

Furthermore, there are limited studies in the literature that focus on teachers' mathematical knowledge and professional noticing skills of the perimeter concept simultaneously. It is well known that teachers' existing knowledge plays a critical role in identifying and responding to students' mathematical thinking (e.g., Dick, 2017). However, as mentioned in the previous paragraphs, preservice teachers and teachers lacked the mathematical knowledge of the perimeter concept (e.g., Doğan-Çoşkun & Işıksal-Bostan, 2019; Reinke, 1997; Yenilmez & Koza-Çiftçi, 2014) and they have difficulty in professionally noticing students' mathematical thinking about the concept of perimeter (e.g., Girit-Yıldız et al., 2022; Güner, 2017). Yet, knowing how students think about the perimeter concept and how they perceive and construct this concept could be essential to improving the quality of perimeter instruction. Thus, more attention should be paid to how to simultaneously support preservice teachers' mathematical knowledge and professional noticing skills regarding the concept of perimeter. Therefore, the findings of the present study could have a significant contribution to the existing literature by providing evidence of such developments. Besides, although researchers have mainly used the cycles of enactment and investigation in their studies to develop preservice teachers' knowledge, skills, and practices on several mathematical concepts such as numbers (e.g., Arbaugh et al., 2019), there is little study in the literature the development of preservice teachers' mathematical knowledge and professional noticing skills of the perimeter concept. Thus, the present study's findings could provide empirical

evidence about the development of preservice teachers' mathematical knowledge and professional noticing skills on the perimeter concept as they participated in the model based on the cycles of enactment and investigation. Overall, this study might be beneficial for teacher educators and policymakers in that the findings promote the development of more effective courses or educational programs for preservice teachers to support the development of their mathematical knowledge and professional noticing skills on the perimeter concept by benefitting from the cycles of investigation and enactment. In a broader sense, designing or redesigning the structures of courses in teacher education programs or designing professional development programs based on the cycle of enactment and investigation could support the bonds between research on teacher education and teacher development.

Another contribution of the present study to the literature relates to the use of the Knowledge Quartet Model. While the knowledge types under the Foundation Dimension could be observed by analyzing preservice teachers' responses, lesson plans, and instructional practices, the knowledge types under other dimensions of the model are only observed by focusing on their planning and teaching practices. Most research studies in the literature using the KQ model have analyzed preservice teachers' knowledge by focusing on their teaching practices through other dimensions of the model, such as Contingency (e.g., Dogan-Coskun et al., 2021b) and Connection (e.g., Mutlu, 2021). There is a limited study in the literature that examines preservice teachers' mathematical knowledge with and without teaching practices by focusing on the Foundation Dimension of the KQ model. Because the current study consisted of multiple phases (examining the knowledge based on their responses to the pretest and examining the development of knowledge during the model through such as the planning, rehearsing, discussing, reflecting, and enacting teaching practices through the model, and examining the knowledge based on their responses to the posttest and post interview), it allowed the researcher to examine preservice teachers' mathematical knowledge of the perimeter concept holistically by focusing on the knowledge types under the Foundation Dimension. At the same time, since the Foundation Dimension forms the basis for other dimensions of the

KQ model, which are Transformation, Connection, and Contingency, it was assumed that the development of the Foundation Dimension could nurture the development of these dimensions. Thus, it was believed that prospective teachers would be given the opportunity to reflect on such developments in other dimensions. Considering these, the findings of the current study could provide more precise information about the developmental process of prospective teachers' mathematical knowledge of the perimeter concept. Additionally, this research might open a new window for researchers to explore the various uses of the KQ model in different settings.

1.3 Definitions of the Important Terms

A description of the meaning of essential terms used in the current study is essential. These terms are defined constitutively and operationally in the following:

Teachers' Professional Noticing: Professional noticing involves teachers' expertise in identifying students' strategies, making inferences about their understanding based on their strategies, and deciding appropriate responses considering their understanding (Jacobs et al., 2010). In this study, the preservice teacher's professional noticing skill was examined under the three components of professional noticing skills: attending, interpreting, and deciding skills, as defined by Jacobs et al. (2010).

Attending Skills: Attending skill is the ability to identify mathematically important details in students' strategies, which is the first component of professional teacher noticing (Jacobs et al., 2010). In this study, the preservice teacher's attending skills involved:

- examining the mathematical details of the student's written strategies and strategies observed in simulated and natural classroom environments,
- producing solutions for problems from students' eyes and evaluating their correctness.

Moreover, discussions about the details of students' strategies were investigated under attending skills since it was claimed they contributed to developing the attending skills.

Interpreting Skills: Interpreting skill is interpreting students' mathematical understanding considering their strategies, which is the second component of professional teacher noticing (Jacobs et al., 2010). In this study, the prospective teacher's interpreting skills involved the investigation of preservice teachers' inferences on students' understanding by considering students' written strategies and examining the possible underlying reasons for them. Moreover, the discussions in the intervention about students' mathematical understanding of the perimeter concept were investigated within this context since it was claimed that they contributed to the development of interpreting skills.

Deciding Skills: Deciding skills involves teachers' decision-making process to respond to students' understanding, which is the third component of professional teacher noticing (Jacobs et al., 2010). In this study, prospective teachers' deciding skills involved investigating their decision-making process for written and produced strategies and the actions to respond to them in the simulated and actual classroom environments. Moreover, the discussions in the intervention about what could be the next move to advance or support students' understanding were investigated within this context since it was claimed that they contributed to the development of the deciding skills.

Cycles of enactment and investigation: The cycle of enactment and investigation includes several pedagogies together to provide preservice teachers opportunities to examine, rehearse, enact, and reflect on practices through instructional activities (Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013). In the current study, the cycles of enactment and investigation were described as the combination of various pedagogies (representation, decomposition, approximation of the practices, enactment, and the pedagogies of investigation and reflection).

Representation of practice: Representations of practice "comprise(s) the different ways that practice is represented in professional education and what these various representations make visible to novices" (Grossman et al., 2009a, p.2058). In this study, observing the given dialogue and examining the preservice teacher's teaching practices were investigated under the representation of the practice.

Decomposition of practice: Decomposition includes "breaking down practice into its constituent parts for the purposes of teaching and learning" (Grossman et al., 2009a, p.2058). In this study, examining the given dialogue, producing dialogue in the lesson plan and rehearsals, and the discussions, where collectively identified the details of students' strategies and made inferences about their understanding, were explored in light of the decomposition of the practice.

Approximation of the practice: Approximations of practice include "opportunities for novices to engage in practices that are more or less proximal to the practices of a profession" (Grossman et al., 2009a, p.2058). In this study, the prospective teacher's intended responses to students' thinking (in written tasks and rehearsal) were examined within the context of approximation of practice.

Pedagogies of investigation and reflection: Pedagogies of reflection and investigation involve examining the teaching practices and reflecting on them (e.g., Grossman et al., 2009b; Grossman & McDonald, 2008). In this study, the discussions on how the prospective teachers should have responded to the students' thinking while examining teaching practice in the simulated and actual classroom environment were explored in the light of pedagogies of investigation and reflection.

Enactment: Enactment involves teaching in real classrooms (e.g., Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013). In this study, enactment meant that the prospective teacher's perimeter teaching practice experience in one of the fifth-grade classrooms.

Knowledge Quartet Model: The Knowledge Quartet (KQ) Model focuses on preservice teachers' mathematical knowledge by considering the content and pedagogical content knowledge together (Rowland et al., 2003; 2005). It consists of four dimensions with several components: Foundation, Transformation, Connection, and Contingency.

Foundation Dimension: The Foundation Dimension combines beliefs, goals, and theoretical knowledge about mathematics and mathematics teaching (Rowland et al., 2005). It consisted of several components or codes such as overt subject knowledge, using terminology, identifying errors, and theoretical underpinning of the pedagogy.

Overt Subject Knowledge: Overt subject knowledge includes investigating preservice teachers' definitions of mathematical concepts and explaining underlying reasons behind the definitions and procedures (Rowland et al., 2003; 2005; 2009). This study examined how the prospective teacher defined the perimeter concept and explained the underlying reasons for the perimeter calculation within the overt subject knowledge component of the Foundation Dimension.

Using Terminology: Using terminology code involves correctly using mathematical language and expressions (Rowland et al., 2003; 2005; 2009). This study examined how prospective teachers used mathematical language and terminology related to perimeter within using the terminology component of the Foundation Dimension.

Identifying Error: Identifying errors involves teachers' knowledge of students' difficulties with a particular concept and the knowledge to take necessary precautions to prevent them (Rowland et al., 2003; 2005; 2009). In this study, what the prospective teacher knew about students' potential difficulties with the concept of perimeter, their sources, how she overcame them in the classroom, and how she planned to overcome them were examined in the context of the identifying errors component of Foundation Dimension.

Theoretical Underpinning of the Pedagogy: The theoretical underpinning of pedagogy includes preservice teachers' knowledge of teaching strategies, methods, tools, manipulatives, and representations for teaching mathematics Rowland et al., 2005). In this study, the theoretical underpinning of the pedagogy component of the Foundation Dimension explored the types of tools, manipulatives, representations, and instructional strategies that prospective teachers used and planned to use in perimeter instruction.

Perimeter: Perimeter is "the distance around the region" (Dickson, 1989, p. 79).

Prospective middle school mathematics teachers: Prospective middle school mathematics teachers are senior-grade students in teacher education programs. These teacher candidates take courses, including mathematics, education courses, and pedagogy, in teacher education programs to teach mathematics to students from 5th to 8th grades.

CHAPTER 2

LITERATURE REVIEW

This study aims to support the improvement of prospective middle school mathematics teachers' mathematical knowledge and professional noticing skills on the concept of perimeter through the development of a model based on the cycles of enactment and investigation within the framework of design-based research. In this regard, the first section presents the underlying theories and related research studies on teacher knowledge. The second section explains the underlying theories and related research studies on professional noticing skills. The following section mentions the underlying theories and related research studies on the pedagogies in teacher education and the model of the current study. The fourth section presents the definitions of the perimeter concept and illustrates the literature on learners' difficulties with the concept. The last section provides conclusions based on the review of the literature that guided this study.

2.1 Frameworks for Teacher Knowledge

Teachers prepare lesson plans, enact the teaching to promote understanding, and respond to students' mathematical needs based on their existing knowledge (Carpenter et al., 1989). Therefore, teachers' knowledge inevitably plays an essential role in shaping instruction and influences their instructional decisions (Jacobs & Empson, 2016). For this reason, several studies in the literature have examined teachers' knowledge throughout the years and classified the knowledge types in different ways. To illustrate, Shulman (1986) categorized the different domains of knowledge that teachers should possess into subject matter knowledge, pedagogical content knowledge, and curricular knowledge. He clarified teachers' subject matter knowledge as "the amount and organization of the knowledge per se in mind of the

teacher" (Shulman, 1986, p.9). Pedagogical content knowledge includes "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented and adapted to the diverse interests and abilities of interests and abilities of learners, and presented for instruction" (Shulman, 1987, p.8). Moreover, it includes "an understanding of what makes the learning of specific topics easy or difficult, the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (Shulman, 1986, p.9). Finally, curricular knowledge includes knowing the sequence of a subject, text, and materials used in lessons and how to integrate them into teaching.

In light of Shulman's (1986) categorization, Ball et al. (2008) introduced a framework called Mathematical Knowledge for Teaching (MKT) to examine mathematics teachers' knowledge. They defined it in the broader sense as the "mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008, p. 395) and "practice-based theory of content knowledge for teaching" (Ball et al., 2008, p. 389). MKT compares two main domains, subject matter knowledge (SMK) and pedagogical content knowledge (PCK) concerning mathematics teaching. Subject matter knowledge is comprised of three sub-knowledge domains: 1) Common content knowledge (CCK), 2) specialized content knowledge (SCK), and 3) knowledge of content and curriculum. In a broader sense, SMK contains "how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Hill et al., 2008, p. 377) and does not include any knowledge connected to teaching (Hill et al., 2008). Specifically, CCK is related to mathematical knowledge teachers should have, such as solving mathematical problems correctly and providing definitions of mathematical terms. SCK is similar to common content knowledge related to teachers' mathematical knowledge. Unlike common content knowledge, specialized content knowledge is also concerned with pedagogical knowledge related to mathematics teaching, such as answering teacher-specific tasks. To illustrate, knowing underlying reasons used

representations, interpreting invented definitions, and solution strategies of their students in terms of mathematical aspects are examined as part of this knowledge. Horizon content knowledge is "awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p.403).

PCK is related to teachers' knowledge about the teaching profession. It consists of three sub-knowledge domains: 1) Knowledge of content and students (KCS), 2) knowledge of content and teaching (KCT), and 3) knowledge of content and curriculum. Specifically, KCS is defined as the combination of knowledge about students and knowledge about mathematics. More specifically, it contains the knowledge of how students think about their possible conceptions, misconceptions, errors, and difficulties and how they learn particular content (Hill et al., 2008). Moreover, it covers trying to find a way to remedy them. It includes predicting students' responses related to subjects or concepts and enjoyable, challenging, and motivating aspects of the topics in terms of students' points of view and understanding while shaping the instruction. KCT includes mathematical knowledge to design mathematical tasks, such as selecting suitable examples and representations, knowing the sequence of particular content, integrating multiple representations of mathematical concepts into lessons, and considering different instructional methods and procedures specific to content (Ball et al., 2008). Knowledge of content and curriculum is a combination of content knowledge and curriculum that builds the relationship between mathematics and other subject areas, designing instruction considering the program materials in specific circumstances (Ball et al., 2008). The model of Ball et al. (2008) differs from the model of Shulman (1986) in that it is specific to the teaching of mathematics and deals with subject matter knowledge and pedagogical content knowledge under subheadings. The domains of the model are presented in Figure 2.1.

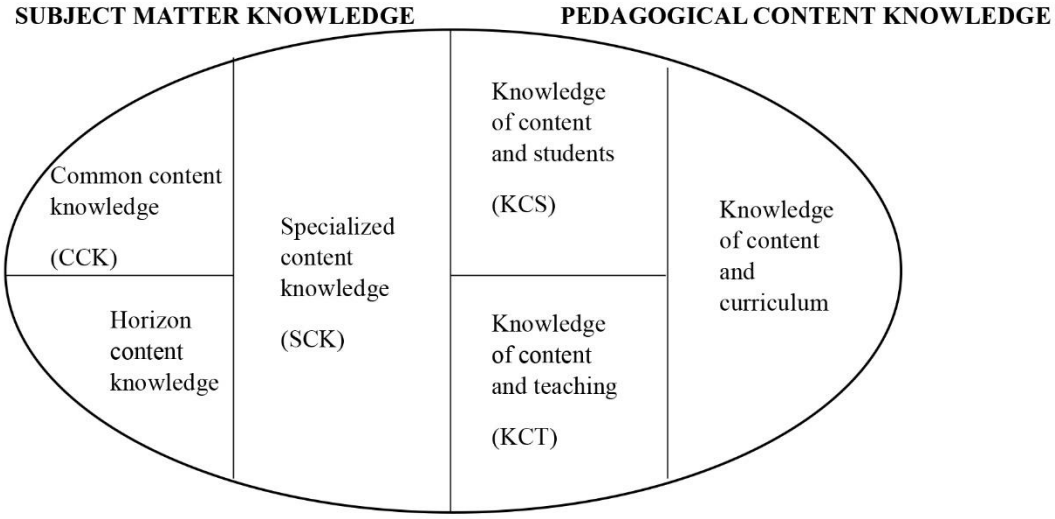


Figure 2.1. Domains of mathematical knowledge for teaching (Ball et al., 2008, p.403).

Furthermore, Carrillo et al. (2018) developed a model called The Mathematics Teacher's Specialised Knowledge - MTSK to provide a new perspective on mathematical knowledge for teaching by extending Shulman's (1986) and Ball et al.'s (2008) frameworks on teacher knowledge. The MTSK framework mainly includes two main domains - just like Shulman's 1986) starting point - Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK), and each domain includes three subdomains. The domains of the model are presented in Figure 2.2.

As seen in Figure 2.2, teachers' beliefs regarding mathematics and its teaching and learning are placed at the center of the model. Moreover, Carrillo and colleagues (2018) divided the teacher's mathematical knowledge into three sub-domains: "mathematics content itself (Knowledge of Topics); the interlinking systems which bind the subject (Knowledge of the Structure of Mathematics); and how one proceeds in mathematics (Knowledge of Practices in Mathematics)" (Carrillo et al., 2018; p. 6). Knowledge of Topics (KoT) includes having knowledge about mathematical concepts, different meanings of the topics, and corresponding phenomenological aspects.

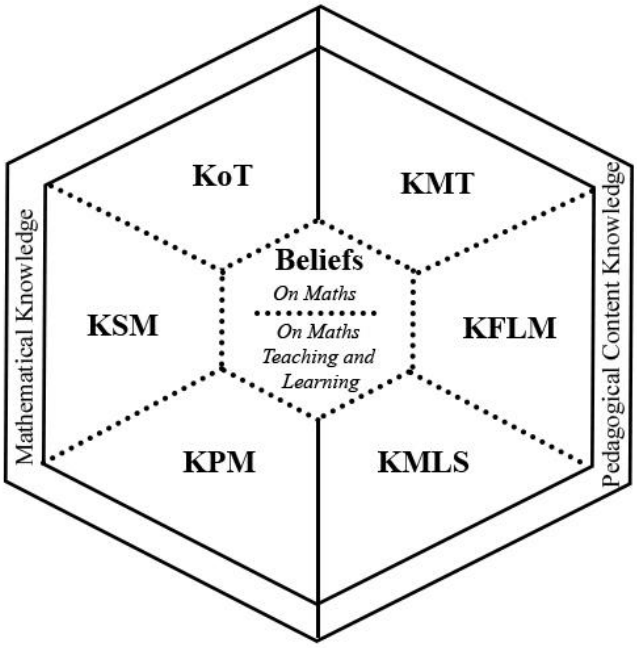


Figure 2.2. The mathematics teachers' specialized knowledge model (Carrillo et al., 2018, p. 6).

More specifically, they define a subdomain of such knowledge as "the type of problems the content can be applied to, with their associated contexts and meanings; properties and their underlying principles, definitions, and procedures, including connections to items within the same topic; and ways of representing the contents" (Carrillo et al., 2018; p. 7). In a broader sense, Knowledge of the Structure of Mathematics (KSM) describes the teacher's knowledge of connections between mathematical concepts. They define categories of KSM as connections based on simplification, connections based on increased complexity, auxiliary connections, and transverse connections, which could be considered as the combination of horizon knowledge and knowledge of content and student subdomains of Ball and colleague's framework (2008). It includes knowledge about the sequence of a particular mathematical concept, mathematical communication, and reasoning. PCK includes the following subdomains: Knowledge of Mathematics Teaching (KMT), Knowledge of Features of Learning Mathematics (KFLM), and Knowledge of Mathematics Learning Standards (KMLS). KFLM includes knowing how students

think and build knowledge (including their difficulties and strengths, both content-specific and broader) when dealing with mathematical activities and different ways students interact with mathematical content. It also focuses on the emotional aspects of learning mathematics. KMT includes an adaptation of teaching resources (including physical and virtual), strategies, techniques, tasks, and examples specific to the content. KMLS includes both expected learning outcomes and levels of conceptual or procedural development and includes information on the sequence of topics based on grade levels. Overall, MTSK focuses on the process of becoming, not being like other frameworks (e.g., MKT developed by Ball et al., 2008). In this regard, it helps researchers recognize the complex dynamics of knowing mathematics.

Another model developed for mathematics teaching is the Knowledge Quartet developed by Rowland and colleagues (Rowland et al., 2003; 2005). Rowland et al. (2003; 2005) proposed a model to examine, evaluate, and support preservice teachers' knowledge of mathematics teaching. This model allows preservice teachers to observe and also improve their teaching (Thwaites et al., 2005). Meanwhile, it allows researchers to examine preservice teachers' pedagogical content knowledge and subject content knowledge holistically without discriminating against each of them. Rowland and colleagues investigated the knowledge types under four dimensions with some codes under each dimension. In the most general sense, these dimensions are Foundation, Transformation, Connection, and Contingency. Before discussing each dimension, the following paragraph provides a brief description of how the model was developed.

To develop the Knowledge Quartet Model (KQ), Huckstep et al. (2003) observed two-hour teaching practices of 12 preservice teachers out of 149 preservice teachers lectures within the scope of the SKIMA (Subject Knowledge in Mathematics) program as a part of the Post-graduate Certificate in Education at Cambridge University Faculty of Education. At the same time, they recorded those preservice teachers' teaching practices to review further. Later, the researchers outlined and took

notes on the critical features of each practice and interviewed each preservice teacher in light of these notes. Then, they examined all data, including the lesson plans, interviews, and teaching practices, to unpack preservice teachers' knowledge of mathematics and pedagogy to teach mathematics. In this study, Huckstep et al. (2003) proposed a model, which was not yet finished and open to changes, with 18 codes under four dimensions: a) *theoretical background and beliefs, b) transformation, presentation, and explanation, c) coherence, and d) contingent actions,* to examine preservice teachers' mathematical knowledge. In light of the study of Huckstep et al. (2003), Rowland et al. (2003) conducted a study and mentioned the Knowledge Quartet (KQ) model for the first time. They examined one preservice teacher's lesson (Naomi's). They developed the model for examining preservice teachers' knowledge in teaching mathematics, and the details of the model were mentioned in the studies of Rowland et al. (2005; 2009). The relationship among dimensions of the KQ Model is shown in Figure 2.3.

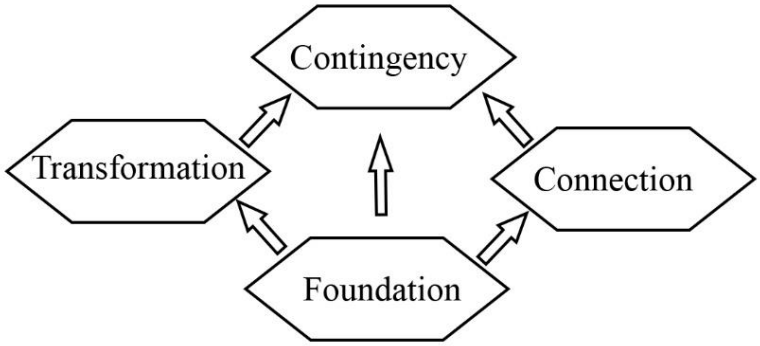


Figure 2.3. The dimensions and relationship among dimensions of the KQ model (http://www.knowledgequartet.org/).

The first dimension of the KQ model is called the Foundation, which encompasses theoretical knowledge that teachers and preservice teachers have beliefs and understanding about mathematics and mathematics teaching (Rowland et al., 2009; Thwaites et al., 2005). In the most general sense, this dimension concerns teachers' and preservice teachers' knowledge of mathematics, mathematical pedagogy, and beliefs about why and how to teach mathematics (Rowland et al., 2005). Rowland et

al. (2009) stated that various questions give information about preservice teachers' knowledge of the Foundation Dimension. Some of these questions are as follows:

- Does the trainee teacher use appropriate teaching strategies to promote the required mathematical understanding in pupils?
- Does the trainee teacher concentrate on developing understanding rather than excessively on procedures?
- Does the trainee teacher show, in his/her planning, knowledge of common errors and misconceptions and take steps to avoid them?
- Does the trainee teacher show care in writing mathematical expressions correctly?
- Does the trainee teacher use mathematical language correctly?
- Does the trainee teacher demonstrate an accurate understanding of mathematical ideas or concepts? (Rowland et al., 2009, p. 35)

The questions show that this knowledge is holistically concerned with teachers' subject matter and pedagogical content knowledge. In this regard, Rowland et al. (2009) emphasized the relationship between Foundation Knowledge and the subject matter and pedagogical content knowledge as follows:

> We saw that the first category in Lee Shulman's typology of knowledge for teaching was subject-matter knowledge. This may be described as knowledge of the facts, concepts, processes, and connections within the subject (substantive knowledge) and how knowledge within that subject is investigated and developed (syntactic knowledge). Both aspects of subject-matter knowledge are essential facets of foundation knowledge, but they do not make up the whole picture. Theoretical, pedagogical content knowledge is also seen as a critical component of foundation knowledge. Teachers need

> to understand how pedagogical strategies relate to the mathematics they are trying to teach to decide which strategies to use. These actual decisions would be considered part of the transformation dimension; however, the theoretical understanding underpinning them is part of the teacher's foundation knowledge… (Rowland et al., 2009, p. 153).

As understood from Rowland et al. (2009) explanations, Foundation Dimension includes teachers' and preservice teachers' "knowledge of mathematics and of mathematical pedagogy" (p.153). In addition, it is stated that the Foundation Dimension covers the other three dimensions under the KQ model and contributes to their development. At the same time, this knowledge may not always be directly observable during teaching.

The codes under the Foundation Dimension are a) *overt subject knowledge,* b) the *theoretical underpinning of the pedagogy*, c) *using terminology*, d) *identifying errors*, e) *concentrating on procedures*, f) *adherence to textbooks*, and g) *awareness of purposes*. *Overt subject knowledge* explains how teachers and preservice teachers can define basic mathematical concepts and whether they have a deep mathematical understanding. *The theoretical underpinning of the pedagogy* gives information about teachers' and preservice teachers' knowledge of mathematical pedagogy. It refers to teachers' and preservice teachers' theoretical knowledge about tools, manipulatives, and teaching strategies to learn and teach mathematics. *Using terminology* informs how teachers and preservice teachers use mathematical language, expressions, and terminology regarding mathematical concepts, rules, and procedures (Rowland et al., 2005; 2009). *Identifying errors* is about teachers' and preservice teachers' knowledge of the mathematical errors and misconceptions that their students may have or develop and taking the necessary steps to prevent them from occurring in the classroom. *Concentrate on procedures* concerns with teachers and preservice teachers' knowledge that focuses on the result and the process by supporting students' conceptual understanding. *Adherence to textbooks* is related to teachers' and preservice teachers' knowledge of using various sources in planning

and implementing lessons. A*wareness of purposes* is related to teachers' and preservice teachers' knowledge that concerns why students should learn mathematics in general or that subject in particular (Rowland et al., 2005; 2009).

The second dimension, the Transformation Dimension, is related to the fact that teachers and preservice teachers present their knowledge during the planning and implementation of the lesson in a way that students can understand the concepts, procedures, rules, and operations (Rowland et al., 2005; 2009). State differently, it concerns how they transform their knowledge through representations, demonstrations, examples, and analogies to teach mathematics to children. Rowland et al. (2009) stated that various questions give information about preservice teachers' knowledge of the Transformation Dimension. Some of these questions are as follows:

- Does the trainee teacher select appropriate forms of representation?
- Does the trainee teacher choose appropriate examples when demonstrating or eliciting an idea?
- Does the trainee teacher give clear explanations of ideas or concepts, possibly making use of analogy?
- Does the trainee teacher demonstrate clearly and accurately how to carry out procedures?
- Does the trainee teacher make use of interactive teaching techniques to develop and assess understanding? (Rowland et al., 2009, p. 36).

The codes under the Transformation Dimension are a) *choice of examples,* b) *choice of representation,* and c) *teacher demonstration*. *Choice of examples* is related to teachers' and preservice teachers' choice of clear and understandable examples for students to make sense of mathematical concepts. Moreover, it is essential to consider the relationship of the examples to the real-life context. *Choice of*

representations includes the usage of appropriate representations such as tables, models, and graphics to make knowledge understandable and to help concept formation. *Teacher demonstration* includes teachers' and preservice teachers' demonstrations and explanations to enable students to understand mathematical concepts (Rowland et al., 2005; 2009).

The third dimension, Connection, is related to establishing a coherence of planning and teaching and connecting previous lessons, a specific part of a lesson, and the following lessons through examples, procedures, concepts, and other disciplines (Rowland et al., 2005; 2009). In other words, it concerns "the decisions about sequencing and connectivity" (Rowland et al., 2009, p.36). Rowland et al. (2009) stated that various questions give information about preservice teachers' knowledge of the Connection Dimension. Some of these questions are as follows:

- Does the trainee teacher make links to previous lessons and make links between the mental and oral starter and the main part of the lesson?
- Does the trainee teacher make appropriate conceptual connections within the subject matter?
- Does the trainee teacher ask questions to elicit children's understanding of connections between mathematical ideas?
- Does the trainee teacher anticipate the complexity of an idea and break it down into steps that can be understood by the children?
- Does the trainee teacher introduce ideas and strategies in an appropriately progressive order? (Rowland et al., 2009, p.36-37).

The codes under the Connection Dimension are a) *anticipation of complexity*, b) *decisions about sequencing*, c) *making connections between procedures*, d) *making connections between concepts*, and e) *recognition of conceptual appropriateness. Anticipating complexity* involves anticipating the points at which students may have

difficulty during the lesson and organizing the lesson (including planning and teaching) with these points in mind. In this way, it is assumed that teachers should be prepared for students' difficulties and arrange their lessons with this in mind, with the aim of facilitating students' mathematical understanding. This knowledge influences teachers' decisions about the sequence of their lessons. *Decisions about sequencing* are related to ordering contents of lessons, examples, representations, and teaching strategies in planning and implementing lesson stages. *Making connections between procedures* includes establishing relations among procedures in mathematics. *Making connections between concepts* includes establishing relations among mathematical concepts considering previous lessons, current lessons, and other lessons. *Recognition of conceptual appropriateness* includes planning and implementing instruction by considering students' grade levels and prior knowledge (Rowland et al., 2005; 2009).

The fourth dimension, Contingency, concerns unexpected and unpredictable cases during teaching (Rowland et al., 2005; 2009). In other words, since teachers cannot estimate what will happen during lessons, they should prepare all cases and adapt their instruction by responding to students' ideas or using resources and tools in the lesson (Rowland et al., 2005). It is that "think on one's feet" (Rowland et al., 2009; p.37). Rowland et al. (2009) stated that several questions give information about preservice teachers' knowledge of the Contingency Dimension. Some of these questions are as follows:

- Does the trainee teacher respond appropriately to children's comments, questions, and answers?
- Does the trainee teacher deal appropriately with children's responses to activities?
- Does the trainee teacher respond appropriately when children give incorrect answers to questions or make incorrect statements during the course of a discussion?

- Does the trainee teacher deviate from their agenda when appropriate? (Rowland et al., 2009, p.37).

The codes under the Contingency Dimension are a) *deviation from the agenda*, b) *use of opportunities*, and c) *responding to children's ideas* (Rowland et al., 2005). *Deviation from the agenda* concerns teachers' and preservice teachers' decisions when they face unexpected situations, such as the emergence of students' misconceptions during instructions. *The use of opportunities* is related to using learning opportunities in the lessons to promote students' mathematical understanding. *Responding to students' ideas* includes providing accurate and informative information about students' unplanned questions and mathematical thoughts during the lesson that teachers and pre-service teachers face in the moment of instruction (Rowland et al., 2005; 2009).

This study aimed to support the improvement of prospective middle school mathematics teachers' mathematical knowledge and professional noticing skills regarding the perimeter concept through the development of the model based on the cycle of enactment and investigation. In this respect, the Foundation Dimension of the KQ model was chosen to analyze how prospective teachers defined the perimeter concept, how they used mathematical language regarding the concept, how they identified students' errors regarding the concept, and what kind of instructional strategies they preferred to teach the concept. At the same time, since the Foundation Dimension forms the basis for the types of dimensions of the KQ model, it was believed that supporting the development of the Foundation Dimension could promote the development of mathematical knowledge holistically. After this review of the general characteristics of various frameworks related to teacher knowledge, the following section provides some details on the studies of teachers' knowledge.

2.1.1 Studies on Teacher Knowledge

In recent years, researchers have focused on the relationship between teachers' knowledge and their instructional decisions as a part of effective teaching to support students' mathematical understanding using the frameworks above (e.g., Dick, 2017; Dogan-Coskun & Isıksal-Bostan, 2019; Jakobsen et al., 2014; Liston, 2012; Maher et al., 2022; Speer & Wagner, 2009; Thomas et al., 2017). At this point, it could be essential to provide a summary of how teachers' knowledge has an impact on their instructional decisions through the use of different theoretical perspectives in mathematics education mentioned above. Consequently, the following paragraphs provide a summary of the international and local studies that have investigated the role of knowledge in their instructional decisions.

Thomas et al. (2017) explored the theoretical links between teachers' responsiveness and their mathematical knowledge for teaching by using the components of the MKT model proposed by Ball et al. (2008). They highlighted that teachers need specialized mathematical knowledge to interpret children's mathematical thinking and adapt their instructional decisions based on such interpretations. They then assessed teachers' responsiveness as the intersection of teachers' knowledge with all components of mathematical knowledge for teaching and noticing skills. They concluded with a representation of mathematical knowledge and responsiveness to students' mathematical thinking with a continuum.

Furthermore, Speer and Wagner (2009) investigated how an experienced mathematics educator utilized his knowledge to respond to whole-class discussions during undergraduate education. The study results showed that the teacher used the subject matter knowledge to understand the students' mathematical ideas and needed the pedagogical content knowledge to adapt these ideas to the class discussions.

In addition, Jakobsen et al. (2014) examined the content and nature of mathematical knowledge of 49 Norwegian prospective primary teachers for teaching interpretive knowledge of fractions. They investigated the role of interpretative knowledge as a

combination of standard and specialized content knowledge in interpreting students' understanding. They designed a set of fraction tasks to investigate this type of knowledge. The findings showed that developing preservice teachers' interpretive knowledge contributed to their ability to give effective feedback to support students' understanding.

Dick (2017) conducted a study to examine the relationships between the development of elementary preservice interns' specific domain of mathematical knowledge for teaching, called specialized content knowledge, and their engagement in professional noticing, considering all three components. She combined professional noticing and mathematical knowledge for teaching frameworks to measure two constructs simultaneously. At the same time, she mapped the properties of subject matter knowledge with professional noticing components, namely attending, interpreting, and decision-making. More specifically, she related “critiquing notation and language" with attending components of the professional noticing framework, "assessing mathematical expressions and students' claims and being familiar with the non-standard methods and common mistakes" with interpreting component of the framework, and finally, "asking constructive mathematical questions" with decision making components of the framework (Dick, 2017, p. 342). Then, she developed three professional learning tasks considering combined frameworks. She applied them consecutively to examine the changes in four preservice teachers' subject matter knowledge and professional noticing their students' mathematical thinking in specific content, multi-digit addition, and subtraction. The study's findings showed that subject matter knowledge should be considered an inevitable part of professional noting, including their decisions.

In addition to teachers' responses, Dogan-Coskun and Isıksal-Bostan (2019) stated that teachers' awareness of students' solutions strategies is linked to their mathematical knowledge. They investigated the cognitive demand levels of tasks an in-service primary teacher prefers during length measurement and perimeter instruction. Also, they examined the relationship between used tasks and teachers'

mathematical knowledge in teaching. The study's findings indicated that the preferred tasks by the teacher in the instructions were mainly categorized as low-level cognitive demand, and the teacher's mathematical knowledge for teaching has a role in the process of implementation of the tasks. The in-service primary teacher's mathematical knowledge did not advance her students' learning during length measurement and perimeter instruction.

Liston (2012) used critical video-based experiences and first-hand observations to develop 29 preservice mathematics teachers' pedagogical content and subject content knowledge through the KQ Model. Indeed, she first gave information about the model and then asked each preservice teacher to conduct teaching. Meanwhile, she recorded preservice teachers' teachings and shared these recordings with them. At the same time, all preservice teachers participated in other preservice teachers' lessons in the role of observer. Later, each preservice teacher was asked to analyze their teaching and their peers' teaching. The study's findings showed that preservice teachers' knowledge about Foundation Dimension could have been better and more procedural. Moreover, the study's findings implied that preservice teachers' knowledge about the Foundation Dimension impacted their knowledge regarding the Contingency Dimension and, consequently, their responses.

In addition, Maher et al. (2022) also examined the teaching of senior secondary mathematics teachers with 27 years of teaching experience through the KQ Model. Data were collected through lesson observations, video recordings, field notes, and post-lesson interviews. In the most general sense, the study's findings showed that the experienced teacher lacked knowledge of all dimensions of the KQ model. Moreover, Maher et al. (2022) concluded that teachers' knowledge of the Foundation Dimension affected their instructional decisions.

Furthermore, using the KQ Model, Doğan-Coşkun (2017) investigated primary school teachers' mathematical knowledge in teaching length measurement and perimeter topics. Two primary school teachers joined the study. The data was collected through observations, semi-structured interviews, video recordings, and

field notes. The study's findings show that even though the primary school teachers were aware of their students' errors located under the Foundation Dimension, they had difficulty responding to students' mathematical ideas.

Besides, Karagöz-Akar (2016) examined prospective secondary mathematics teachers' mathematical knowledge through the KQ Model. Data was collected from two prospective teachers through teaching practices in two different contexts (one geometry and one algebra), lesson plans, and self-reflection. The study's findings showed that the preservice teachers' knowledge of the Foundation Dimension played an essential role in their teaching.

Additionally, Dogan-Coşkun et al. (2021b) examined the triggers of unexpected moments during a preservice elementary teacher's teaching of length measurement and perimeter and the teacher's responses to these moments at the moment. They found that students' ideas, questions, and incorrect answers were some of the triggers of unexpected moments. The teacher's responses to such moments were examined under four titles: a) ignore or dismiss, b) acknowledge but continue, c) pursue student thinking, and d) extend or make connections proposed by Stockero and Van Zoest (2013). Dogan et al. (2021b) found that when the teacher ignored or acknowledged students' ideas, it was seen that she did not attend to these ideas, interpret them, and adapt them to the flow of length and perimeter instruction. When she tended to display responses to extend or make connections to students' ideas, she sometimes did not support students' understanding due to a lack of knowledge. Dogan-Coşkun et al. (2021b) concluded that teachers' mathematical knowledge in the classroom played a role in their current responses.

After mentioning particular characteristics of frameworks related to teachers' knowledge and the studies of teachers’ knowledge, the details of noticing skills are mentioned in the following section.

2.2 Frameworks on Noticing Skills

In recent years, researchers in mathematics education have been increasingly interested in teachers' noticing skills due to their vital role in teaching and learning (e.g., Mason, 2002; van Es & Sherin, 2002; Star & Strickland, 2008; Jacobs et al., 2010). Studies have shown that different situations, such as students' thinking, interactions, behaviors, and problems related to classroom management and classroom events, co-occur in a classroom environment, and it is not possible for teachers to be able to make sense of every situation in the classroom. For this reason, teachers need to have special skills to filter classroom situations in terms of mathematically significant details on students' explanations (Mason, 2002; Jacobs et al., 2010).

The literature does not define noticing since it varies based on the researchers' foci and theoretical approaches. For example, Van Es and Sheerin (2002) defined noticing as a professional vision concerning a cognitive-psychological perspective with three main components: "(a) identification of what is important or noteworthy about a classroom situation; (b) the ability of making connections between the specifics of classroom interactions and the principles of teaching and learning they represent; and (c) the ability of what it knows about the context to reason about classroom events" (p. 573). As it is understood from these components, Van Es and Sheerin (2002) generally focused on what teachers pay attention to in classroom interactions and how they interpret them. In this perspective, researchers have mainly focused on the mental processes of teachers or preservice teachers by analysis of video recordings of teaching sessions (Sherin et al., 2011; Sherin & Star, 2011). Furthermore, the cognitive-psychological perspective emphasizes that teachers' perception as a human is limited; for this reason, teachers focus on specific aspects of teaching while ignoring other aspects. In light of this perspective, van Es and Sherin (2021) have recently reshaped the dimensions of teacher noticing skills they put forward (2002) by adding a new dimension. Specifically, van Es and Sherin (2021) redescribed the constructs of teachers noticing under-attending, interpreting,

and shaping dimensions in their study. Notably, the attending dimension involves identifying noteworthy features in a teaching situation and ignoring less critical aspects of classroom interactions. The interpreting dimension encompasses using teachers' knowledge to make sense of them with an inquiring stance. Shaping dimension, the novel one involves "the act of creating interactions that provide increased opportunities to attend to and interpret noteworthy mathematical interactions" (van Es &Sherin, 2021, p. 17). In this study, van Es and Sherin (2021) emphasized the role of teachers' dialogues in teaching, and they believed teachers shape the interactions to obtain additional information about students' understanding.

Furthermore, some other approaches to defining teacher noticing place a similar emphasis on the classroom environment by including teachers' decision-making process. For example, Kaiser et al. (2015) handled noticing with the PID model with three components: a) perceiving, which includes specific events in a teaching setting; (b) interpreting, which includes "the perceived activities in the classroom," and (c) decision-making which includes "either as anticipating a response to students' activities or as proposing alternative instructional strategies" (p.374). Blömke et al. (2015) also defined these constructs, including perception, interpreting, decision-making, and intersection, as the situation-specific skills between teachers' disposition and performance.

In contrast to these researchers, Jacobs et al. (2010) put forward a different perspective and narrowed their focus to students' mathematical thinking rather than focusing on classroom environments. They define teachers' noticing skills as professional noticing of children's mathematical thinking and proposed their framework as The Professional Noticing of Children's Mathematical Thinking. Their framework consists of a set of three interrelated skills: (a) *attending* to children's strategies, (b) *interpreting* children's understandings, and (c) *deciding* how to respond on the basis of children's understandings" (Jacobs et al., 2010, p. 172).

The first skill of the framework, *attending*, is related to focusing on the mathematical essence in students' solutions strategies. Jacobs et al. (2010) define attending as "the

extent to which teachers attend to a particular aspect of instructional situations: the mathematical details in children's strategies" (p.172). Identifying mathematically important details of students' strategies is essential since it enables teachers to view relevant content through children's eyes and establishes a base for teachers to make inferences on students' mathematical thinking (Jacobs et al., 2010). In addition, focusing on students' strategies opens a window for teachers to know students' mathematical understandings since teachers tend to underestimate students' mathematical thinking due to some reasons, such as the intensity of the curriculum (e.g., Tan-Sisman & Aksu, 2012) and a lack of content knowledge (e.g., Doğan-Coşkun, 2017).

The second dimension of the framework is *interpreting*, related to making sense of children's mathematical understandings of relevant concepts or procedures. Interpreting is described by Jacobs et al. (2010) as how teachers make sense of students' mathematical understandings as reflected in their strategies. They stated that there is no best interpretation of students' understanding, or teachers are not expected to discuss drawing the big picture of students' understanding of the related concept. The critical question in the framework is whether teachers' interpretations of students' mathematical knowledge relate to the strategies students reflect on and to detail and research strategies regarding children's mathematical development. State differently, Jacobs et al. (2010) emphasized that interpreting students' mathematical understanding needs to reflect (a) identifying the specific connections among mathematical concepts mentioned by children, (b) explaining underlying reasons for their understanding in detail rather than using broad terms, (c) making sense of their preferred strategies in several ways, and (d) being able to make inferences on their understanding considering research on children's mathematical development. Evaluating students' mathematical understanding is critical since it shapes teachers' decision-making processes.

The third component skill of the framework, *deciding*, is related to deciding how to respond considering students' mathematical understanding of the concepts or

procedures. In other words, teachers identify questions that support or advance students' mathematical understanding based on the data obtained in the first two steps of the framework. Similar to the main idea behind evaluating the second component of the framework, Jacobs et al. (2010) expressed that there need not be only one answer to be the best answer. The crucial question in the framework is to focus on students' understanding and research on mathematical development while deciding how to respond to them (Jacobs et al., 2010). Besides, their extensions and suggestions are based on students' understanding and strategies used on relevant content. As mentioned before, the three-component skills of the framework, attending, interpreting, and deciding, complete each other conceptually and occur almost simultaneously in the decision-making process (Jacobs et al., 2010).

In the light of Jacobs et al.'s (2010) framework, researchers in education have a chance to investigate teachers' or preservice teachers' intended decisions rather than their instructional decisions. State differently, the decision-making process is generally based on teachers' intended responses about what should be the following problem. Nevertheless, researchers have recently emphasized that attending to children's strategies, interpreting children's understandings, and deciding how to respond based on children's understandings establish a base for teachers' or preservice teachers' in-the-moment decisions. Thus, in-the-moment responses of teachers, especially preservice teachers, who lack teaching experience should also be investigated. Considering this situation, Amador et al. (2021) suggest further investigating preservice teachers' instructional decisions and extending the professional noticing framework proposed by Jacobs et al. (2010) by adding an "enact" component as a fourth component of the framework. Figure 2.4 represents the cyclical relations among the suggested framework's components.

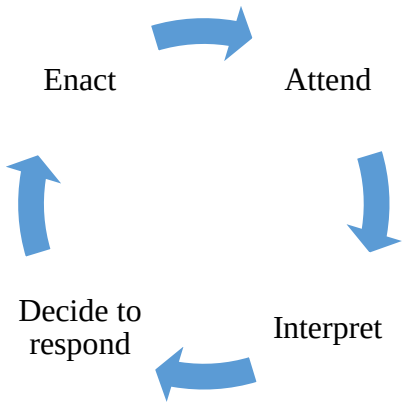


Figure 2.4. The cyclical relations among the components of the suggested framework (Amador et al., 2021, p. 19).

To sum up, the current study aimed to support the development of preservice teachers' mathematical knowledge and professional noticing skills on the perimeter concept through the development of the model grounded on the cycle of enactment and investigation. In light of this aim, Professional Noticing of Children's Mathematical Thinking, proposed by Jacobs et al. (2010), was chosen to analyze preservice teachers' professional noticing skills. Following this review of the general characteristics of various frameworks related to noticing skills, the following section provides details on the studies of teachers' professional noticing skills of children's mathematical thinking.

2.2.1 Studies on Professional Noticing of Children's Mathematical Thinking

Given their critical role in effective teaching and student learning, the ability to notice is accepted as an essential component of the teaching profession (e.g., Jacobs et al., 2010; Kaiser et al., 2015; Sherin et al., 2011; van Es & Sherin, 2021). Therefore, over the last decades, researchers have carried out an increasing number of studies on professional noticing skills. Findings of studies have shown that professional noticing is a developable skill (e.g., Fernández et al., 2012; Jacobs et al., 2011; Star & Strickland, 2008) through the appropriate professional development

programs and interventions (e.g., Amador et al., 2017; 2021; Girit-Yildiz et al., 2022; Osmanoğlu et al., 2012; Llinares & Valls, 2010; Schack et al., 2013; Star & Strickland, 2008; Wieman & Webel, 2019). However, most of them were primarily conducted to focus on attending skill and interpreting skills of professional noticing (e.g., Amador et al., 2017; Star & Strickland, 2008), and there were limited studies to focus on deciding skills (e.g., Amador et al., 2021). These studies suggested that considering its crucial impact on their responses, studies should focus more on deciding skills (Amador et al., 2021). The examples of national and international studies regarding professional noticing skills are mentioned in the following.

Özel (2019) investigated thirty-two prospective middle school mathematics teachers' ability to notice students' algebraic thinking in the context of pattern generalization, considering three components of the framework. Questionnaires and semi-structured interviews were used as data collection tools. The questionnaire included examples of students' solutions and explanations considering pattern generalization. The results of the study showed that while a large majority of the preservice teachers could attend to the details of students' strategies, they had difficulty interpreting students' algebraic thinking based on their solutions. Additionally, they generally preferred to follow up on incorrect solutions and generally did not extend students' thinking when students answered the questions correctly. In such circumstances, they preferred to give drills or general answers to respond to students' mathematical understanding.

LaRochelle (2018) examined the characteristics of the professional expertise of three groups of secondary teachers within the domain of proportional reasoning. Study participants included 30 prospective teachers, 30 experienced teachers with at least four years of teaching experience, and 12 emerging teacher leaders with at least four years of sustained professional development. She used written artifacts of student thinking to collect data. Results indicated that the findings of experienced secondary teachers and emerging secondary teachers were quite similar in terms of attending and interpreting component skills. In contrast, the experienced secondary teachers

were better at deciding how to respond than the emerging secondary teachers. However, prospective secondary teachers provided more information about the three subskills of attending, interpreting, and deciding. As a result, she concluded that teaching experience is not a significant variable in the development of noticing skills and that teachers need ongoing professional development to develop these skills.

Fernández et al. (2020) investigated the development of prospective mathematics teachers' noticing in an online teacher education program, focusing on the role of feedback in an online forum and the task of writing and sharing narratives about one's teaching during the traineeship period. The program does not require face-to-face learning or a blended learning environment. Instead, it was created as a virtual learning environment supported by the institutional Moodle virtual platform. The study participants, who were five prospective mathematics teachers, were expected to participate in classroom observation by investigating example narratives and theoretical documents and attending online forums at Moodle in the first part of the study. In the second part, they wrote narratives and attended online discussions by sharing narratives on the same platform under practice in the classroom part. They got feedback from peers and tutors. Fernández et al. (2020) concluded that sharing narratives in online forums and getting feedback from their peers and the university tutor contributed to developing prospective mathematics teachers' professional noticing skills.

Schack et al. (2013) investigated ninety-four prospective elementary school teachers (from three institutions) professional noticing of children's early numeracy within the context of a method course via a researcher-developed five-session module. The module contains video excerpts of diagnostic interviews of children doing mathematics. They measured the changes in three interrelated components of professional noticing-attending, interpreting, and deciding altogether thorough pre- and post-assessment. Results showed that prospective teachers can develop their professional noticing skills considering all components with the help of these modules.

Çaylan-Ergene and Işıksal-Bostan (2021) examined three preservice mathematics teachers' professional noticing skills to measure the length before and after participating in the intervention, based on the Levels of Sophistication in Students' Reasoning about Length framework (Battista, 2006). They were asked to examine students' written solutions to length measurement during the pre- and post-interview. The results showed that engaging in the framework supported the development of preservice teachers' professional noticing skills with all components (attending, interpreting, and deciding).

Amador et al. (2021) examined preservice teachers' instructional decisions through animation. In fact, in their study, preservice teachers examined a video recording of a classroom lesson with the goal of engaging students' mathematical thinking about fractions. Then, preservice teachers were asked to approximate their instructional decisions, considering the students' mathematical understanding observed in the video, through technological media such as Go Animate or Lesson Sketch. The study involved 89 preservice mathematics teachers from four different universities in the United States. Amador et al. (2021) defined components of professional noticing skills situated in the pedagogy of practice components described by Grossman and colleagues (2009a). They prepared a noticing illustration task and began the task by watching a segment of one of the publicly available instructional videos. The topic in the video was fractions for fourth graders, and a small group of students tried to find a strategy to solve the problem. Then, during the ten minutes, the video was stopped at the end of the instructional moment without the group of students coming up with a strategy to answer the problem. Up to this point, the pre-service teachers were supposed to pay attention to the content of the lesson, make interpretations, and decide what would happen next (Jacobs et al., 2010). They were then asked to illustrate their decisions through animations or representations using questions in the task. In fact, while making instructional decisions, preservice teachers were allowed to create motion, people, and objects and insert text and speech into the animations. The results of the study showed that preservice teachers generally explored students' strategies and ensured students' understanding of the problems. More specifically,

the findings indicated that preservice teachers most often focused on teacher moves "(a) explored what the student(s) had already done or suggested" or "(b) ensured that the student(s) understood the problem" or "had an entry point" (p. 1). They found notable differences in the teacher decisions according to the different platforms. Amador and colleagues concluded that noticing should have the fourth sub-skill called "enactment" to notice the framework described by Jacobs et al. (2010) to understand teachers' decision process as mentioned above. They offered it should be given a chance for preservice teachers to approximate practice to make sense of how the decisions were put into practice.

Through a case study, Güner and Akyüz (2017) investigated four preservice middle school mathematics teachers' knowledge about students' mathematical thinking related to the concepts of perimeter and area. Their findings indicated that the preservice teachers' predictions and expectations about students' mathematical thinking were limited. They were grouped into three themes: expected and happened, expected but did not happen, and not expected but happened. They concluded that preservice teachers did not adequately make sense of students' mathematical ways of the perimeter concept.

Estapa et al. (2018) conducted research with 126 elementary preservice teachers at six universities to investigate preservice teachers' responses to the activity focused on the representation, decomposition, and approximation of practice for teaching and learning mathematics, which are components of the framework for teaching practice developed by Grossman et al. (2009a). The researchers developed The Noticing in Action Task grounded in the construct of professional noticing developed by Jacobs and her colleagues (2010) and Grossman et al. (2009a) pedagogies of practice. Preservice teachers took up the task of articulating their professional noticing within video representations, written decompositions, and animated approximations of practice. Findings showed that most of the preservice teachers attended student thinking in written accounts. However, when they were asked to show their noticing

through animation, their focus shifted to the classroom teacher rather than the students' thinking.

Girit-Yıldız et al. (2022) investigated the professional noticing skills of 30 preservice mathematics teachers regarding measurement concepts (including length, area, and volume). In this regard, the preservice teachers were asked to watch video cases about students' misconceptions of the measurement concepts in a video case-based professional development environment. Their responses and discussions were examined using content analysis. The results showed that while the preservice teachers' responses tended to describe the students' misconceptions, their responses in a video-case-based professional development environment also supported the interpretations of the student's mathematical understanding and the determination of the following steps to promote the student's mathematical understanding.

Through a lesson study professional development model, Güner (2017) investigated two preservice middle school mathematics teachers' noticing of students' mathematical thinking about measurement concepts. A case study design was used in this study. Data were collected through transcripts of video recordings, interviews, observations, lesson plans, and field notes over two months during the spring semester of 2014-2015. Results indicated that preservice teachers had the opportunity to develop their noticing skills on the measurement concepts during the lesson study professional development model.

So far, brief information about the studies on professional noticing skills has been provided. The analysis of the existing literature highlights the need for further research on developing preservice teachers' mathematical knowledge and professional noticing skills through professional development during teacher education programs. Current studies have shown that one way to support the development of mathematical knowledge and professional noticing skills is to integrate various pedagogies into teacher education programs. For this purpose, the following section gives information about several of the pedagogies of teacher education and their adaptations into courses via various models.

2.3 Pedagogies in Teacher Education

Current research studies have shown that preservice teachers have difficulty transferring their university learning into teaching practices (e.g., Dilberoğlu, 2017; Grossman et al., 2009a). One reason behind this difficulty could be related to the fragmented structure of the traditional teacher education programs, which mainly provided knowledge-based experiences for preservice teachers rather than authentic experiences (Darling-Hammond, 2006; Ghousseini, 2009; Grossman et al., 2009a; Lampert et al., 2013). To prepare preservice teachers with authentic experiences, researchers have proposed to restructure teacher education programs, aiming to strengthen the bond between research on teacher education and teaching (e.g., Franke et al., 2007; Grossman et al., 2009a; Grossman & McDonald, 2008; Kazemi et al., 2009; Smith, 2001), teaching and practices (e.g., Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013). Before doing this, they have clarified the practices that are key to the work of teaching. They refer to these practices as core practices, such as eliciting and responding to students' thinking, planning and evaluating students' work, and orienting students' ideas (Ball & Forzani, 2011; Grossman et al., 2009a; 2009b; Kazemi et al., 2015; Lampert et al., 2013; McDonald et al., 2013). McDonald et al. (2013) emphasized the role of core practices in teacher education as follows:

> The move toward core practices is an attempt to weave together novices' development of meaningful knowledge for teaching with their capacity to actually enact ambitious teaching in particular disciplines in the classroom. The emerging community of teacher educators, practitioners, and researchers focused on core practices attend to the problem of enactment by specifying aspects of teaching practice that are essential to the work of teaching, and which novices can learn to enact in their early years. Their intent has been to support novices to develop a vision of high-quality teaching that is content-rich, rigorous, and meaningful to students and which novices can enact in their classrooms (p. 226).

In light of the above research, it could be said that it is essential to find ways to empower preservice teachers' knowledge and skills to implement core practices into courses of teacher education programs. To achieve this, different pedagogies should be integrated into the courses of teacher education programs, and cooperation with these pedagogies should be established (Kazemi et al., 2009). State it differently; researchers have recently sought ways to integrate core practices into teacher education programs by taking advantage of several pedagogies in teacher education (e.g., Grossman et al., 2009a; 2009b; Kazemi et al., 2009; Lampert et al., 2013; Silver et al., 2007; Steele, 2006). To illustrate, Grossman et al. (2009a) offered a framework called *the pedagogies of practice* "for thinking about the teaching of practice in the context of the university" (p. 2058). Indeed, they proposed a framework to introduce several new pedagogies in teacher education: a) Representation of practice, b) decomposition of practice, and c) approximations of practice.

Representation of practice is the first step to preparing preservice teachers for teaching practice by examining written dialogues, narrative cases, or video recordings of classroom videos "to develop ways of seeing and understanding professional practice" (Grossman et al. 2009a, p. 2065). The second step, *decomposition of the practice*, involves dismantling teaching and learning to examine the practice in more detail. In this way, it aimed to clarify teaching practices for novices. The third and last step, *approximation of practice,* encompasses providing "opportunities for novices to engage in practices that are more or less proximal to the practice of a profession" (Grossman et al., 2009a, p. 2058) with different degrees of authenticity, such as continuing the lesson by using technological tools such as Go Animate & Lesson Sketch; (e.g., Amador et al., 2021), coach rehearsals (e.g., Arbaugh et al., 2019), and rehearsals (e.g., Lampert et al., 2013; McDonald et al., 2013) as part of approximation of practice.

Furthermore, some researchers took this framework one step further by giving preservice teachers the chance to enact teaching practice in real settings. They were called this pedagogy as the *enactment* (e.g., Grossman & McDonald, 2008; Kazemi

et al., 2009; Lampert et al., 2013; McDonald et al., 2013). The *enactment* includes the implementation of teaching practice in actual classroom settings. Furthermore, some researchers emphasized the importance of the role of community and reflection of the teaching in developing their knowledge, skills, and practices and introduced other pedagogy as the *pedagogies of reflection and investigation* (Grossman et al., 2009b; Lampert & Ball, 1998).

Recently, researchers have combined these pedagogies under the *cycle of enactment and investigation*, aiming to investigate and enact teaching practices through the use of multiple pedagogies such as observation, approximation of teaching practices, enactment of teaching, and reflection on those practices (Kazemi et al., 2009). The root of cycles of enactment and investigation has originated the community of practice framework (Lave & Wenger, 1991; Wenger, 1998). The goal of the framework was to enable preservice teachers to learn and teach practices through being part of the community. State it differently; one of the most influential learning cycles in teacher education adopted the community practices is the cycle of enactment and investigation (e.g., Arbaugh et al., 2019; Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013).

Communities of practice are defined as communities of people "who share a concern, a set of problems, or a passion about a topic, and who deepen their knowledge and expertise in this area by interacting on an ongoing basis" (Wenger et al., 2002, p. 4). State it differently; they work together regularly to share information, improve their skills, learn together, and increase their general knowledge of the field. Lave and Wenger (1991) defined learning in communities as follows:

> Learning involves the whole person; it implies not only a relation to specific activities but a relation to social communities – it implies becoming a full participant, a member, a kind of person. In this view, learning only partly – and often incidentally – implies becoming able to be involved in new activities, to perform new tasks and functions, to master new understandings.

> Activities, tasks, functions, and understandings do not exist in isolation; they are part of broader systems of relations in which they have meaning (p. 53).

Considering this explanation, it could be inferred that there are three aspects of the participation of the community in practice: a) mutual engagement, b) joint enterprise, and c) shared repertoire (Wenger, 1998). Firstly, mutual engagement includes the developing of a shared idea through becoming total participants in the community. It is generally related to the participant's role in the community. In the cycles of enactment and investigation, the participants have several roles, such as observers, teachers, and students. Secondly, joint enterprise refers to "com(ing) together with a shared goal and work toward a joint product that is meaningful to all participants" (Stein & Brown, 1997, p. 162). In the cycles of enactment and investigation, participants have an opportunity to examine the activities collectively to construct individual and shared mathematical knowledge, skills, and practices through productive discussions and reflections. Finally, a shared repertoire involves "routines, words, tools, ways of doing things, stories, gestures, symbols, genres, actions, or concepts that the community has produced or adopted in the course of its existence, and which have become a part of its practice" (Wenger, 1998, p. 83). Shared repertoire belongs to the community (Wenger, 1998). In the cycles of enactment and investigation, participants have an opportunity to develop common mathematical goals and discourse, lesson plans, and to conduct perimeter teaching in multiple settings. The participants of this study are, therefore, part of learning communities through the cycle of enactment and investigation.

So far, several pedagogies, their importance in teacher education programs, and the roots of cycles of enactment and investigation have been mentioned. In light of the mentioned pedagogies, as Lampert et al. (2013) state, researchers offered learning cycles to enable preservice teachers to “go back and forth between investigating teaching and enacting it” (p. 4) and to facilitate the application of the pedagogies in teacher education courses. Consequently, the following section presents detailed

information on learning cycles grounded on the cycles of enactment and investigation.

2.3.1 The Cycles of Enactment and Investigation (CEIs)

Lampert et al. (2013) focus on the cyclical process of investigating and enacting teaching practices by integrating several pedagogies into the learning cycles. As mentioned before, these cycles aim to provide preservice teachers chances to learn in, from, and for the teaching through using representation, decomposition, and approximation of practice components of the pedagogies of practice framework (Grossman et al., 2009a), the pedagogies of reflection and investigation (Grossman et al., 2009b; Grossman & McDonald, 2008); and *enactment* (Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013). In this way, Lampert et al. (2013) embedded such pedagogies into the learning and teaching cycles called the *Cycles of Enactment and Investigation* (CEIs). Figure 2.5 shows the process of cycles of investigation and enactment.

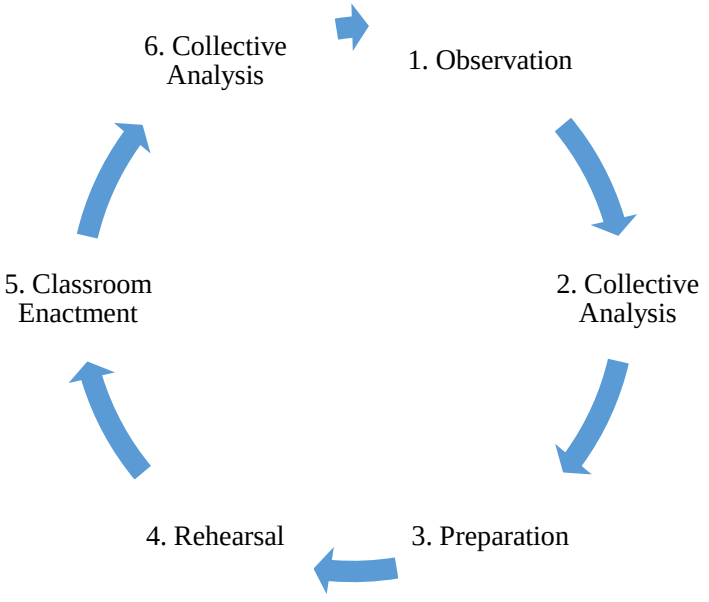


Figure 2.5. Cycles of Enactment and Investigation (CEIs) (Lampert et al., 2013, p. 229).

As Figure 2.5 shows, Lampert et al.'s (2013) learning cycles consist of six iterative nodes and the same instructional activity used in each node. Before defining each node, what the instructional activities mean is mentioned. The instructional activities are defined in the study of Kazemi et al. (2009, p.13) as follows:

> Tasks enacted in classrooms that structure the relationship between the teacher and the students around content in ways that consistently maintain high expectations of student learning while adapting to the contingencies of particular instructional interactions (Lampert & Graziani, 2009).

The first node, *observation*, includes observing the implementation of the instructional activity in the university classroom in live, video, or written form. In this way, preservice teachers could examine crucial practices embedded in the instructional activity. Hence, in this node, one of the pedagogies in teacher education called the representation of the practice component of pedagogies of practice framework (Grossman et al., 2009a) was used.

The second node, *collective analysis*, includes collectively analyzing the principles, practices, and mathematics embedded in the instructional activity within the guidance of mathematics educators. Thus, in this node, one of the pedagogies in teacher education called the decomposition of the practice (Grossman et al., 2009a) was used.

The third node, *preparation,* includes preparation for the rehearsal to teach the same instructional activity. At that stage, the preservice teachers use their experiences regarding the instructional activity gained in previous cycles as a planning tool. Hence, in this node, one of the pedagogies in teacher education called the decomposition of the practice (Grossman et al., 2009a) was used.

The fourth node, *rehearsal*, includes conducting the instructional activity teaching in the simulated classroom environment. Indeed, one preservice teacher acted like a teacher while others behaved, questioned, and queried like students. Hence, in this

node, one of the pedagogies in teacher education called the approximation of the practice (Grossman et al., 2009a) was used.

The fifth node, *enactment*, includes that while one preservice teacher implemented the instructional activity with the actual students in the classroom settings, other preservice teachers and the instructor attended his/her lesson in an observer role. They were implementing the instructional activity in the actual classroom setting. Thus, in this node, one of the pedagogies in teacher education called the enactment (Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013) was used. Moreover, this implementation is also recorded for further analysis.

The sixth and final node, *collective analysis*, includes the collective analysis of video recordings of the classroom enactment. In this stage, preservice teachers collectively analyze the principles, practices, and mathematics regarding the instructional activity implemented in the classroom. Meanwhile, the instructor guides them to question what could be done to improve the quality of the teaching. In this node, one of the pedagogies in teacher education called the pedagogy of reflection and investigation (Lampert & Ball, 1998; Grossman et al., 2009b) was used as a pedagogical practice.

So far, it has generally been mentioned how preservice teachers could obtain opportunities to learn in, from, and for the practice of teaching through the iterative and interactive nature of the learning cycles based on the cycles of enactment and investigation. Given some limitations and the usefulness of this learning cycle in teacher education programs, mathematics educators have developed other models that integrate pedagogies to "articulate a common language for specifying practice..." (McDonald et al., 2013, p. 379). One limitation might be that the steps of the learning cycles should not be sequential (McDonald et al., 2013) since researchers and mathematics educators should start using the model where they want, taking into account their purposes. Another limitation is to talk about the need to position the core practices in the model. Researchers have claimed that mathematics educators and researchers should be able to use one or more core practices throughout the learning cycle according to their purposes. Thus, researchers offered complementary

frameworks and models considering the studies above. In this regard, the details of the Learning Cycle Framework and the Three Minute Rehearsal Cycle of Enactment and Investigation (TMR-CEI) model, which adopted the cycle of enactment and investigation, were mentioned in the following sections.

2.3.2 A Learning Cycle

In light of the pedagogies of practice (Grossman et al., 2009) and the cycles of enactment and investigation (Kazemi et al., 2009; Lampert et al., 2013), McDonald et al. (2013) propose a supplementary framework called a Learning Cycle "for orienting the pedagogy of teacher education" (p.382) by placing the cycles of enactment and investigation and different pedagogies within this learning cycle. More specifically, McDonald and colleagues (2013) proposed it to:

> Learn particular practices by introducing them (participants) to the practices as they come to life in meaningful units of instruction, preparing them to enact those practices, requiring them to enact the practices with real students in real classrooms, and then returning to their enactment through analysis (p. 382).

Thus, they claim that learning is a process, and preservice teachers should have teaching experiences in different settings (controlled, designed, and authentic). Moreover, McDonald et al. (2013) designed the nodes of the model not sequentially, as mentioned before. Put differently, they allowed researchers to start from any stage of the model according to their purposes. They structured their model around one of the core practices, eliciting and responding to students' thinking through the instructional activity. Besides, McDonald et al. (2013) offer to use various pedagogies for each quadrant of their cycles. A Learning Cycle model is shown in Figure 2.6.

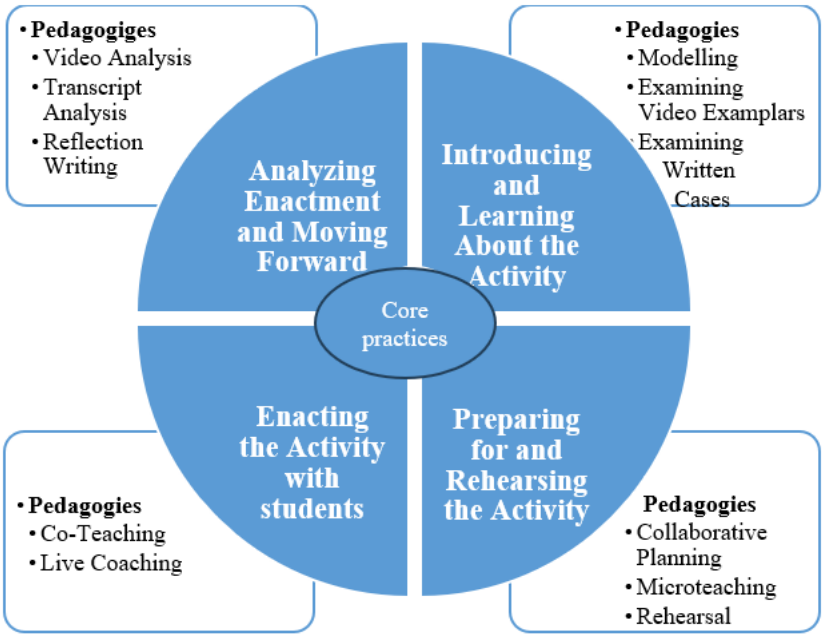


Figure 2.6. A Learning Cycle (McDonald et al., 2013, p. 382).

In the first quadrant of the cycles, *introducing and learning about the activity*, the instructor introduces the instructional activity for preservice teachers by implementing, modeling, watching, or examining written cases about the implementation of the activity within the context of representations of practice (Grossman et al., 2009a). Put differently, McDonald and colleagues offer to use various representations, such as modeling and video analysis, to represent teaching practice in this quadrant. This way, they claim that preservice teachers could obtain information about the activity by observing the principles, practices, and mathematics embedded in the instructional activity. Moreover, preservice teachers engage in the instructional activity and collectively break down the principles, practice, and mathematics in the activity as a part of the practice decomposition. This quadrant is similar to observation and collective analysis nodes of the cycles of enactment and investigation developed by Lampert et al. (2013).

The second quadrant, *preparing for and rehearsing the activity*, might start with the collaborative or individual lesson planning processes depending on the mathematics

teacher educator's purpose. In this process, preservice teachers anticipate possible student responses regarding the instructional activity and construct pathways to respond to such thoughts. Then, they implement the instructional activity in a controlled environment (with the instructor's guidance) through microteaching or rehearsal, which were used in the context of approximations of the practice (Grossman et al., 2009a). In this way, they get a chance to prepare themselves "for enactment through planning and sheltered practice" (McDonald et al., 2013, p. 383) as in the preparation and rehearsal nodes of the cycles of enactment and investigation developed by Lampert et al. (2013).

The third quadrant, *enacting the activity with students*, includes implementing the instructional activity with the actual students within the context of enactment (Lampert et al., 2013). The authenticity of the implementation could depend on the mathematics teacher educators' purposes. In other words, preservice teachers could enact the activity by interviewing students or in natural classroom settings. This quadrant provided opportunities for preservice teachers to gain experience responding to students' expected and unexpected questions, and the enactment is recorded with the video. This quadrant is similar to the enactment node of the cycles of enactment and investigation developed by Lampert et al. (2013). During the implementation of the instructional activity, mathematics teachers' educators could use pedagogies called co-teaching or live coaching to improve the quality of their teaching.

The fourth and final quadrant, *analyzing enactment and moving forward*, includes collectively analyzing the enactment recordings to improve their teaching quality. In this way, they collectively reason about in what ways to enhance student's understanding within the context of pedagogies of investigation and reflection (Grossman et al., 2009b; Grossman & McDonald, 2008) and have a chance to learn their practice as in the collective analysis node of the CEIs developed by Lampert et al. (2013).

To sum up, the cycles of enactment and investigation developed by Lampert et al. (2013) and a Learning Cycle developed by McDonald et al. (2013) were designed to allow preservice teachers to improve their knowledge, skills, and practices through authentic and collaborative experiences. Both studies structured their cycles by using the same instructional activity in various settings. However, researchers have proposed that mathematics, pedagogy, skills, and practices could be handled separately in the learning cycles (e.g., Arbaugh et al., 2019) rather than embedding them in one instructional activity as in the cycles of Lampert et al. (2013) and McDonald et al. (2013). Moreover, recent literature emphasizes the necessity of individual analysis before collective analysis within the learning cycle (e.g., Arbaugh et al., 2019). To improve these cycles, Arbaugh and colleagues (2019) offered the model called The Three-Minute-Rehearsal Cycle of Enactment and Investigation (TMR-CEI), aiming to improve preservice teachers' knowledge, skills, and practices. The details of the model are discussed in the following section.

2.3.3 The Three-Minute-Rehearsal Cycle of Enactment and Investigation (TMR-CEI)

In light of the mentioned models above, Arbaugh et al. (2019) offered a model called The Three-Minute-Rehearsal Cycle of Enactment and Investigation (TMR-CEI). The differences between these models and other mentioned models are as follows. Firstly, Arbaugh et al. (2019) preferred to use the same type of mathematical tasks in different nodes of their model. In addition to using the same type of tasks, they dealt with mathematics and pedagogies in the tasks separately rather than embedding them in the instructional activity as in the models of Lampert et al. (2013) and McDonald et al. (2013). Another difference from other mentioned models was that Arbaugh et al. (2019) added individual analysis before collective analysis, considering the role of individual analysis in developing knowledge, skills, and practices. Finally, they offered to replace the enactment node, which occurred the implementation of the instructional activity in the actual classroom setting with the

real students, with coach rehearsal "by engaging PSTs (preservice teachers) in a low-risk approximation of practice that afforded opportunities for them to make decisions in response to student thinking" (Arbaugh et al., 2019, p. 24). They focused on implementing the mathematical tasks with the preservice teachers, graduate students, and teacher educators during the university course hours. Indeed, in this process, the preservice teachers acted like teachers to implement one of the tasks in the list distributed by mathematics teachers' educators in a simulated classroom environment.

Meanwhile, the graduate students acted like students by using partial responses distributed by the educators, and the teacher educators coached the dialogue between the preservice teachers and graduate students. Since implementing each task in the classroom could last around three minutes, Arbaugh and colleagues (2019) called their model The Three-Minute-Rehearsal Cycle of Enactment and Investigation (TMR-CEI). The details of the model are shown in Figure 2.7.

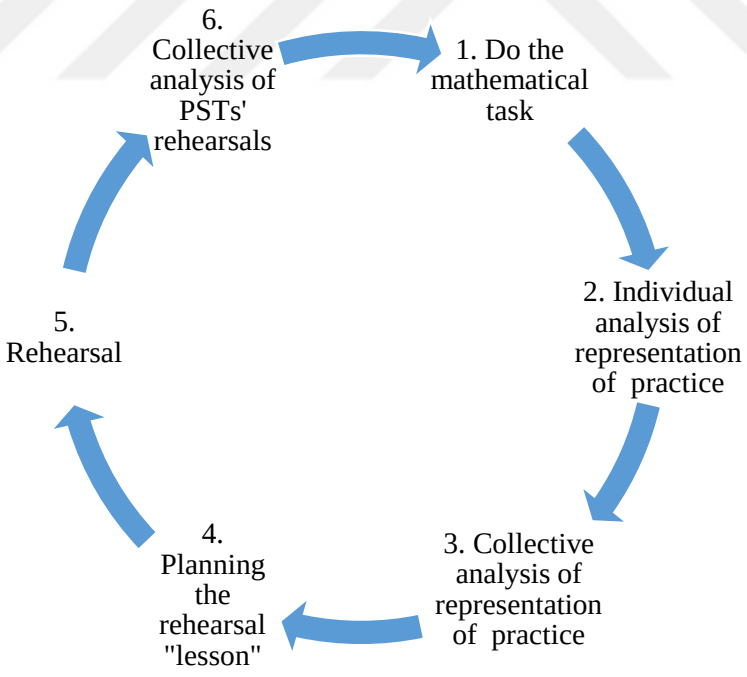


Figure 2.7. TMR-CEI model offered by Arbaugh et al. (2019, p. 25).

As shown in Figure 2.7, TMR-CEI consists of six nodes. The first node, *Do the Mathematical task,* included implementing two activities. The first task involved working individually on the mathematical tasks to establish criteria to decide whether a mathematical argument serves as proof and then sharing their thinking with the class. The second task involved analyzing ten students' responses mathematically regarding the same mathematical tasks. Arbaugh and colleagues (2019) claimed these tasks were vital since they shaped the preservice teachers' mathematical thinking for further steps. Notably, preservice teachers engaged the tasks in the first node for developing knowledge of students and mathematical content. The second and third nodes, *Individual and Collective Analysis of a Representation of Practice,* were included to examine the narrative cases regarding implementing the tasks in one of the eighth-grade classrooms. This way, the researchers allowed the preservice teachers to examine such cases concerning students' and teachers' actions. This node was grounded on the representation and decomposition component of the pedagogies of practice framework (Grossman et al., 2009a). The fourth node, *planning the rehearsal "lesson,"* included planning the rehearsal to implement one of the tasks they have studied in previous nodes of the model. In this process, the preservice teachers were assigned to plan the tasks collaboratively to rehearse them and decompose the teaching practices by focusing on students' possible thinking (Grossman et al., 2009a).

The fifth node, *rehearsal,* includes implementing the tasks in a middle school classroom simulation. During this node, one of the preservice teachers acts as a teacher, and one of the graduate students acts as a student. The instructor behaves like a coach to manage the simulated teacher and student dialogue. Meanwhile, the rehearsal was video recorded. This node is grounded on the approximation of the practice component of the pedagogies of the practice framework (Grossman et al., 2009). The sixth and final node, *collective analysis of PST's rehearsal,* included analysis of the rehearsal by focusing on teachers' and students' actions through one of the qualitative analysis software. This node is grounded on pedagogies of investigation and reflection of the teaching practices (Grossman et al., 2009b;). To

summarize how some different pedagogies have been integrated into the learning cycles mentioned through the cycles of enactment and investigation and to allow readers to compare their similarities and differences, Table 2.1 is presented.

Given the models mentioned above, the model of the current study was designed to support the development of the preservice teachers' mathematical knowledge and professional noticing skills on the perimeter concept through the model based on the cycle of enactment and investigation. There were several reasons for the development of the model used in this study. First of all, the model was designed around the core practices called eliciting and responding to students' thinking. To develop preservice teachers' responsiveness, knowledge, and skills that affect their practices were dealt with separately through several activities similar to the model of Arbaugh et al. (2019). This feature of the current intervention distinguishes it from the models of Lampert et al. (2013) and McDonald et al. (2013) since these researchers embedded the principles, practices, and mathematics in the instructional activity and used the same instructional activity across the nodes of their models. Secondly, considering the importance of individual analysis before collective analysis, the preservice teachers were expected to complete the activities individually at first. Then, they were asked to discuss their reasoning collectively, similar to the study of Arbaugh et al. (2019). Thirdly, the models of Lampert et al. (2013) and McDonald et al. (2013) involved rehearsing the teaching and then implementing it in an actual classroom setting. Unlike these models, in this study, the analysis of the video recordings of rehearsal steps was added to improve the quality of the teaching in real classrooms. At the same time, different than other studies, it enabled preservice teachers to review their teaching plan in almost every node of the model, aiming to improve their knowledge and noticing skills simultaneously. The detail of the model called the Cycles of Enactment and Investigation for Developing Knowledge and Professional Noticing Skills, is presented in the Methodology Section.

Table 2.1 The summary of the features of the mentioned models.

Model Name	The Cycle of Enactment and Investigation (CEI) by Lampert et al. (2013)	A Learning Cycle by McDonald et al. (2013)	The Three-Minute-Rehearsal Cycle of Enactment and Investigation (TMR-CEI) by Arbaugh et al. (2019)
The number of nodes	Six nodes	Four nodes	Six nodes
The types of study in the nodes of the models	Preservice teachers were allowed to study collectively in all nodes of the models.	Preservice teachers were allowed to study collectively in all nodes of the models.	Preservice teachers were allowed to study individually at first and then collectively in the first and second nodes of the models. Preservice teachers were allowed to study collectively in the models' third, fourth, fifth, and sixth nodes.
Used materials in nodes	The same instructional activity was used in all model nodes, with embedded principles, practices, and mathematics.	The same instructional activity was used in all model nodes, with embedded principles, practices, and mathematics.	The same types of tasks were used in the model's whole nodes, in which principles, practices, and mathematics were handled separately.
The integration of the representation of the practice (Grossman et al., 2009a) into the model	Either live, video recordings, or written documents regarding the implementation of the instructional activity could be used in the first node of the model.	Either modeling, written cases, and video exemplars regarding the implementation of the instructional activity could be used in the first node of the model.	The narrative case regarding the implementation of the mathematical tasks studied in the first node of the model could be used in the second node of the model.

Table 2.1 (Continued)

Model Name	The Cycle of Enactment and Investigation (CEI) by Lampert et al. (2013)	A Learning Cycle by McDonald et al. (2013)	The Three-Minute-Rehearsal Cycle of Enactment and Investigation (TMR-CEI) by Arbaugh et al. (2019)
The integration of the decomposition of the practice (Grossman et al., 2009a) into the model	Preservice teachers were allowed to analyze the practices and prepare lesson plans considering students' and teachers' actions in the second and third nodes of the model.	Preservice teachers were allowed to analyze the practices and prepare lesson plans considering students' and teachers' actions in the second node of the model.	Preservice teachers were allowed to analyze the practices and prepare lesson plans considering students' and teachers' actions in the model's second, third, and fourth nodes.
The integration of the representation of the approximation of practice (Grossman et al., 2009a) into the model).	Preservice teachers were allowed to rehearse the instructional activity in a simulated classroom environment in the fourth node of the model.	Preservice teachers were allowed to implement the instructional activity through either collaborative planning, micro-teaching, or rehearsal, in a simulated classroom environment in the second node of the model.	Preservice teachers were allowed to rehearse the mathematical tasks in a simulated classroom environment in the fifth node of the model.
The integration of the enactment (e.g., Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013) into the model.	Preservice teachers were allowed to implement the instructional activity in a natural classroom environment in the fifth node of the model.	Preservice teachers were allowed to implement the instructional activity through either co-teaching or live coaching in a natural classroom environment in the third node of the model.	None.
The integration of the pedagogies of investigation and reflection (Grossman et al., 2009b, Lampert & Ball, 1998) into the model.	Preservice teachers were allowed to examine and analyze the recordings of teaching practices in the sixth node of the model.	Preservice teachers were allowed to examine and analyze the recordings of teaching practices in the fourth node of the model.	Preservice teachers were allowed to examine and analyze the recordings of teaching practices in the sixth node of the model.

2.4 Length Measurement and Perimeter Concept

The length measurement and the perimeter concept are milestone concepts in measurement (NCTM, 2000). Before mentioning the length measurement and perimeter concept, it is essential to explain what measurement means. In the most general sense, measurement is "assigning a numerical value to an attribute of an object" (NCTM, 2000, p.44). As understood from the definition, measuring is a process that consists of a few sequential steps. First, the attribute of the object to be measured should be decided. For example, if you are asked to measure the bucket, consider what attributes of the bucket are being measured, such as perimeter, surface area, volume, or mass (Van de Walle et al., 2013). Therefore, the attribute of the object is critical to the measurement process. Second, it should be the determined unit to measure that attribute. Indeed, it is appropriate to measure length with units of length, area with units of area, and volume with units of volume. This unit could be considered a non-standard and standard unit of measurement. Finally, the selected units should be compared with the attributes of the objects and the number obtained as a result of this process. To do this, it is necessary to determine the comparison strategy, such as filling, covering, and iterating (Van de Walle et al., 2013).

Lehrer (2003) noted that "measurement is an enterprise that spans both mathematics and science yet has its roots in everyday experience" (p. 179). In addition to its involvement in everyday life, measurement is recognized as a bridge between different mathematical domains, such as number and operations and algebra, and as an opportunity for students to learn about topics in domains such as fractions, operations, functions, and statistics (Clements & Battista, 2001; NCTM, 2000). In addition, it was generally present in curricula under the sub-concepts of length, perimeter, area, volume, temperature, time, fluid, and angle from first through eighth grade. While these concepts occur in the elementary curriculum under the learning area of measurement, they occur in the middle school curriculum under the learning area of geometry and measurement.

The perimeter is a concept that is a particular form of length measurement since it is the total length around a figure, enclosed shape, or region. Furthermore, the concept of the perimeter is defined as "the distance around the region" (Dickson, 1989, p. 79). In other words, it is a measure of length and distance. Clements and Stephan (2004) also defined the length as "a characteristic of an object and can be found by quantifying how far it is between the endpoints of the object" and the distance as "the empty space between two points" (p.300). In addition, Szilágyi et al. (2007) also addressed the one-dimensional nature of length measurement.

Furthermore, Clements and Stephan (2004) highlighted the critical properties of length measurement. The first fundamental property is partitioning, which is concerned with breaking the length of an object into pieces of the same size units. It is essential to develop the idea that length is continuous. The second one is iteration, the process of determining how many times the unit takes place successively until the length of an object fills up. The third one is transitivity. Transitivity occurs when the students make sense of the following relations:

> (a) if the length of object 1 is equal to the length of object 2 and object 2 is the same length as object 3, then object 1 is the same length as object 3;
>
> (b) if the length of object 1 is greater than the length of object 2 and object 2 is longer than object 3, then object 1 is longer than object 3, and
>
> (c) if the length of object 1 is less than the length of object 2 and object 2 is shorter than object 3, then object 1 is shorter than object 3" (Stephan & Clements, 2003, p.5).

The conservation is the fourth property. It is related to being aware that the length of an object does not change when it is moved. The conservation is valid when the parts of the objects are reorganized. Students need to gain the concept of conservation to perform length measurement and many other concepts in geometry and measurement. The fifth one is an accumulation of distance, which determines how

often this unit is used due to adding the specified unit end to end to find the object's length. It should be addressed that adding units or the number of iterations gives the object's length. The sixth and last property of length measurement is the relationship between number and measure. Stephan and Clements (2003) addressed that this crucial property involves understanding that measurement:

> Is related to numbers in that measuring is simply a case of counting. However, measuring is conceptually more advanced since students must reorganize their understanding of the very objects they are counting (p.7).

As mentioned earlier, understanding these properties is necessary for constructing knowledge of the concept of perimeter because it is a particular form of length measurement. The perimeter can be measured using any object that has a length. The perimeter of a polygon can be calculated by adding the lengths of its sides around the shape as long as they do not overlap. In addition to the critical properties related to length measurement, the perimeter, figure, or region must be enclosed.

In addition, it is essential to focus on students' reasoning about length measurement to deepen their understanding of length measurement for meaningful perimeter instruction. To focus on students' understanding, Battista (2006) mentioned two types of reasoning about length measurement: non-measurement reasoning and measurement reasoning. Non-measurement reasoning does not use numbers. Students compare objects or figures in terms of length using holistic strategies such as visual judgments, direct comparisons, part correspondences, and transformations. This type of reasoning focuses on the whole object rather than a part of it.

Measurement reasoning involves "determining the number of unit lengths that fit end to end along an object, without gaps or overlaps" (Battista, 2006, p.141). It involves iterating unit lengths. It also includes "the use of measuring instruments, but also the mental manipulation of numerical measurements - for example, adding side lengths to find the perimeter of a rectangle, or inferring that opposite sides of a rectangle are

of equal length" (Battista, 2006, p.141). Table 2.2 summarizes the levels of reasoning about length measurement.

Table 2.2 Levels of sophistication in students' reasoning about length (Battista, 2006, p.141).

	Non-measurement Reasoning
N0	Holistic Visual comparison
N1	Comparison by decomposing or recomposing
1.1	Rearranging parts for direct comparison
1.2	One to One matching pieces
N2	Comparison by property-based transformations
	Measurement Reasoning
M0	Use of numbers unconnected to unit iteration
M1	Incorrect unit iteration
M2	Correct unit iteration
M3	Operating on iterations
M4	Operating on numerical measurements

Furthermore, as shown in Table 2.2, students typically begin with measurement using non-measurement reasoning. The M4 level is the most advanced, and students can use the procedures in the N2 non-measurement in this process. The difference between non-measurement level N2 and measurement level M4 is that students at the M4 level infer numerical measures of objects rather than the objects themselves. The levels in the framework can help teachers identify their students' difficulties and misconceptions during and after instruction.

As Clements and Stephan (2004) stated, "Measurement… is a complex combination of concepts and skills that develops slowly over the years" (Clements & Stephan, 2004, p. 307–308). Numerous studies have shown that learners (including students, preservice teachers, and teachers) struggle with learning the concepts of length measurement and perimeter (Bragg & Outhred, 2000; Lehrer, 2003; Rayn & Williams, 2007). In the following section, the studies on learners' difficulties with the concept of perimeter are mentioned.

2.4.1 Studies about Learners' Difficulties with the Perimeter Concept

Although it has a practical side, studies have shown that students have difficulty understanding what perimeter means (Fuller, 1997; Martin & Strutchens, 2000; Menon, 1998; Reinke, 1997; Simon & Blume, 1994; Steele, 2006; Tierney et al., 1990). One reason for students' poor understanding of perimeter may be their difficulties with length, as it is a particular form of length measurement, as noted earlier by Martin (2009). Thus, the length measurement provides a foundation for the other measurement concepts, especially the perimeter.

One of the most common difficulties with length measurement is that students leave gaps between units or overlap units (Lehrer, 2003) and are unaware of the need to iterate the unit without leaving or overlapping along the object (Bragg & Outhred, 2000; 2004). These errors often occur when calculating the perimeter of irregular objects. For example, Tan-Şişman and Aksu (2012) examined the conceptual and procedural knowledge and problem-solving skills of 445 sixth-grade students in length measurement. The results showed that students were more successful in common problem types. However, when students are given the lengths of the sides of a figure, they typically calculate the perimeter by summing all the values. It is difficult for students who do not understand the meaning of perimeter when the lengths of the sides are not given directly or when students are asked to find the perimeter of irregular shapes.

In addition, a pioneering study is Fuller (1997), which examined preservice teachers' mathematical knowledge of perimeter. The study was conducted with twenty-six preservice teachers. They were asked to draw a cage for the dog using a 64-meter fence of various sizes and to determine the drawing with the most significant area for the cage. Almost all preservice teachers drew geometric shapes with a perimeter of 64 m to find the most significant area. The results showed that most of the pre-service teachers' knowledge of perimeter was mainly at the procedural level, which was limited to the area of quadrilateral regions.

Another seminal study conducted with preservice teachers is Reinke's (1997) study. Reinke examined the strategies used by preservice teachers to find perimeters of geometric shapes. The study was conducted with seventy-six preservice teachers. As a data collection tool, the preservice teachers were asked to find ways to calculate the perimeters of the given shapes in which a semicircle is drawn inside a rectangle, with the short side of the rectangle being the diameter. The results showed that only nine pre-service teachers correctly calculated the circumference of the figure. The other preservice teachers incorrectly calculated the perimeter of the given shapes. When the incorrect strategies were examined, it was found that the preservice teachers commonly considered subtracting the perimeter of the rectangle from the circumference of the semicircle, measuring, ignoring the semicircle, and finding the perimeter of the rectangle.

Menon (1998) conducted a study to examine preservice teachers' understanding of perimeter, which was conducted with fifty-four preservice teachers. They were asked to prepare a problem to assess students' understanding of the perimeter. They were then given three problems to examine whether the data in the problems were sufficient to solve the problem. The problems written by the preservice teachers were examined at three levels. The first was the low level of understanding. At this level, students can solve the problem by following a single step and using memorization. The second is the intermediate level. Students can solve problems at this level by following multiple steps and using previously learned procedures. The third is a high level of understanding. At this level, students can achieve deep and conceptual understanding. The results showed that forty-five pre-service teachers wrote problems that required a low level of understanding, such as calculating the perimeter of the rectangle given the side lengths, and a medium level of understanding, such as calculating the other side length of the rectangle given the perimeter and one side length. However, six preservice teachers wrote problems that included a high level of understanding in calculating the values that the side lengths of the rectangle can take given the perimeter in the problem given to the preservice teachers to determine whether data were sufficient to calculate the perimeter of an

L-shaped polygon. Thirteen of the pre-service teachers stated that the perimeter could not be calculated because all the side lengths of the figure were not given on the figure. They stated that the data were not sufficient to solve the problem. In general, the results indicated that the preservice teachers had a procedural understanding of the perimeter, for which, conceptually, it was not easy to teach perimeter topics.

Additionally, Livy et al. (2012) conducted a study with 17 preservice teachers to examine their understanding of the concepts of perimeter and area. Their study revealed that preservice teachers lacked understanding of the perimeter concept and provided incomplete definitions of the concept, such as "perimeter is the outside of an object" or "perimeter is length and width" (Livy et al., 2012, p. 101). As can be understood from the sample responses, pre-service teachers generally overestimated the total length of their responses to emphasize them.

Yenilmez and Koza-Çiftçi (2014) investigated the mathematical knowledge of preservice high school teachers regarding the concepts of perimeter and area. Thirty-six preservice teachers participated in this study. The study results showed that preservice teachers tended to think that shapes that are not sufficiently closed do not have a perimeter. Meanwhile, most preservice teachers justified their definition of the concept by going back to where they started. Thus, the findings indicated that most preservice teachers did not have adequate knowledge of the perimeter concept.

Furthermore, studies have shown that students confuse the meaning of area and perimeter (Clement & Stephan, 2004; Danielson, 2005; Hirstein et al., 1978; Kouba et al., 1988; Steele, 2006). At this point, students perform only operations and apply rules. For example, students multiply edges rather than adding side lengths around the shapes to find their perimeter (Kenney & Silver, 1997). This difficulty may stem from understanding the meaning of perimeter and area measurements (Martin, 2009). Danielson (2005) emphasized that this misunderstanding may be due to the formula. Teachers typically give the formula for calculating the perimeter of regular shapes, such as rectangles, without constructing the relationships between length

measurement and perimeter. The formula is based on adding two sides of a rectangle and multiplying by two. The multiplication operation is also used in the formula for area, which is usually taught immediately after perimeter. When students need to understand the meaning of perimeter and then apply the perimeter formula, they confuse the perimeter and area formulas (Mulcahy, 2007). In this regard, Sherman and Randolph (2004) conducted a study with twenty-seven fourth-grade students to examine their understanding of perimeter and area. They found that students could perform area and perimeter formulas to solve problems without understanding the underlying meanings of the formulas.

Similarly, Woodward and Byrd (1983) examined 258 eighth-grade students' understanding of the concept of perimeter. They constructed several gardens with a 60-foot fence and asked the students to choose the garden with the largest perimeter. The results showed that most students said that if the fence used was the same and the perimeters were the same, then the area of the gardens was the same.

In addition, students generally confused the unit of measurement related to perimeter and area (e.g., Martin & Strutchens, 2000; Tan-Şişman &Aksu, 2009; Yeo, 2008). To illustrate, Tan-Şişman and Aksu (2009) examined the performance of 134 seventh-grade students on the perimeter concept. Data were collected through four open-ended questions. The study results showed that students used unit squares when calculating the perimeter. In addition, students could use different units of measurement to calculate the perimeter. Their properties should be introduced to students to make sense of units of measurement.

Another difficulty is assuming a linear relationship between perimeter and area (e.g., Livy et al., 2012; Ma, 1999). Students tend to construct linear relationships between perimeter and area such that all rectangles with the same area must have the same perimeter, and the rectangle with the most significant area must have the largest perimeter (D'Amore & Fandino Pinilla, 2006). Barrett et al. (2006) investigated the strategies of 2nd, 3rd, 5th, 6th, 8th, and 10th-grade students in calculating the perimeter. This study was conducted with a total of thirty-eight students. The

students were given a plastic strip 24 units long and asked to form all the triangles and quadrilaterals they could. The research results show that second and third-graders focus more on where to count and the units that make up the edges of the figures drawn. In addition, it was observed that students at this level generally refrain from using a specific strategy and try to third-graders that can be formed by trial and error.

On the contrary, it was observed that 5th and 6th-grade students systematically decreased or increased the numbers on the rectangle to obtain 24 units after writing the side lengths on the rectangle they drew. It was observed that upper-grade students questioned whether the shapes they found could be formed using strategies. The following section summarizes the literature referenced in the current study.

2.5 Summary of Literature Review

Research studies have shown that the quality of teacher education programs should be enhanced by providing preservice teachers with multiple experiences to increase their effectiveness (e.g., Darling-Hammond, 2006; Ghousseini, 2009; Grossman et al., 2009a; Lampert et al., 2013). In this respect, researchers have suggested that one way to provide these experiences is to incorporate different pedagogies into courses in teacher education programs (e.g., Forzani, 2014; Grossman et al., 2009a; Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013). The cycle of enactment and investigation is one way to achieve this aim (e.g., Lampert et al., 2013; McDonald et al., 2013). Indeed, it allows preservice teachers to deconstruct teaching practices as part of the decomposition of practice and also approximate teaching practices through rehearsals (Grossman et al., 2009a). In addition, it allows preservice teachers to have the opportunity to enact perimeter teaching in an actual classroom setting as part of enactment (Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013) and to examine video recordings of their teaching as part of the pedagogy of investigation and reflection (Grossman et al., 2009b). To increase the applicability of this approach in university courses, researchers have proposed

various models, such as a Learning Cycle (McDonald et al., 2013) and the Three Minute Rehearsal Cycle of Enactment and Investigation (Arbaugh et al., 2019). Considering the above studies and their mentioned limitations, the current study aimed to promote preservice teachers' mathematical knowledge and professional noticing skills regarding the perimeter concept through the model grounded on the cycle of enactment and investigation.

The perimeter concept was chosen since preservice teachers and even teachers often lack knowledge about the perimeter concept (e.g., Fuller, 1997; Livy et al., 2012; Menon, 1998; Reinke, 1997; Yenilmez & Koza-Çiftçi, 2014). Meanwhile, they tend to overlook the details of students' strategies during instruction and have difficulty supporting and responding to students' mathematical ideas about the perimeter concept (e.g., Girit-Yıldız, 2022). However, teachers and preservice teachers are expected to support students' understanding of the perimeter conceptually and to respond to their mathematical thinking about the perimeter concept. In this regard, they should first grasp the concepts related to measurement and length measurement (e.g., Barret et al., 2006; Martin, 2009; Outhred et al., 2003) and note that perimeter is a particular form of length measurement. In addition, they need to a) grasp the meaning of the concept (e.g., Reinke, 197; Menon, 1998), such as the total length around a two-dimensional shape, b) use mathematically correct language about the perimeter, c) know how to enhance perimeter instruction (e.g., Güner, 2017), and d) focus on different ways students think about the perimeter, including misconceptions and difficulties (e.g., Girit-Yıldız et al., 2022).

Although there are several frameworks in the literature about teacher knowledge (e.g., Ball et al., 2008; Shulman, 1986; 1987), in this study, the Knowledge Quartet Model (KQ) proposed by Rowland et al. (2003; 2005) was used. Unlike other models, the reason for using this model is that teacher content knowledge and pedagogical content knowledge are assessed together rather than separately. Furthermore, prospective teachers' noticing skills were examined through the Professional Noticing of the Children's Mathematical Thinking framework put

forward by Jacobs et al. (2010) since it has a particular focus, namely students' mathematical thinking. Therefore, the three components of the framework (attending, interpreting, and deciding) were utilized in the current study.

CHAPTER 3

METHODOLOGY

This study aimed to support the improvement of prospective middle school mathematics teachers' mathematical knowledge and professional noticing skills on the perimeter concept through the development of a model based on the cycles of enactment and investigation within the framework of design-based research. In light of this aim, designed-based research consisting of phases (i.e., preliminary analysis, prototyping, and evaluation) was employed in the current study. The following sections describe this methodological approach, the context and participants of the study, data collection procedures and instruments, and data analysis methods. This chapter also explains the issues related to the study's trustworthiness and the researcher's role.

3.1 The Design of the Study

Design-based research was used as the methodological framework of the current study to investigate prospective middle school mathematics teachers' mathematical knowledge and professional noticing skills on the perimeter concept. In the most general sense, design-based research is defined as "a systematic but flexible methodology aimed to improve educational practices through iterative analysis, design, development, and implementation, based on collaboration among researchers and practitioners in real-world settings, and leading to contextually sensitive design principles and theories" (Wang & Hannafin, 2002, p.6-7). Furthermore, it is

> the systematic study of designing, developing and evaluating educational interventions1 - such as programs, learning processes, learning environments, teaching-learning materials, products and systems (Plomp, 2013, p.11).

In defining design-based research, van den Akker et al. (2006) highlighted five essential characteristics: interventionist, iterative, process-oriented, utility-oriented, and theory-oriented. The interventionist characteristic emphasizes designing and implementing interventions in an authentic context. Because design-based research is iterative, it follows a cyclical design approach through continuous, flexible, and interactive interactions between cycles. In this way, researchers can make necessary revisions in cycles through ongoing analysis. Van den Akker (2013) emphasized that iterations continue "until a satisfying balance between ideals and realization has been achieved " (p. 63). In addition, a process-oriented approach focuses on a process rather than a product to understand and improve interventions. Design-based research is utility-oriented, aiming to produce practical products or outcomes in authentic contexts. It is theory-oriented, as mentioned above; one of its aims is to test and contribute to developing (sub)theories. In addition to these characteristics, Bakker (2018) emphasized that several research approaches, such as surveys, case studies, and experiments, can be used while conducting a design-based study.

For the methodology of the current study, the design-based research was chosen with several objectives in mind. Firstly, the design-based research incorporates theories and previous research findings. As a result, previous designs can be adapted to create a new design to understand better the teaching and learning process in a particular context. This study investigated the teaching and learning processes developed and implemented, considering the literature on the perimeter concept, teacher education, teacher knowledge, and professional noticing skills in the most general sense. Secondly, it arises from the need to answer theoretical questions about the nature of learning phenomena in authentic contexts rather than laboratory environments (e.g., Bakker, 2018) and to reduce the gap between theory, practice, and educational policy (e.g., Brown, 1992; van den Akker et al., 2006). In this context, the present study aimed to support the development of prospective teachers' mathematical knowledge and professional noticing skills on the perimeter by incorporating the course into their undergraduate education and teaching experiences in the field. Thirdly, in design-based research, the term intervention allows researchers to design and test

innovations such as instructional materials, tasks, activities, computer tools, and learning environments to support learners' mathematical understanding. In the current study, the activities and the intervention for the current study were designed and applied to promote prospective teachers' mathematical knowledge and professional noticing skills on the perimeter. Finally, design-based research allows researchers to test and develop theories by evaluating interventions and to understand how theory and practice feed each other over time (Design-Based Research Collective, 2003). Therefore, the purpose of this study was to simultaneously measure and support the development of preservice teachers' mathematical knowledge and professional noticing skills on the perimeter concept rather than focusing on the relationship between mathematical knowledge and professional noticing skills. In this way, the results of the current study may enable the researcher to synthesize such frameworks to reduce existing frameworks on teachers' knowledge and skills (Prediger et al., 2008). So far, the design-based research methodology has been mentioned by explaining its purposes and general characteristics and why design-based research the appropriate methodology for this study is. The following paragraphs will mention the phases of design-based research and the principles and procedures followed in each stage.

There are several frameworks in the literature to clarify the phases of design-based research. For example, Bannan (2013) described the design-based process with four phases: informed exploration, enactment, evaluation (local), and evaluation (broader impact). In addition, Gravemeijer and Cobb (2006) mentioned these three phases: preparing for the experiment, conducting the experiment, and conducting retrospective analyses. Moreover, McKenney and Reeves (2012) expressed the three phases of the design research: Baseline, intervention, and retrospective analysis. In another study, Plomp (2013) highlighted three essential phases of design-based research: preliminary research, development or prototyping phase, and assessment phase. Hence, considering these studies, it could be said researchers agree that three essential phases should be followed to conduct design-based research (Bakker, 2004; Bannan, 2013; Gravemeijer & Cobb, 2006; McKenney & Reeves, 2012; Plomp,

2013). Given the purpose of this study, the phases identified by Plomp (2013) for conducting the design-based research study were followed. Figure 3.1 summarizes the phases of the design-based research.

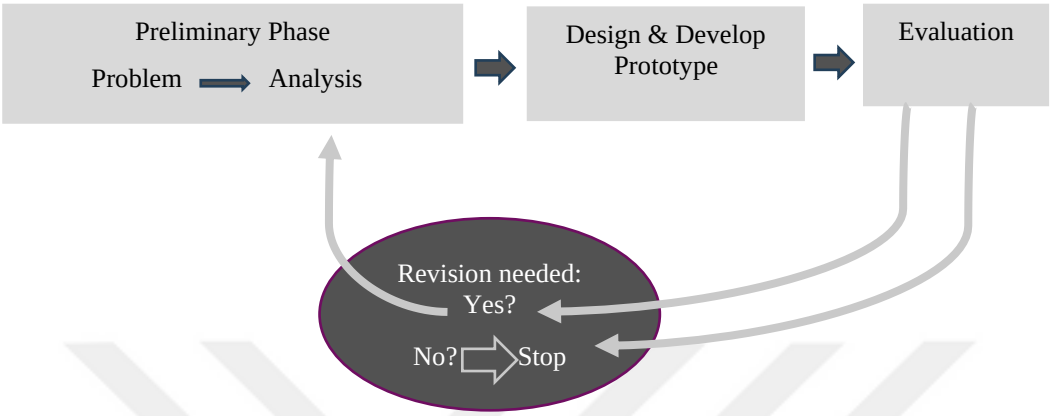


Figure 3.1. The general design process of the design-based research adapted from Plomp (2013, p.17).

In the most general sense, the first phase, preliminary research, includes "needs and context analysis, literature review, and development of a conceptual or theoretical framework for the study" (Plomp, 2013, p. 19). Thus, the preliminary phase begins with identifying and analyzing the research problem through needs, context analysis, and literature review. In addition, researchers should identify or develop their assessments using available items and instruments such as written tests, interviews, or whole-class performance assessments (Gravemeijer & Cobb, 2006). At this point, the design team's role comes to the fore. The design team might include mathematics education researchers, subject matter experts, graduate students, and middle school mathematics teachers. The design team might anticipate the need for revisions to the assessment in the plan and work in an advanced manner. The next step is to formulate a conceptual and theoretical framework, considering the relevant literature and the purpose of the study. A detailed explanation of the preliminary research for the current study is provided in the Data Collection Procedures section.

Secondly, the prototyping phase involves the design, redesign, and ongoing analysis of the instructional activities (Plomp, 2013). The prototyping phase aims to “test and improve the conjectured local instruction theory that was developed in the preliminary phase and to develop an understanding of how it works” (Gravemeijer & Cobb, 2006, p. 24). In other words, by the nature of research, instructional activities and learning objectives or models can be modified based on the ongoing analysis of activities to improve current conditions. As Confrey et al. (2017) stated, it begins with "what students bring to their early understanding of target concepts and identifying landmarks and obstacles students are likely to encounter as they proceed from a naive to a more sophisticated understanding" (p. 718). The design team met before and after implementing the micro-cycles and conducted the necessary session evaluations. Clements and Sarama (2007) state that the frequency and length of iterative design cycles depend on the intervention's development stage. The combination of micro-cycles forms macro-cycles that parallel the learning objectives. The details of the prototyping phase of the current study are explained in the section on Data Collection Procedures.

The third, the evaluation phase, involves “summative evaluation to conclude whether the solution or intervention meets the pre-determined specifications” (Plomp, 2013, p. 19). Because the design-based research aims to contribute to the development of a local instructional theory and ontological innovations (Gravemeijer & Cobb, 2006), the data sets should include multiple sources of data, such as video recordings of all classroom lessons, video recordings of pre- and post-interviews, student written work, field notes, and recordings of design team meetings. The feasibility and effectiveness of the models are evaluated through retrospective analysis to improve educational practice (Collins et al., 2004). In addition, this phase includes removing activities that did not work as planned and modifying practical tasks. Considering both the ongoing analyses conducted during the experiment and the retrospective analyses, the conclusions drawn from these findings may necessitate conducting a follow-up experiment. In this section, general characteristics of stages

of the design-based research have been mentioned. The following section provides information about the context and participants of the current study.

3.2 Context and Participants of the Study

The study participants were senior-grade prospective middle school mathematics teachers enrolled in the Elementary Mathematics Education (EME) program at one of the state universities in Istanbul, Türkiye. Indeed, these prospective teachers who enrolled in this course were expected to graduate from the EME program in the spring semester of the 2020-2021 academic year if they were successful in the spring semester courses. The underlying reason for selecting the fourth-grade prospective teachers as the study participants was related to their experience in the program, as they have a high potential to gain in-depth information about knowledge and noticing skills compared to other grades. This study was also conducted in one of this program's sections of the Teaching Practice 2 course. Forty-five seniors enrolled in the Teaching Practice 2 course. However, not all prospective teachers enrolled in the same sections. During the spring semester, the course had six sections. The first three (A, B, and C for these sessions) had eight prospective students enrolled, and the last three (D, E, F) had seven prospective students enrolled. All students would enroll in the same section with the same instructor and in the same school classes where they observed the fall semester. This study was conducted in the F section purposefully because the majority of the prospective teachers enrolled in the F section had the highest cumulative GPA among the senior-grade prospective teachers who enrolled in the program. Although all of the prospective teachers completed the pre-and post-tests in the F section, five of them attended the courses from the beginning to the end of the study voluntarily. Therefore, the participants of the current study consisted of five prospective teachers. One was male, and four were female. The pseudonyms for these prospective teachers were Maya, Preservice Teacher 2, Preservice Teacher 3, Preservice Teacher 4, and Preservice Teacher 5.

Among them, only (Maya) had the opportunity to have perimeter teaching experiences in various classroom settings as a part of this study. Therefore, the findings regarding the development of Maya's mathematical knowledge and professional noticing skills on the perimeter concept through the model grounded on the cycle of enactment and investigation were shared within the context of the current study. Besides, when considering the roles of the other prospective teachers in developing Maya's mathematical knowledge and professional noticing skills on the perimeter concept, the demographic information about the other participants, namely Preservice Teacher 2, Preservice Teacher 3, Preservice Teacher 4, and Preservice Teacher 5, were also shared in the following paragraphs, respectively.

Maya, a 22-year-old student, had taken almost all of the courses related to mathematics and the teaching profession. She completed the Foundations of Mathematics I and II courses with BA and BB grades. Maya completed the Analytic Geometry, Approaches to Learning and Teaching Mathematics, Geometry, and Measurement Teaching courses with a BA grade, Teaching Experience I, and Misconceptions in Teaching Mathematics courses with AA grades. She had a high cumulative GPA and wanted to be a teacher since childhood, so Maya chose the field willingly and lovingly. At the same time, she had teaching experience. Maya gave private mathematics lessons at various levels during her university education. Besides, she was enthusiastic about participating in the study because she felt uncomfortable teaching concepts in geometry. She shared her thoughts about geometry before participating in the present study as follows:

> I chose my department with pleasure; in all my preferences, it was elementary school mathematics teaching. I have chosen a profession that I have wanted since my childhood... I like calculus and algebra courses. It seemed closer to me because I am familiar with it. So it was easy for me. I liked the misconceptions in math classes because when we looked at those misconceptions, there were some misconceptions that I had made. So, I enjoyed seeing them and learning about them. There is no course I dislike,

> but I struggled with geometry in the courses. This was also the case during the university preparation process. I never succeeded in geometry because I needed a conceptual base regarding geometry. Therefore, I always needed help in geometry class. However, I needed to gain well-developed knowledge, so I focused on the perimeter formula when solving the problem or teaching the concept. However, I want to be something other than a teacher who always follows the formula. I want to be a teacher who gives the logic behind the concept. I want to participate in this research because I believe I would learn a lot in the process that would affect my teaching.

As could be understood from her explanations, she said she was interested in designing her lessons based on students' understanding but was unsuccessful in geometry. She was willing to participate in the current study to develop her knowledge and teaching of the perimeter and could express her opinions comfortably. Additionally, she had a talkative and moderate personality, which was important because this study was based on collaborative work, such as discussing ideas. Furthermore, among the participants of the current study, she only volunteered to practice and enact perimeter teaching in simulated and actual classroom environments. In this way, the researcher obtained a chance to examine her instructional decisions and strategies during the teaching. Thus, the findings about her mathematical knowledge and professional noticing skills on the perimeter concept were presented in the current study.

Preservice Teacher 2 was a 22-year-old student. She completed the courses Fundamentals of Mathematics I and II courses with grades of BB and BA, respectively. Meanwhile, she completed Analytic Geometry, Approaches to Learning and Teaching Mathematics, Geometry and Measurement Teaching courses with CB, CC, and CB grades, respectively. Teaching Experience I course and Misconceptions in Teaching Mathematics courses with BA and AA grades, respectively. At the same time, she had teaching experience through private lessons. She stated that she willingly completed the intervention of the current study. In

addition, she participated in almost all of the discussions and made valuable comments that brought the discussion to the next level.

Preservice Teacher 3 was a 38-year-old male student. He graduated from another program at the university level before enrolling in this program. He completed Foundations of Mathematics I and II, Analytic Geometry, Teaching Geometry and Measurement, Teaching Experience I, and Misconceptions in Teaching Mathematics courses with AA grades. He completed Approaches to Learning and Teaching Mathematics courses with a CB grade. As indicated by his grades, he was strong in mathematics and mathematics education. Like Preservice Teacher 2, he participated in almost all the discussions and made valuable comments that brought the discussion to the next level.

Preservice Teacher 4 was a 22-year-old female student. She completed the Foundations of Mathematics I and II courses and the Analytic Geometry courses CB, CC, and CC grades, respectively. She also completed Approaches to Learning and Teaching Mathematics and Geometry and Measurement Teaching courses with CB grades. In addition, she completed Teaching Experience I and Misconceptions in Teaching Mathematics with AA and BA grades, respectively. She studied hard to prepare for the Public Personnel Selection Examination (KPSS). Although she willingly participated in the study, she did not contribute to all discussions in the current study.

Preservice Teacher 5 was a 22-year-old female student. She completed Fundamentals of Mathematics I and II and Analytic Geometry courses with CC, CB, and DD grades, respectively. She also completed Approaches to Learning and Teaching Mathematics and Geometry and Measurement Teaching courses with CC grades. In addition, she completed Teaching Experience I and Misconceptions in Teaching Mathematics with grades of AA and BB, respectively. Like Preservice Teacher 4, she was studying for the KPSS and did not contribute to all discussions in the current study. All participants willingly participated in the study but were less eager than Maya. They are all regular students and have completed all courses

offered by the EME program. The information regarding the academic background of the participants is summarized in Table 3.1.

Table 3.1 Participants' academic background.

Participants	Fundamentals of Mathematics I	Fundamentals of Mathematics II	Analytical geometry	Approaches to Learning and Teaching Mathematics	Geometry and Measurement Teaching	Misconceptions in Teaching Mathematics	Teaching Experience I
Maya	BA	BB	BA	BA	BA	AA	AA
Preservice Teacher 2	BB	BA	CB	CC	CB	BA	AA
Preservice Teacher 3	AA	AA	AA	CB	AA	AA	AA
Preservice Teacher 4	CB	CC	CC	CB	CB	BA	AA
Preservice Teacher 5	CC	CB	DD	CC	CC	BB	AA

The grades' coefficients are as follows: DD-1, DC-1.5, CC-2; CB-2.5, BB-3, BA-3.5, and AA-4 out of 4.00.

This study was conducted in the Teaching Practice 2 course within the context of undergraduate education, as mentioned before. The course was established as a required course in the EME program. Prospective teachers take this course in the eighth and final semester of the program. In the course, they were expected to observe lessons conducted in one of the middle school classrooms during the semester and teach some mathematics subjects for at least two hours. However, although the preservice teachers took the courses to improve their content and pedagogical knowledge by observing hypothetical situations, they only had the courses related to doing practice related to teaching geometry and measurement in this course. The general catalog description of the Teaching Practice 2 course includes observing teaching methods and techniques, doing microteaching practices with individuals and groups of people, developing mathematical activities and materials, having teaching experiences in the actual classroom, and reflecting on their experiences. As part of this course, preservice teachers attended eight hours of

lectures during the semester, with two hours of theory and six hours of practice. Due to the COVID-19 pandemic, the theoretical and practical courses were conducted through the Zoom program when the current study was conducted. As a result, the preservice teachers did not have the opportunity to take courses in classrooms, schools, and universities. That is why they did not conduct face-to-face teaching and establish direct communication with students.

Middle school classes lasted 40 minutes, with ten-minute breaks between classes. There were about 20 students in each class. Aslı, the cooperating teacher with fifteen years of teaching experience, was willing to help the prospective teachers improve the quality of their teaching. She said she was always ready to review the preservice teachers' lesson plans and give them feedback when necessary. During the semester in which the current study was conducted, she had classes with only 5th and 6th graders. At the same time, she used questioning and constructed her lessons with students' thinking in mind. Therefore, her students were used to sharing their ideas comfortably during the lesson. Only preservice teachers enrolled in the F section of the Teaching Experience Course in the same teacher education program attended Aslı Teacher's lesson as observers and did active teaching. This was essential because it meant that the current study participants knew each other and took the same courses and interventions from the same faculty member.

Meanwhile, the researcher taught the theoretical hours of the Teaching Practice 2 course as part of my university education. The preservice teachers and the researcher met twice a week to share and discuss the observations and teaching experiences of the preservice teachers, which they gained in the practical hours of the course. Meetings were usually designed for two consecutive lessons, each lesson hour lasting approximately 40-50 minutes. The researcher has worked as a research assistant in the same department since the beginning of 2017. In addition, she taught several courses, such as Fundamentals of Mathematics II, Mathematical Connections in Teaching, Misconceptions in Mathematics, and Teaching Experience I and II, throughout the preservice teachers' participation in the program. Thus, the researcher

and the participants knew each other and had strong communication with them. In addition, the researcher used the Zoom program in my courses at the beginning of the COVID-19 pandemic. Thus, she was familiar with using the Zoom program and conducting lessons through the program.

3.3 Data Collection Tools

In design-based research, different data collection tools with different research techniques are used to evaluate intervention outcomes and review the intervention process (Design-Based Collective, 2003). In this study, the data collection tools consisted of the pre-posttest of the perimeter concept, individual responses to the written activities used in the intervention, the video recordings of the implementation process, the lesson plans, the post-interview, and the field notes. All of these data helped the researcher to understand the development of the preservice teachers' mathematical knowledge of the perimeter concept and their professional noticing skills before, during, and after participating in the current study's model. The following sections provide information about each of these data collection tools.

3.3.1 Pre-Post Test on the Perimeter Concept

Pre- and post-tests are mainly used to evaluate the development of the preservice teachers' mathematical knowledge and professional noticing skills on the perimeter concept. In addition to this purpose, the preliminary analysis of the pretest results was used while designing the activities used in this study. To this end, the questions in the pretest and post-test were designed to unpack the preservice teachers' mathematical knowledge of the perimeter concept and their professional noticing skills. The pre and post test contained the same questions and consisted of eight open-ended structured questions, some of which had sub-dimensions. In the most general sense, the pre-post-test included three types of questions. The first type of question was designed to examine the preservice teachers' mathematical

understanding of the concept of perimeter and the measurement process. In addition, they were asked to examine their mathematical terminology and language related to the perimeter concept. In preparing such questions, the *overt subject knowledge* and *using terminology* knowledge of the Foundation dimension of the KQ model (Rowland et al., 2005; 2009) were considered. The second type of questions were designed to examine the preservice teachers' representational preferences and how they planned to adapt them to their teaching. In addition, they were asked to explore their knowledge of students' (mis)conceptions of the perimeter concept and the possible sources of them. In preparing these questions, *the theoretical underpinnings of pedagogy* and the *identifying of error* knowledge of the Foundation dimension of the KQ model (Rowland et al., 2005; 2009) were considered. The third type of questions measured the preservice teachers' professional noticing skills regarding the perimeter concept. The specification table of the items in the pretest based on the knowledge types under the Foundation Dimension of the KQ model (Rowland et al., 2005; 2009) and the Professional Noticing Framework (Jacobs et al., 2010) are given in Table 3.2.

Table 3.2 Table of specification of the items in the pre-and post-test based on the used frameworks.

Questions in Pre/Post Test	Evidence of the *Foundation Dimension* of the KQ Framework (Rowland et al., 2005; 2009).	Dimensions of Professional Noticing Framework (Jacobs et al., 2010).
1. Define the concept of the perimeter in detail.	*Evidence of overt subject knowledge* and *Evidence of using terminology*	
2. Describe the perimeter measurement in detail.	*Evidence of overt subject knowledge* and *Evidence of using terminology*	
3. What information do you need to calculate the perimeter of any shape? Please explain.	*Evidence of overt subject knowledge* and *Evidence of using terminology*	
4. How would you describe the perimeter to middle school students? Does your definition change according to the grade level? Please explain.	*Evidence of the theoretical underpinning of pedagogy* and *Evidence of using terminology*	
5. Which representation(s) (Table, model, diagram, picture, verbal problem, real-life situations, concrete materials, etc.) do you prefer to use in teaching the perimeter concept? How? Please explain.	*Evidence of the theoretical underpinning of pedagogy*	
a. Do you prefer to use technology use in teaching the perimeter concept? If your answer is yes, How? Please explain.	*Evidence of the theoretical underpinning of pedagogy*	
6. What could be middle school students' potential difficulties with the perimeter concept? Please explain in detail.	*Evidence of identifying errors*	
a. What could be the source of these difficulties? Please explain.	*Evidence of identifying errors*	
b. What could you do to overcome these difficulties? Please explain.	*Evidence of identifying errors* and *Evidence of the theoretical underpinning of pedagogy*	

Table 3.2 (Continued).

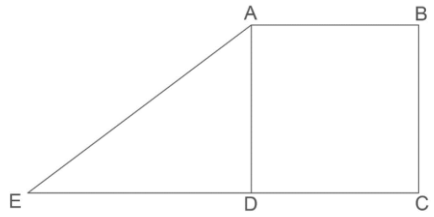
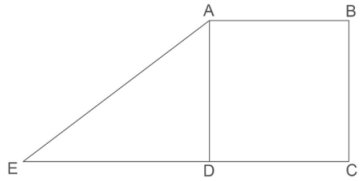
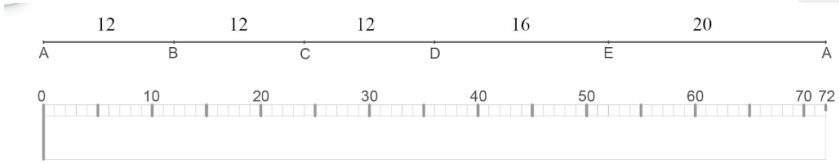
Questions in Pre/Post Test	Evidence of the *Foundation Dimension* of the KQ Framework (Rowland et al., 2005; 2009).	Dimensions of Professional Noticing Framework (Jacobs et al., 2010).
7. ABCD is a square, and ADE is a right triangle. Since the perimeter of triangle ADE is 48 cm, IECI=28 cm, and IABI=12, calculate the perimeter of trapezoid ABCE as much as possible ways. 	*Evidence of overt subject knowledge* and *Evidence of using terminology*	
8. Let us imagine that you are teaching the concept of the perimeter to 5th grades, and you started the lesson by asking the following question: ABCD is a square, and ADE is a right triangle. Since the perimeter of triangle ADE is 48 cm, IECI = 28 cm, and IABI = 12, calculate the perimeter of trapezoid ABCE. 		

Table 3.2 (Continued).

	Questions in Pre/Post Test	Evidence of the *Foundation Dimension* of the KQ Framework (Rowland et al., 2005;2009).	Dimensions of Professional Noticing Framework (Jacobs et al., 2010).
	 While solving this question, your student took the ruler on the board and made some measurements. Meanwhile, the following solution appeared on the board, and she told the class: "I found it in a very different way. My result is 72."		
a.	Explain how the student solved this problem (her strategy, etc.). Do you think her solution is correct? Why?	*Evidence of overt subject knowledge*, *Evidence of using terminology*, and *Evidence of identifying errors*	*Evidence of attending skill*
b.	Based on the student's approach to the question, what can you say about her mathematical understanding of the perimeter concept? Explain in detail.	*Evidence of overt subject knowledge*, *Evidence of use of terminology*, and *Evidence of identifying errors*	*Evidence of interpreting skill*
c.	What kind of path would you follow after the student's solution? What question or questions would you ask her? Why? Figure out how you would continue the lesson in detail.	*Evidence of theoretical underpinnings of pedagogy*	*Evidence of deciding skill*

Two mathematics educators reviewed the questions and a specification table to determine pre- and post-test content validity. In addition, an assessment and evaluation specialist in education also reviewed the format of the questions. As mentioned earlier, since the courses were held on the Zoom program, the pre-and post-tests were administered to the preservice teachers through Google Forms. The prospective teachers and the researcher met during the university course period and the time allotted for the course during the last week. Each application lasted approximately 90 minutes through the Zoom program and was recorded. In addition, the responsiveness of Google Forms was turned off at the end of each application. At the same time, since the same questions were asked in the pretest and post-test, the prospective teachers were not allowed to record the screen, take pictures, etc., during the administration of the pre-and post-test. For this reason, it was assumed that the possibility of prospective teachers remembering the questions and answers was eliminated.

3.3.2 Individual Responses to the Activities

The prospective teachers' responses to the activities were used as a data collection tool because it was essential to understand the role of fruitful discussion in developing their mathematical knowledge and professional noticing skills on the perimeter concept. In other words, to understand the preservice teachers' ways of developing their knowledge and noticing skills, it was focused on both individual and social aspects of learning. These activities were shared with prospective teachers through Google Forms during the current study and followed a process similar to the pre-and post-test applications. The details of these activities are mentioned in the preliminary section.

3.3.3 Post Interview

As Kvale and Brinkman (2009) stated, interviews were essential tools "to understand the world from the subjects' point of view, to unfold the meaning of their experiences, to uncover their lived world before scientific explanations" (p.1). This study used a semi-structured interview to allow for the triangulation of information from multiple sources. In addition, the interview was conducted to gain an in-depth understanding of the development of mathematical knowledge and their professional noticing skills regarding the perimeter concept. Moreover, it was conducted to evaluate the effectiveness of the model nodes and the general process. In other words, the interview questions were based on assessing the prospective teachers' comments on the intervention, whether the nodes of the model contributed to the development of their knowledge and noticing skills separately, and if so, how they developed their knowledge and skills in this process. Sample interview questions are presented in Table 3.3. The whole post-interview questions are presented in Appendix A. The interviews were conducted using the Zoom program and recorded and transcribed by the researcher. The interviews lasted approximately 75 and 120 minutes.

3.3.4 Video Recordings of the Implementation of the Model

The entire implementation of the model, including the classroom discussions, the rehearsal, the enactment, and the analysis of the rehearsal and enactment, was videotaped to observe and examine the development of the prospective teachers' mathematical knowledge and professional noticing skills on the perimeter concept, their tool use, gestures, and drawings. Because the intervention took place in the Zoom program, the recordings were stored in the program's cloud. After completing each intervention node, the researcher transcribed the recordings line by line.

Table 3.3 Question number 6 from the interview and sub-questions.

No	Question
6	In this step, some of the pre-service teachers played the role of 5th-grade students, and one of them played the role of a teacher and rehearsed teaching.
a	In general, what was your experience at this stage?
	During the lesson rehearsal,
b	Did it have an impact on your definition of the concept of perimeter? If so, in what way?
c	Did it have an impact on the mathematical language (use of terminology, etc.) you used for the concept of perimeter? If so, in what way?
d	Did it have an impact on the teaching strategies you used for the concept of perimeter? If so, in what way?
e	Did it have an impact on the identification of students' difficulties with the concept of perimeter? If so, in what way?
f	Did it have an impact on the strategies you used/planned to use to overcome students' difficulties with the concept of perimeter? If so, in what way?
g	Did it have an impact on your identification of mathematical details in students' strategies for the concept of perimeter? If so, in what way?
h	Did it affect your interpretation of students' mathematical understanding of the concept of perimeter? If so, in what way?
i	Did it affect your response to students' mathematical ideas about the concept of perimeter? If so, in what way?
j	Did you encounter any student thinking that you expected and that you responded to or did not respond to during the lesson? Can you give an example?

3.3.5 Planning Materials

Lesson plans (initial, revised, and re-revised) were used as data sources. The primary purposes of the lesson plans were to a) prepare the preservice teachers to examine the problems mathematically, b) anticipate possible correct and incorrect student strategies, and c) generate possible responses to support student understanding. The initial and revised lesson plans were designed to allow preservice teachers to revise their teaching in light of their experiences in the current study. The lesson plans allowed the researcher to obtain more information about the preservice teachers' mathematical knowledge and professional skills related to the perimeter concept.

The lesson plan format was adapted from a four-column lesson plan that Matthews et al. (2009) developed to help prospective teachers deconstruct instructional practices and approximate their decisions (Grossman et al., 20009a). The lesson plan format is provided in Table 3.4.

Table 3.4 The lesson plan format.

The questions and the dialogues regarding questions (Including teachers' and students' actions)	The Possible Reasons for The Student (s)'s Strategy	Important Points
…	…	…
…	…	…
…	…	…
…	…	…

3.3.6 Field Notes

According to Frankel and Wallen (2005), field notes are "researchers' written account of what they hear, see, experience, and think in the course of collecting and reflecting on their data" (p.516). Thus, it was essential to benefit from the researcher's notes to examine the developmental trajectories of preservice teachers' mathematical knowledge and noticing skills related to the perimeter concept.

I participated in the current study process as an instructor and researcher. As the instructor, I introduced the tasks used in each intervention node and facilitated classroom discussions. As a researcher, I collected data from the preservice teachers who participated in the current study. I was involved throughout the research process. I always took field notes related to each node of the model and reviewed them after each node was completed. I took these notes using a piece of paper with two columns. The first column contained descriptive information, such as the participants' names, implementation dates, etc. The second column contained my reflective notes, such as my ideas and observations about the study participants' development or my inferences about their thoughts while explaining their ideas.

3.4 Data Collection Procedures

As mentioned earlier, the design-based research study consisted of three phases: preliminary research, prototyping, and assessment. The following sections provide information on the procedures followed in this study's preliminary and prototyping phases. The details of the procedures followed in these sections are summarized in Figure 3.2.

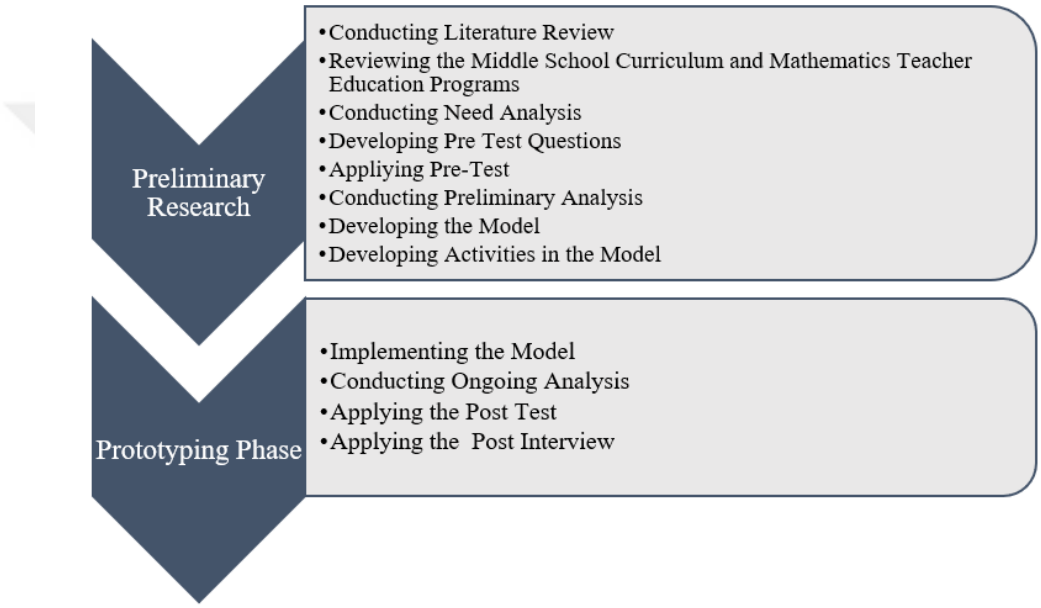


Figure 3.2. The procedures followed in the first two phases of the current design-based research.

3.4.1 Preliminary Research

As mentioned before, it is essential to identify the problem, review the literature, and conduct a needs analysis in this stage. In this regard, firstly, the problem was identified: preservice teachers and even teachers have difficulties responding to students' mathematical ideas regarding the concept of perimeter (e.g., Girit-Yıldız et al., 2022). The literature review showed that teachers' knowledge and professional

noticing skills were essential to their responsiveness (e.g., Jacobs & Empson, 2016). With the aim of the current study, the literature was first reviewed by using the following keywords: "teacher knowledge," "professional noticing skills of the perimeter concept," "preservice teachers' knowledge of the perimeter concept," "teachers' knowledge of the perimeter concept," "professional noticing of the perimeter concept difficulties related to the perimeter concept."

Furthermore, research studies were examined to unpack the learners' (including students and preservice teachers) difficulties with the perimeter concept. Given the purpose of the current research, the analysis of the literature review was grouped under four headings: a) lack of understanding of the perimeter concept (e.g., Bray et al., 2006; Carpenter et al., 1975; Kouba et al., 1988; Reinke, 1997), b) confuse the concepts of perimeter and area (e.g., Carpenter et al., 1975; Cavanagh, 2007; Clement & Stephan, 2004; D'Amore & Fandino Pinilla, 2006; Danielson, 2005; Hart, 1884; Hiebert, 1981; Kouba et al., 1988; Reinke, 1997; Steele, 2006), c) confusion about units of measurement (e.g., Chappell & Thompson, 1999; Hiebert, 1981; Lappan et al., 1998; Martin, 2009; Martin & Strutchens, 2000; Moyer, 2001; Steele, 2006), and d) difficulty in determining the relationship between perimeter and area (e.g., D'Amore & Fandino Pinilla, 2006; Ferrer et al., 2001; Ma, 1999; Reinke, 1997).

In addition, the mathematics curriculum of Turkish Middle Schools (grades 5-8) (Ministry of National Education [MoNE], 2018) was examined to get an idea about perimeter teaching in middle schools and to determine the most relevant, essential, valuable, and critical learning goals related to the perimeter concept (Gravemeijer & Cobb, 2006). As a result of the investigation, it was found that there were four objectives directly related to the perimeter concept in the middle school mathematics curriculum (MoNE, 2018), and two of them belonged to the fifth grade. These objectives are presented in Table 3.5.

Table 3.5 Objectives related to the perimeter concept in the middle school curriculum.

Code of the objectives	Objectives
M.5.2.3.1.	-Students should be able to recognize units of length measurement, convert meter-kilometer, meter-decimeter-centimeter-millimeter units into each other, and solve related problems.
M.5.2.3.2.	-Students should be able to calculate the perimeter of triangles and quadrilaterals and construct different shapes with a given perimeter. a) Students should be able to solve problems related to estimating the perimeters of the shapes.
M.7.3.2.5.	-Students should be able to solve the problems related to area measurement. b) Students should be able to associate the perimeter of a rectangle with its area. Students should be able to examine the perimeters of different rectangles with the same area and the areas of different rectangles with the same perimeter.
M.7.3.3.2.	-Students should be able to calculate the length of the circle and the circle segment.

The objectives in the curriculum focused primarily on constructing the concept's meaning by calculating the perimeter of the shapes that form the quadrilaterals in fifth grade, the circumference of the circle, and the length of the circle segment in seventh grade. In addition, the NCTM (2000) Standards for Measurement in Instructional Programs were reviewed to gain insight into the program's expectations for students. Based on these standards, a) knowing attributes of the perimeter, b) knowing types of units based on each attribute, c) knowing standard and nonstandard units, d) knowing estimation, e) calculating the perimeter of irregular shapes, d) using the formula of the perimeter, and e) knowing why students used such formula are essential concerns to make sense of the perimeter concept.

After examining the literature and middle school mathematics curriculum, an investigation of the contents of the courses in the EME programs was initiated. It was found that most universities in Türkiye, including the university where this study was conducted, follow the four-year elementary mathematics education program published by the Higher Education Council in 2018, which aims to equip teachers with skills and knowledge related to mathematics education. Graduates of the

program have a diploma as elementary mathematics teachers who work in schools from grades 5 to 8 (ages 10 to 14) in Türkiye. The courses offered by the Council of Higher Education in 2018 are presented in Appendix B.

In the most general sense, to graduate from this program, preservice teachers need to take (a) content courses (e.g., mathematics courses, algorithms, and programming, probability, statistics, English, Turkish, and history) and (b) educational science courses (e.g., classroom management, instructional technologies, educational psychology, educational philosophy, guidance in schools, and special education and inclusion) mainly in the first two years of their education, and (c) mathematics education courses related to teaching (e.g., approaches to learning and teaching mathematics), learning domains (e.g., teaching number, teaching algebra), and skills (e.g., problem-solving in mathematics), and (d) elective courses mainly in the third and fourth years of university education.

In terms of scope and scope of teaching, the program included the Foundations of Mathematics II course in the second semester and the Geometry and Measurement Teaching course in the fifth semester. In addition, this program includes a single course that provides pre-service teachers with teaching experience in perimeter concepts in the eighth semester. Table 3.8 provides brief information about the credits, contact and laboratory hours, ECTS, and catalog descriptions of the mentioned courses according to the Higher Education Council 2018.

Table 3.6 shows limited courses in the teacher education program equip preservice teachers with mathematical knowledge about the perimeter. In addition, even if they had taken the courses to improve their mathematical knowledge of the perimeter concept, they only had the courses related to doing practices in the field in the last semester, which could not nurture their professional noticing skills and responsiveness. Considering these issues, the literature review was extended considering teacher education frameworks. Some models mentioned in the literature review section were found in this regard. At the same time, a design team including the researchers (the Ph.D. student, the advisor, and the jury members) was formed to

identify the most relevant, essential, and valuable teaching and learning objectives aimed at developing preservice teachers' mathematical knowledge and professional noticing skills on the perimeter concept.

Table 3.6 Information about the courses that involved perimeter teaching.

Course	Catalog Description of the Course
Foundations of Mathematics II (Credit 2, Contact 2, Lab 0, ECTS, 4)	The basic information concepts in mathematics related to the learning areas of geometry, statistics, and probability (basic geometric concepts and drawings, triangles and quadrilaterals, triangles, length and time measurement, area measurement, geometric objects, angles, lines and angles, circle, circle and circle, fluid measurement, transformation geometry, polygons, the relationship of these concepts to each other, the discussion of mathematical concepts and their transformation into each other with multiple representations, and the purpose, principle, and middle school mathematics and teaching of these subjects.
Geometry and Measurement Teaching (Credit 3, Contact 3, Lab 0, ECTS, 4)	Van Hiele's levels of thinking; basic geometric concepts, geometric structures, geometric objects; congruence and similarity; transformation geometry, projection, patterns and ornaments, fractals; Pythagorean theorem; nature of measurement, time, length, area, volume, and angle measurement (organizing the course content - using appropriate teaching materials and strategies, etc.); student knowledge of these topics; the relationship of these topics to everyday life and other lessons.
Teaching Practice 2 (Credit 5, Contact 2, Lab 6, ECTS, 12)	Making observations about discipline-specific teaching methods and techniques; micro-teaching using discipline-specific techniques; ability to plan a lesson independently; developing course-related activities and materials; preparing classroom environments; classroom management; measurement.

Furthermore, the needs analysis was conducted with an in-service middle school mathematics teacher with nine years of teaching experience and a senior grade preservice mathematics teacher. In this context, they were mainly asked whether they needed help responding to students' mathematical thinking in general and concerning the concept of perimeter during the lesson. If they mentioned difficulties, they were

asked to identify possible reasons. Both indicated they had difficulties responding to students' mathematical thinking and pointed out the deficiencies in teacher education programs. For example, an in-service middle school mathematics teacher with nine years of teaching experience stated that such difficulties stem from a lack of mathematical knowledge. She stated the following:

> Although I have many students and have encountered many thoughts, I still need help responding to students' mathematical thoughts. The main reason is that I lack mathematical knowledge about the concepts, including perimeter. Mathematics teaching was given little importance in my university education. Educators focused more on pedagogy and how to teach mathematics. I need well-developed content knowledge of mathematical concepts. As a result, I need to help understand what students are saying or their strategies, which affects my responses to students.

Meanwhile, the preservice teacher emphasized that the difficulty could be due to the lack of teaching experience in this area during her undergraduate education. She explained her reasoning as follows:

> Our undergraduate courses are generally theoretical. There are few opportunities during undergraduate education to work with students face-to-face or in the classroom. Therefore, in my first-semester practicum, I needed help responding to students' concerns about mathematical thinking, whether I should manage my excitement, focus on what the student was saying, or keep the class under control. We should have more classroom experience, in my opinion.

By bringing all this evidence together, the researcher's own teaching experiences, and the opinions of the mathematics educators, the questions in the pretest were developed and applied at the beginning of the model. Detailed information about the questions in the pre-and post-test is presented in the 3.4.2 heading. Moreover, to develop a model and direct activities in the model, the preliminary analysis of the

pretest was conducted. In this regard, the following section outlines the preliminary findings of the pretest results in general terms.

3.4.1.1 Preliminary Analysis

The initial analysis of the pretest in the current study revealed that the participants of the current study lacked mathematical knowledge of the perimeter concept. They generally followed the formula and based their definition on operations rather than the concept's meaning. They tended to misuse mathematical language and terminology regarding the concept. They did not clearly clarify students' potential difficulties with the perimeter concept and tended to follow unclear teaching strategies without supporting manipulatives in teaching the concept. At the same time, they had difficulty in attending, interpreting, and responding to the student's mathematical understanding of the concept. Therefore, in addition to the literature review, need analysis, etc., the initial analysis of the pretest results gave the researcher an idea of how to design a model and activities in the current study, which aimed to support preservice teachers' mathematical knowledge and professional noticing skills on the perimeter concept. In the following sections, the details of the structure of the model and activities used in the current study are mentioned, respectively.

3.4.1.2 Developing the Structure of the Current Study's Model

This study aimed to support the improvement of prospective middle school mathematics teachers' mathematical knowledge and professional noticing skills on the perimeter concept through the development of the model based on the cycle of enactment and investigation. In structuring the model of the current study, the focus was on the competencies that prospective teachers should have to elicit and respond to students' mathematical understandings. In this regard, several assumptions were considered. Furthermore, considering the characteristics and limitations of the

studies mentioned in sections 2.3.1, 2.3.2, and 2.3.3, the needs analysis, the preliminary results, etc., the general structures of the model of the current study were outlined. The following paragraphs provide detailed information on the assumptions of the study and the details of the model.

First, it was assumed that as prospective teachers' content knowledge develops, they could respond to students' mathematical understanding from a conceptual aspect. In other words, preservice teachers need to grasp the mathematical meaning of concepts to identify the details of students' strategies and focus on the processes rather than the outcome. Otherwise, they might miss or ignore students' high-level and unusual ways of understanding due to a lack of knowledge. Thus, it was considered that it is essential to support their content knowledge at the beginning of the model. While constructing or reconstructing the content knowledge, the role of collective experiences was also considered.

Second, it was assumed that preservice teachers could elicit and respond to students' mathematical thinking if they had an idea about students' (mis)conceptions with relevant mathematical concepts. In other words, it was assumed that if they had ideas about students' thinking about the relevant concepts, they might have a chance to extend students' mathematical understanding. Therefore, it was considered that it is essential to enable prospective teachers to be more aware of students' (mis)conceptions about concepts before actual teaching took place, with the aim of responding to students' mathematical understanding during teaching.

Third, it was assumed that diversifying instructional strategy and instructional decisions requires a strong knowledge of students and mathematical knowledge of relevant concepts. More specifically, when prospective teachers' mathematical understanding of the relevant concepts develops, and they have ideas about students' (mis) conceptions, they could feel more comfortable designing student-centered instruction. Moreover, if they have a chance to demonstrate actions through authentic and reflective experiences, they could elicit and respond to students' mathematical

understanding. Therefore, it was considered that it is essential to enable prospective teachers to demonstrate instructional decisions and strategies in authentic settings.

Fourth, it was assumed that when prospective teachers' professional noticing skills develop, they could elicit and respond to students' mathematical understanding. Therefore, it is essential to support the development of their professional noticing skills to increase these assumptions. Overall, considering these assumptions and Mason's (2011) perspective on noticing, including the pre-prepare and post-prepare stages, the model of the current study was structured with four nodes: *Pre-Preparation*, *Preparation*, *Implementation*, and *Post-Preparation* for developing their mathematical knowledge and professional noticing skills. Figure 3.3 describes the general structure and the nodes of the model.

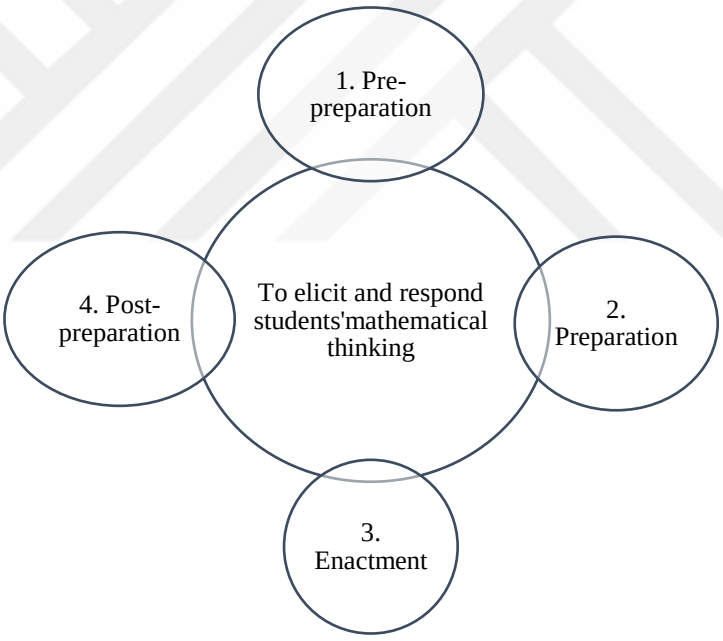


Figure 3.3. The general structure of the current study's model.

3.4.1.3 Preparing Activities of the Current Study

When preparing the activities, it was considered that they should include "a presumed learning process that anticipates how students think and understand" (Gravemeijer & Cobb, 2006, p. 48). Moreover, the main aim of the current study,

which was to support the development of mathematical knowledge and professional noticing skills on the perimeter concept simultaneously but separately, was also considered. In addition, the purpose of each node of the current study's model, the objectives in the curricula, the literature review, and the needs analysis results were considered. In this regard, the goals of the first node, called *Pre-Preparation*, of the model were to enable preservice teachers to improve their knowledge of the concept of perimeter and to become familiar with how to notice students' mathematical thinking about the concept professionally before rehearsal perimeter teaching since the preliminary analysis of the pretest result showed that the prospective teachers had lack of mathematical knowledge of the perimeter concept. Therefore, it was decided to examine in-depth and support the development of their mathematical knowledge about the concept in the context of Activity 1: *What is the Perimeter? Let Us Look at the Concept through Your and the Student's Eyes.* In this regard, the preservice teachers were asked to draw shapes to represent and not to represent the perimeter as the first and second questions of the first activity. Since one of the goals of this node was to support preservice teachers' mathematical knowledge of the perimeter concept, such questions were appropriate to discuss the concept's meaning and provide opportunities for preservice teachers to reconstruct their existing knowledge of the concept collectively. In addition, it aimed to provide preservice teachers with opportunities to justify their reasoning and develop mathematical language regarding the perimeter concept. The first and second questions proposed in the first activity are presented in Table 3.7.

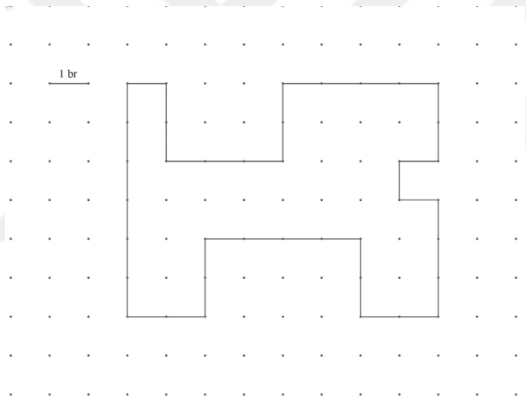
Table 3.7 First two questions from the first activity.

Activity 1: *What is the Perimeter? Let us Look at the Concept through your and the Student's Eyes.*
1. Draw shapes that represent the perimeter concept.
2. Draw shapes that do not represent the perimeter concept.

In addition to these questions, the preservice teachers were asked to calculate the perimeter of the given shape as much as possible since solving a question with multiple solutions is a robust indicator of content knowledge (Rowland et al., 2005).

In this regard, the third question was included in this activity. Moreover, they were expected to think about potential student difficulties with the problem and how they planned to overcome them to develop their knowledge of students and teaching strategies for the perimeter concept since the preliminary analysis of the pretest result showed that they had difficulty in identifying students' misconceptions on the perimeter. State it differently, to familiarize them with students' difficulties with the concept and to enable them to think about possible sources of such difficulties and how to deal with them before perimeter instruction occurred. Fourth and fifth questions were added to the first activity. The remaining questions in the same activity mentioned above are presented in Table 3.8.

Table 3.8 Third, fourth, and fifth questions from Activity 1.


3. Calculate the perimeter of the given shape as much as possible. Please indicate your solutions ways. Then, share your solution ways with your friends.
4. What mistakes do students make when solving the above question? What are the possible sources of them? Please think individually and then share your thoughts with your friends.
5. How do you handle such mistakes or prevent them from occurring such mistakes during the instruction? Please think individually and then share your thoughts with your friends.

In addition to the first activity, Activity 2: *Let us Examine the Dialogue* used in the same node of the model was designed mainly to improve the preservice teachers' professional noticing skills on the perimeter concept. The dialogue in the activity was formulated in light of the learners' conceptions of the perimeter concept mentioned in the literature, the preliminary analysis of the pretest results, and the opinions of mathematics researchers. The prompts related to the given dialogue were

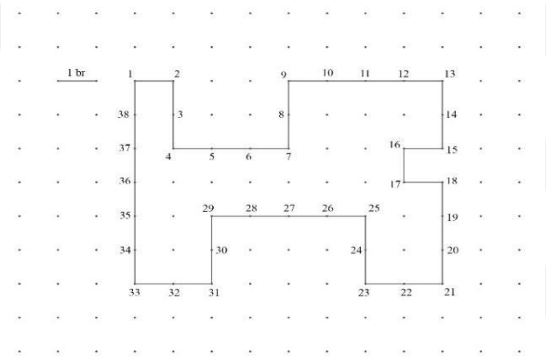
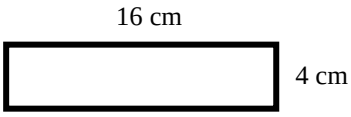
structured considering the study of Jacobs et al. (2010). Meanwhile, the dialogue was formed, so the fifth-grade student solved the third question of the first activity, as mentioned in Table 3.8.

Within this activity, the prospective teachers were expected to examine the given written dialogue between the teacher and the fifth-grade student. The written dialogue was intentionally chosen as the data collection method for the following reasons. First, research has shown that prospective teachers maintain discipline in the classroom rather than focusing on students' thoughts or mathematically significant events to prove their presence (van Es & Sherin, 2008). As a result, it was expected that preservice teachers would have more difficulty noticing student thinking in the classroom environment and the videotaped lessons than experienced teachers (Amador & Weiland, 2015). In addition, the written dialogue was used to represent the practice component of the pedagogy of the practice framework (Grossman et al., 2009a). Table 3.9 provides information about the details of the activity.

As shown in Table 3.9, in the given dialog, the student calculated the perimeter of the irregular shape by counting the points around the shape instead of units. When the student noticed this situation, the teacher asked the student to explain her solution again. Then, the teacher directed the student to another question to check the student's solution in calculating the perimeter of a rectangle whose perimeter could be calculated using the formula. After the student explained her solution, the dialogue ended. Afterward, the prospective teachers were asked how to continue the lesson to further the student's thinking about the concept of perimeter, considering her understanding of the concept.

The professional noticing framework was used to decompose the dialogue into pieces and approximate students' responses as part of the pedagogies of the practice framework (Grossman et al., 2009a).

Table 3.9 The second activity used in the study's first node.

Activity 2: *Let us Examine the Dialogue*	
Representation of Practice for the Perimeter. *Student:* Teacher, the answer is 38. *Teacher:* How did you calculate Aleyna? Can you explain? *Student:* Teacher, I chose a reference point for myself, so I started *from one point and reached that point while passing 38 points.* *Teacher:* Can you show your solution on the board? *Student:* Of course, my teacher. I counted the points to calculate the shape's perimeter and got 38.  *Teacher:* How do we calculate the perimeter of the shape, Aleyna? *Student:* We count the dots around the shape. Specifically, we select a starting point around the shape and count the points until we reach the same point, teacher. *Teacher:* So, do we count the points when calculating the perimeter, Aleyna? *Student*: Yes, teacher. *Teacher:* How would you calculate if I did not place the dotted paper under the shape and asked you this question?  *Student:* (16 + 4) x 2 = 40 cm teacher. *Teacher:* How did you calculate? *Student:* This shape is rectangular. To calculate the perimeter of the rectangle, I use the formula (a + b) x 2.	Prompts regarding the Dialogues. a. What do you think about the student's first strategy? Do you think it is true? b. What do you think about the student's second strategy? Do you think it is true? c. What do you think about the students' mathematical understanding of the perimeter concepts? D. Why do you think the teacher asked the student the second question? e. If you were the student's teacher, how would you continue the lesson? Which questions could you ask the student to advance her learning of the concept?

More specifically, the prospective teachers examined the dialogue between the teacher and student as a part of the representation of the practice component of pedagogies of practice framework (Grossman et al., 2009a). Then, they identified the mathematical details of the student's strategies in the given dialogue, and they made inferences based on her understanding of the decomposition of the practice component of the pedagogies of the practice framework (Grossman et al., 2009a). In other words, they used the professional noticing framework's first two components (attending and interpreting) (Jacobs et al., 2010) to decompose the teaching practice in the given dialogue. As part of the activity, the prospective teachers were also asked to show their intended decisions in responding to this student's thinking. Consequently, the approximation of practice (Grossman et al., 2009a) was used to show their decisions. Overall, the purpose of this node was to support the development of the preservice teachers' mathematical knowledge of the perimeter concept and professional noticing skills before responding to students' thinking during the rehearsal teaching. This node also included examining these two activities separately and then discussing them together, similar to the work of Arbaugh et al. (2019), to support individual progress in the community.

The second node of the model, *Preparation*, was primarily designed to provide preservice teachers with instructional and reflective opportunities before actual teaching and make them more familiar with the details of their strategies. The preliminary analysis of the pretest result showed that the preservice teachers had difficulty in attending, interpreting, and deciding how to continue the lesson considering students' mathematical understanding and did not have well-developed teaching strategies to support students' mathematical understanding of the perimeter concept. At the same time, the ongoing analysis of the first node of the model showed that their mathematical knowledge and professional noticing skills were developing. However, they needed more support to strengthen this development. Consequently, this node included four activities: Activity 1: *Prepare for the Rehearsal*, Activity 2: *Rehearse the Perimeter Teaching*!, Activity 3: *Let us Examine the Rehearsal*, and Activity 4: *Prepare for Enactment*. Now, the details of these activities are mentioned.

First, the preservice teachers were asked to prepare a lesson plan by following the steps in the lesson plan format. This activity was called *Prepare for the Rehearsal.* Indeed, it was designed to enable preservice teachers to plan to diversify their teaching strategies, use materials to teach the concept, produce various solution strategies regarding the perimeter concept from students' perspectives, reason possible students' mathematical understanding considering such strategies, and decide how to respond them as a teacher of these students before conducting the lesson.

The second activity in this node was called *Rehearse the Perimeter Teaching*! As part of this activity, one of the preservice teachers (Maya) volunteered to conduct a perimeter lesson in a simulated classroom environment where the other preservice teachers acted and asked questions like students, similar to the work of Lampert et al. (2013) and McDonald et al. (2013). This activity was included in this node of the model to provide preservice teachers with the experience of attending to the details of possible student strategies and responding to such student thinking in the moment of simulated teaching as part of the approximation of practice (Grossman et al., 20009a). Meanwhile, the goal was to enable preservice teachers to implement instructional strategies, use materials to teach the concept, attend to students' strategies, and respond to such strategies at the moment in simulated classroom settings. Similar to the second quadrant of the model proposed by McDonald et al. (2013), this node of the model dealt with the planning and rehearsing stages together. Since the lesson planning stage included breaking down practices into pieces, such as producing possible students' strategies, this stage was grounded on the decomposition of the practice component of the pedagogies of practice framework (Grossman et al., 2009a). Besides, producing possible responses to fictitious students' strategies in the lesson plan and the rehearsal were designed to approximate the practice (Grossman et al., 2009a).

The third activity in this node, called *Let us Examine the Rehearsal,* included the preservice teachers collectively analyzing the video recordings of the practice by

focusing on the student's mathematical thinking and the teacher's (Maya) actions. While preparing this activity, it was considered to provide opportunities for preservice teachers to review the teacher's actions, who acted as the teacher role in the rehearsal, intending to identify the details of students' strategies in more detail. Moreover, it was aimed to improve the quality of their actions and respond to students' mathematical understanding from a conceptual aspect as part of the pedagogy of reflection (Grossman et al., 2009b). Furthermore, the analysis of the rehearsal activity distinguished the current study's model from the other mentioned models.

The name of the fourth and last activity in the second node of the model was *Prepare for Enactment.* The activity was included in this node to allow preservice teachers to review the dialogues or activities in the lesson plan, students' solution strategies, and their responses to such strategies in light of their experiences in the rehearsal and the analysis of the rehearsal stages and, if necessary, in light of the decomposition and approximation of the practice components of the pedagogies of practice framework (Grossman et al., 20009a). Furthermore, the replanning lesson plans activity distinguished the current study's model from the other mentioned models.

The third node of the model, called *Implementation*, involved the conducting perimeter teaching in the actual classroom by the preservice teacher who conducted the rehearsal. The goal of this node was to enable the preservice teachers to make instructional decisions in the classroom environment and implement perimeter teaching in the actual setting. Thus, Activity 1: *Enact the Perimeter Teaching*! was designed to enable preservice teachers to identify students' difficulties with the perimeter concept during instruction, respond to students' thinking in the actual classroom environment, and use conceptual teaching strategies and materials. This activity was designed in light of enactment (Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013).

The fourth node of the model, called *post-Preparation*, was designed to enable preservice teachers to examine, reflect, and revise their teaching practices. It

included Activity 1: *Let us Examine the Enactment* and Activity 2: *Prepare for Further Teaching*. First, the preservice teachers were asked to examine the classroom enactment's video recordings collectively. This activity was called *Let us Examine the Enactment*. The purpose of this activity was to allow the preservice teachers to identify the details of the students' strategies when the teacher was absent during the lesson, to review the teacher's actions to improve the quality of their instructional decisions and to improve their instructional strategies and materials as part of the pedagogy of investigation and reflection (Grossman et al., 2009b). The second and final activity in this node was to revise the lesson plan that the preservice teachers prepared in the second node of the model, considering the experiences they had gained and whether the revisions were necessary. The activity was called *Prepare for Further Teaching* and was included in this node to enable preservice teachers to diversify students' solution strategies and responses to such strategies. At the same time, it was included to enable preservice teachers to diversify their teaching strategies and use materials to teach the concept as part of the practice decomposition (Grossman et al., 20009a).

In conclusion, the figure below outlines the current study's model, The Cycle of Enactment and Investigation for Developing Knowledge and Professional Noticing Skills, and the activities within each node. Then, Table 3.10 summarizes the purposes and pedagogies used in the activities in each node of the model. Then, the following section provides information about the implementation of the model process.

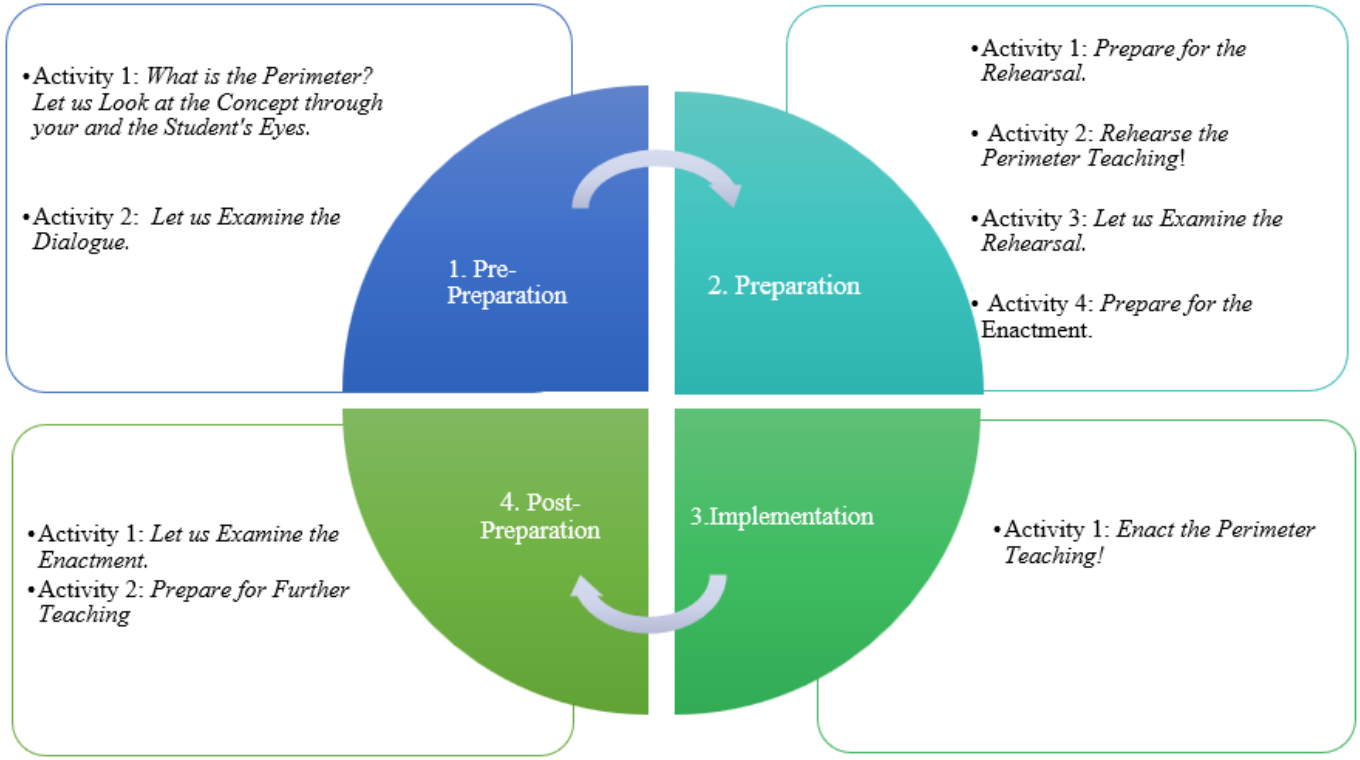


Figure 3.4. The current study's model is called the Cycle of Enactment and Investigation for Developing Knowledge and Professional Noticing Skills.

Table 3.10 Summary of the purposes and pedagogies used in the activities in each node of the model.

Nodes or parts of the model	Activities and their names	Purposes	Pedagogies
The first node of the model, called *Pre-Preparation*	Activity 1: *What Is the Perimeter? Let Us Look at The Concept Through Your and The Student's Eyes*	-To enable preservice teachers to examine and co-construct the mathematical meaning of the perimeter concept. -To enable preservice teachers to develop their knowledge of students' difficulties with the perimeter concept. -To enable preservice teachers to develop their knowledge of teaching strategies and material used on the perimeter concept.	
	Activity 2: *Let Us Examine the Dialogue*	-To enable preservice teachers to identify the mathematical details of the student's strategies. -To enable preservice teachers to interpret the student's mathematical understanding of the concept. -To enable students to decide how to respond to the student's understanding of the perimeter.	-Representation of the practice (Grossman et al., 2009a). -Decomposition of the practice (Grossman et al., 20009a). -Approximation of the practice (Grossman et al., 20009a).

Table 3.10 (Continued).

Nodes or parts of the model	Activities and their names	Purposes	Pedagogies
The second node of the model called the *Preparation*	Activity 1: *Prepare for the Rehearsal*	-To enable prospective teachers to produce possible solution strategies for problems in the lesson plan. -To enable prospective teachers to reason about the details of the possible students' strategies. -To enable prospective teachers to make inferences on the student's mathematical understanding of the perimeter concept. -To enable prospective teachers to respond to such students' thinking in a simulated classroom environment. -To enable prospective teachers to reason with students' possible mistakes and reason how they could overcome them. -To enable prospective teachers to decide on teaching strategies and materials for the perimeter concept.	-Decomposition of the practice (Grossman et al., 20009a). -Approximation of the practice (Grossman et al., 20009a).
	Activity 2: *Rehearse the Perimeter Teaching!*	-To enable prospective teachers to produce possible student strategies for the given problems. -To enable prospective teachers to respond to students' thinking in the simulated classroom environment. -To enable prospective teachers to implement conceptual teaching strategies and to use materials to teach the concept.	-Approximation of the practice (Grossman et al., 20009a).
	Activity 3: *Let us Examine the Rehearsal*	-To enable prospective teachers to review the preservice teacher's actions who acted as the teacher to improve the quality of her decisions. -To enable prospective teachers to identify the details of students' strategies. -To enable prospective teachers to improve teaching strategies and materials to teach the concept.	-Pedagogies of investigation and reflection (Grossman et al., 2009b).

Table 3.10 (Continued).

Nodes or parts of the model	Activities and their names	Purposes	Pedagogies
The third node of the model, called the *Implementation*	Activity 4: *Prepare for the Enactment*	-To enable prospective teachers to diversify students' solution strategies and their responses to such strategies. -To enable prospective to diversify their teaching strategies and use materials to teach the concept.	-Decomposition of the practice (Grossman et al., 20009a). -Approximation of the practice (Grossman et al., 20009a).
	Activity 1: *Enact the Perimeter Teaching!*	-To enable prospective teachers to identify students' strategies with the perimeter concept during teaching. -To enable prospective teachers to respond to students' mathematical thinking. -To enable prospective teachers to use conceptual teaching strategies and materials to teach the perimeter concept in the classroom.	-Enactment (Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013).
The fourth node of the model, called *post-preparation*	Activity 1: *Let us Examine the Enactment*	-To enable prospective teachers to review the preservice teacher actions who acted as the teacher to improve the quality of her actions. -To enable prospective teachers to identify the details of students' strategies. -To enable preservice teachers to improve teaching strategies and materials to teach the concept.	-Pedagogies of investigation and reflection (Grossman et al., 2009b).
	Activity 2: *Prepare for Further Teaching*	-To enable prospective teachers to diversify students' solution strategies and their responses to such strategies. -To enable prospective teachers to diversify their teaching strategies and use materials to teach the concept.	-Decomposition of the practice (Grossman et al., 20009a). -Approximation of the practice (Grossman et al., 20009a).

3.4.2 Prototyping Phase

As mentioned earlier, data was collected from the senior preservice elementary mathematics teachers enrolled in the Teaching Practice 2 course, considering the practical application of the study in one of the state universities in Türkiye. The data was collected in the spring semester of the 2020-2021 academic year. The instructor and the preservice teachers met every Monday from 13:30 to 15:30 for two hours weekly as a requirement of the theoretical classes. Meanwhile, as a requirement of the mentioned course, the preservice teachers were allowed to visit one of the middle schools during the spring semester of the academic year. They observed this school's mathematics classes for six hours every Tuesday and Thursday during the semester. At the same time, they were expected to teach at least two lessons in such schools as part of this course. Both theoretical and practical lessons were held on Zoom due to the COVID-19 pandemic during the academic year.

Before the implementation of the model, the study participants had information about the simulated classroom environment and, consequently, the rehearsal, which was one of the applications of the approximation of the practice component proposed by Grossman et al. (2009a) since the researcher used the same method in her course during the "Connections in Mathematics Teaching" in the spring semester of 2019-2020. The participants of the study were in their third year of undergraduate education. Therefore, no information was given about the simulated classroom environment. In addition, the prospective teachers were reminded that the time they spent and the activities they completed would not be reflected in their scores. During the intervention, data were collected through the prospective teachers' responses to the activities, the videotaped classroom discussions, rehearsal, enactment, and interview, and the instructor's field notes.

During the sixth week of the semester, the pretest was administered to the preservice teachers via Google Forms in the theoretical classroom sessions of the course as a

part of preliminary design-based research. In the context of the course requirement, the prospective teachers and the researcher came together through the Zoom program. Then, the researcher shared the form, and they were expected to respond individually to the questions in the pretest. This process took about 90 minutes. The detailed information was shared in the 3.3.1 heading. After that, the researcher immediately completed the initial analysis of the activities to give the final forms of the activities used during the study. Due to the midterm week, there were no university classes in the seventh week of the semester. Therefore, the intervention implementation could not be continued that week.

The implementation of the model started in the eighth week of the semester. Firstly, in Activity 1, *What is the Perimeter? Let Us Look at the Concept through your and the Student's Eyes* was given to the preservice teachers, who were expected to read and complete the questions in this activity individually for 25 minutes. The lesson was held on the Zoom program and recorded through the program. Later, they were asked to read and complete the questions in the second activity, called *Let us Examine the Dialogue,* individually for 25 minutes. The activities were shared with preservice teachers through Google Forms due to conducting the distance education. If they had difficulty drawing shapes or explaining their ideas, they were asked to use blank paper, take photographs, and upload them into Google Forms as a picture in the space reserved for the question. Afterward, the researcher, the lesson instructor, asked the preservice teachers to explain and discuss their reasoning about the questions in these activities. These explanations and discussions lasted 50 minutes. One focus of such discussion was to co-construct the knowledge of the perimeter concept using mathematically correct language and terminology. Besides, it was seen that they reasoned with potential students' (mis)conceptions of the perimeter concept and potential strategies to overcome them from a procedural and conceptual aspect.

Moreover, during this discussion, the preservice teachers learned how to identify the students' strategies, make inferences based on their understanding, and respond to

mathematical thinking. At the end of the lessons, the instructor asked the preservice teachers whether they volunteered to rehearse and enact the perimeter teaching in a simulated and actual classroom setting. Only Maya volunteered for these tasks for the following intervention nodes. Later, Maya was asked to prepare a lesson plan as homework within the context of Activity 1, called *Prepare for the Rehearsal*, in the second node of the model, Preparation. This activity focused on creating dialogues between fictitious students and the teacher regarding the concept of perimeter. In this way, the first node, called *Pre-Preparation*, was implemented. As soon as the implementation regarding the first node of the model ended, the researcher started to examine the preservice teachers' responses and the video recordings of the discussions to get an idea about their developmental processes.

Before implementing the second node, Maya uploaded the lesson plan to a Google Drive document as Lesson Plan Version 1. Then, the implementation of the second node started with the review of Maya's lesson plan for the rehearsal. This process took 30 minutes in the eighth week of the semester as a part of an additional meeting, or it could be said that this activity was conducted in the first lesson hour of implementing the second node. The focus was on reviewing the lesson plan together, considering the appropriateness of the questions and Maya's actions as a teacher. Then, Maya was given two days (weekends) to revise her lesson plan, taking into account the feedback from her friends.

Maya then uploaded the lesson plan as Lesson Plan Version 2 to the Google Drive document. After that, the second activity in this node, called *Rehearse the Perimeter Teaching*, was implemented in the second and third lesson hours of the node's implementation during the ninth week of the semester. During the rehearsal teaching, Maya acted as a teacher; the others behaved and asked questions like middle school students, intending to simulate the fifth-grade classroom environment. This way, Maya could conduct perimeter teaching and make instructional decisions in the simulated classroom environment. The rehearsal was also videotaped and lasted 58 minutes. Afterward, the researcher examined the first and second versions of the

lesson plans and the video recordings of the rehearsal to determine critical points about Maya's developmental process. The pre-service teachers then moved on to the third activity, *Let Us Examine the Rehearsal*, and watched the rehearsal recordings in the fourth and fifth lesson hours of implementing the node during the tenth week's first lesson. During the recording, when they encountered the students' strategy or Maya's explanation of the concept, they stopped the recording and shared their ideas. The focus was on reviewing Maya's teaching strategies for the perimeter concept, discussing the conceptual and mathematical appropriateness of Maya's discourse in the lesson, and possible ways to improve the quality of her teaching and instructional decisions. The viewing and discussions lasted 115 minutes. At the end of this node, Maya was expected to revise the lesson plan as a homework assignment, considering the experience she had gained so far in the context of Activity 4: Prepare for Enactment. She then uploaded the document to Google Drive as Lesson Plan Version 3. This was the end of the activities for the second node of the model. The researcher then reviewed the lesson plans and the video recordings of the rehearsal and their analysis.

As part of the third node of the model, called *Implementation*, Maya conducted the perimeter lesson in one of the real fifth-grade classrooms during the tenth week of the semester. This enactment occurred during the first and second practical hours of the Teaching Practice 2 course, not as part of the theoretical hours within the context of Activity 1: *Enact the Perimeter Teaching*. This activity allowed her to respond to the student's thinking and teach the concept in the classroom. Meanwhile, the other preservice teachers, the cooperating teacher, and the instructor observed her lesson. The implementation of the lesson consisted of two consecutive lessons. Each lesson lasted 40 minutes, and there was a 10-minute break between the two lessons. The lessons were conducted through the Zoom program because of the COVID-19 pandemic. During the lessons, Maya and the students used a whiteboard program to show their explanations. However, due to internet problems, Maya and the students struggled to use a whiteboard. During this time, the researcher completed the ongoing intervention analysis by examining her obtained data source.

The prospective teachers first watched the recordings of the enactment as part of the first activity in the first- and second-class periods of the implementation of the fourth node of the model, called *Post-Preparation*, during the eleventh week of the semester. The activity was called *Let Us Examine Enactment*. When confronted with the student's strategy, they paused the recordings, examined it, and considered how Maya would respond appropriately to the student's thinking. They also reviewed Maya's teaching strategies to improve the quality of their perimeter teaching. This activity took 120 minutes. In addition, Maya was expected to revise the lesson plan for further implementation, taking into account the experiences she gained throughout the intervention, as part of Activity 2: *Prepare for Further Teaching*. Then, she uploaded the final version of her lesson plan as Lesson Plan Version 4 through Google Drive, and the intervention nodes ended here.

During the fifteenth week of the semester, the posttest was administered to the preservice teachers through Google Forms during the time allotted for this course during the last week. In fact, as part of the course requirement, the preservice teachers and the researcher met through the Zoom program. Then, the researcher shared the form, and they were expected to respond individually to the questions in the posttest. This process took about 90 minutes. Finally, the researcher tried to arrange the interview time by communicating with the preservice teachers. Then, the interview schedule was determined. The interviews lasted approximately 75 and 120 minutes and were conducted using the Zoom program. They were also recorded. This completed the implementation of the model. The researcher shared the ongoing analysis with the research team. Considering the satisfactory data obtained in this process, it was decided to end the intervention here. Table 3.11 summarizes the process of this research in a timetable. Later, the following section gives information about the data collection tools used in the current study.

Table 3.11 The schedule of the research process of the current study.

Periods	Activities and Events
January 2019 – January 2021	The preparation for the intervention was completed. (Literature review, development of the instruments and interventions).
6th week of the spring semester of the 2020-2021 academic year (05.04.2021)	The pretest was applied during the theoretical hours of the course.
7th week of the spring semester of the 2020-2021 academic year (12.04.2021)	There was no course due to midterm week. An ongoing analysis of the pretest results was completed.
8th week of the spring semester of the 2020-2021 academic year (19.04.2021)	Activity 1: *What is the perimeter? Let us Look at the Concept through your and the Student's Eyes*, and Activity 2: *Let Us Examine the Dialogue* were implemented during the first and second lesson hours of the first node of the model, *Pre-Preparation.*
8th week of the spring semester of the 2020-2021 academic year (19-22.05.2021)	Activity 1: *Prepare for Rehearsal* in the *Preparation* was given as homework while implementing the second node of the model, *Preparation*.
8th week of the spring semester of the 2020-2021 academic year (23.05.2021) (Additional meeting)	Activity 1: *Prepare for Rehearsal* was collectively reviewed during the first lesson hour of the second node of the model, *Preparation.*
9th week of the spring semester of the 2020-2021 academic year (26.04.2021)	Activity 2: *Rehearse the Perimeter Teaching* was implemented during the second and third lesson hours of the second node of the model, *Preparation.*
10th week of the spring semester of the 2020-2021 academic year (03.05.2021)	Activity 3: *Let us Examine the Rehearsal* was implemented during the fourth and fifth class hours of the second node of the model, *Preparation*.
10th week of the spring semester of the 2020-2021 academic year (03-05.05.2021)	Activity 4: *Prepare for Enactment* was assigned as the homework as part of the second node of the model, *Preparation.*

Table 3.11 (Continued).

Periods	Activities and Events
10th week of the spring semester of the 2020-2021 academic year (06.05.2021)	Activity 1: *Enact the Perimeter Teaching* was implemented during the course's first and second practical hours as part of the third or *Implementation* node of the model.
11th week of the spring semester of the 2020-2021 academic year (10-13. 05.2021)	Activity 2: *Prepare for Further Teaching* was assigned as the homework while *post-Preparation*.
15th week of the spring semester of the 2020-2021 academic year (07.06.2021)	The post-test was applied in the time allotted for this course during the final week.
June 2021-March 2022	Interviews were conducted at the appropriate times for the participants.

3.4.3 Assessment Phase

The data was analyzed through continuous and retrospective analysis. The ongoing analysis occurred during the teaching episodes, and the retrospective analysis focused on the whole process. Ongoing analysis was the basis for spontaneous and planned interventions with the preservice teachers to generate and modify models of their mathematics teaching knowledge, professional noticing skills, and how they developed such knowledge and skills during this process. In particular, Simon (2000) described the ongoing analysis as "the basis for spontaneous and planned interventions with students, interactions that gather additional information, test hypotheses, and promote further development" (p. 341). Ongoing analysis was used in this study to guide the research flow in designing the tasks used, as shown in Figure 3.5.

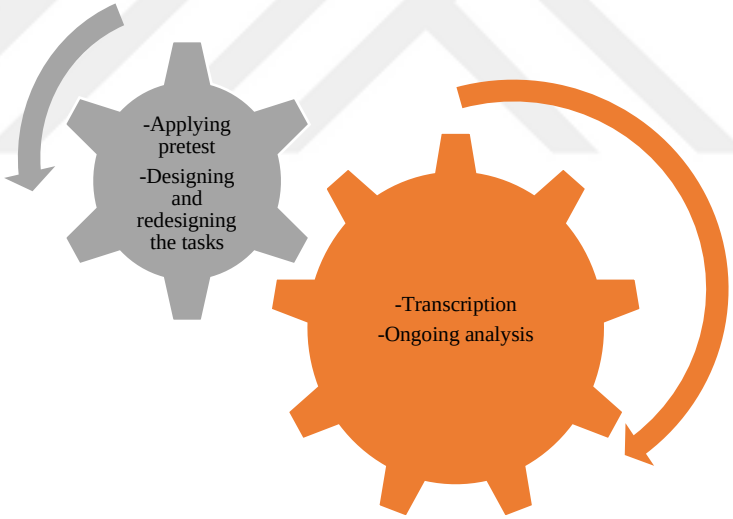


Figure 3.5. The process of ongoing analysis.

Both open coding and predetermined coding were used during the ongoing analysis phase based on the purpose of the data analysis. Notably, the data on preservice teachers' mathematical knowledge of the perimeter concept were analyzed considering the descriptions of the KQ model proposed by Rowland et al. (2005; 2009) regarding the Foundation Dimension.

Table 3.12 Components of the Foundation Dimension used in the study and description for each (Rowland et al., 2005; 2009).

Components	Description
Overt subject knowledge	Having well-developed and connected mathematical knowledge regarding the concept/subject that the teacher would teach.
Using terminology	Using mathematically correct language and terms during the planning and teaching process.
Identifying Errors	Being aware of the difficulties the students could have or develop in the planning process and taking the necessary precautions to prevent or overcome such difficulties during the instruction.
Theoretical underpinnings of the pedagogy	Having the theoretical knowledge necessary to teach mathematics, including the tools, materials, manipulatives methods, strategies, etc., that the teacher prefers to use.

Table 3.12 provides information about the description of each knowledge that is part of the Foundation dimension of the KQ model used in this study (Rowland et al., 2005; 2009). Table 3.12 shows that the *overt subject knowledge*, *using terminology*, *identifying errors,* and *theoretical underpinnings of pedagogy* knowledge of the Foundation Dimension of the KQ model were used as predetermined codes to analyze preservice teachers' mathematical knowledge of the perimeter concept throughout the study.

The components of the Professional Noticing Framework (Attending, Interpreting, and Deciding) (Jacobs et al., 2010) were used as predetermined codes during the ongoing analysis of preservice teachers' professional noticing skills. Essentially, three levels were achieved for each component of the professional noticing framework (attending, interpreting, and deciding): Level 0 (no evidence), Level 1 (limited evidence), and Level 2 (robust evidence). However, the ongoing analysis process played an essential role in determining the creation of the descriptors for each level. In addition, preservice teachers' moment-to-moment decisions and suggestions during the rehearsal and enactment analysis about how to respond to student thinking were evaluated under the decision component of the framework to examine their decision-making processes in different settings. Some codes from the responding framework proposed by Milewski and Strickland (2016) and some codes

that emerged during the ongoing analysis of the deciding component were used to evaluate instructional decisions or suggestions about instructional decisions. Detailed explanations are provided in the following sections. All coding was done using MAXQDA 2022 coding software. Many codes emerged during the open coding process, and the constant comparative method was used to verify and create more detailed categories (Merriam, 1998). Tables 3.15, 3.16, and 3.17 contain the codes that emerged during the analysis related to the components of professional noticing skills. Following the ongoing analysis, all data were transcribed for retrospective analysis to "place classroom events in a broader theoretical context" (Cobb, 2000, p. 325- 326). This analysis was used to develop and refine the models of preservice teachers' professional noticing skills. All the data were considered whole and read line by line to highlight the portions that focused on the preservice teachers' discourse during the intervention. Specifically, during the retrospective, preservice teachers' talk during the intervention was divided into "idea units" to code their discourse, which is defined as "a distinct shift in focus or change in the topic" (Jacobs et al., 1997, p. 13). Each idea unit was coded through open coding and predetermined codes obtained in the ongoing analysis during the retrospective analysis.

All levels related to attending, interpreting, and deciding skills were structured considering the ongoing analysis of the current study and the study of Jacobs et al. (2010). In fact, under the attending skill, the subcodes of “evaluating the correctness of the student's strategy,” “focusing on the mathematics in the student's strategy,” “focusing on the pedagogy,” and “invitation to attend” in the student's strategy emerged. Evaluating the student’s strategy subcode involved that the preservice teachers focused on the outcome rather than the process. When they failed to identify the details of the students' strategy or just evaluated the correctness of the students’ strategy, their responses or actions were coded under this code. In addition, when the preservice teachers questioned the mathematical correctness of the student's strategy or the mathematical meaning of the concept in light of the student's strategy, their responses were coded under focusing on the mathematics in the student's strategy

subcode. In addition, focusing on the pedagogy in the student's strategy subcode included examining the correctness of the student's strategy from a pedagogical perspective. Invitation to attend subcode involved the preservice teachers encouraging each other to attend to the details of students' strategies.

For interpreting skills, the subcodes "describing the student's strategy," "making inferences without considering the student's understanding," and "making inferences considering the student's understanding" emerged during the ongoing analysis. Describing the student's strategy involved paying attention to the details of the student's strategy rather than interpreting the student's understanding. When the preservice teachers mentioned their personal experiences with the concept or made general comments independent of the student's strategy, their responses were coded under making inferences without considering the student's understanding subcode. If the preservice teachers made interpretive inferences about the student's understanding of the perimeter concept, their responses were coded under making inferences considering the student's understanding subcode.

For deciding skills, the subcodes of "displaying unresponsive actions," "displaying partially responsive actions, "displaying responsive actions," invitation to display unresponsive actions," "invitation to partially responsive actions," and "invitation to display responsive actions" emerged during the ongoing analysis. When the preservice teachers described the general explanations/specific questions without considering the student's thinking and directed the student's correct answer by ignoring her/his solution, they displayed several actions categorized under displaying unresponsive actions.

Table 3.13 describes the unresponsive actions of the preservice teachers observed in this study. The actions included the "*" symbol that appeared during the ongoing analysis of the current study, while the others were taken from Milewski and Strickland's (2016) study.

Table 3.13 Preservice teachers' unresponsive actions.

Codes	Description
Ignore	Preservice teachers do not focus on the student's thinking.
Evaluate	Preservice teachers focus on the result rather than the process.
Evaluative revoice	Preservice teachers restate the students' saying to evaluate their understanding.
Invite to evaluate	Preservice teachers invite students to evaluate the other student's ideas.
Explain*	Preservice teachers directly explain the solution to the problem.
Invite to explain*	Preservice teachers explain the solution to the problem but get approval from other students or the student.
Remind*	Preservice teachers remind the definition of the concepts or formulas regarding the problem's solution.
Insist*	Preservice teachers do not understand the students' solutions ways despite trying to understand them and insist on their way.
Hesitate*	Preservice teachers hesitate about the correctness of the student's ideas.

When preservice teachers asked questions to make students aware of their error, it was assumed that they were demonstrating actions to understand students' strategies; however, their actions did not successfully extend students' understanding. Therefore, they were referred to as partially responsive actions.

Table 3.14 Preservice teachers' partially responsive actions.

Codes	Description
Explore*	Preservice teachers ask probing questions to understand the students' ideas.
Invite to explore*	Preservice teachers encourage the preservice teacher (who implemented perimeter teaching) to explore the details of the student's strategy.
Drill*	Preservice teacher asks similar questions to make the students realize their mistake or poses the same problem with number context modification to make it more understandable.
Invite to produce an alternative solution*	Preservice teachers encourage the preservice teacher (who implemented perimeter teaching) to ask the students to find other solutions to the questions. Preservice teachers invite the students to produce alternative solutions.
Invite to Scaffold*	Preservice teachers encourage the preservice teacher (who implemented the perimeter teaching) to scaffold the students' understanding when they explain their thinking.

Table 3.14 describes the partially responsive actions observed in the context of this study. The actions included the "*" symbol during the current study's ongoing analysis, while the other was considered in Milewski and Strickland's (2016) study.

When preservice teachers displayed actions that extended or promoted student understanding, they were referred to as responsive actions. Table 3.15 describes the responsive actions that occurred at this level. The actions shaped were considered in Milewski and Strickland's (2016) study.

Table 3.15 Preservice teachers' responsive actions.

Codes	Description
Connect	Preservice teachers encourage students to make connections among and within mathematical concepts or procedures.
Invite to connect	Preservice teachers encourage the preservice teacher who implemented the perimeter teaching to make connections among and within the mathematical concepts or procedures considering the student's discourse.
Advance	Preservice teachers ask students comprehensive questions to advance their understanding.
Invite to advance	Preservice teachers encourage the preservice teacher who implemented the perimeter teaching to ask probing questions to extend students' understanding.

Given the ongoing analysis process and emerging codes, the creation of descriptors for each level of professional noticing skills was structured during the retrospective analysis. For example, "displaying unresponsive actions" was associated with Level 0: Lack. "Displaying partially responsive actions" was associated with Level 1: Limited, and "displaying responsive actions" was associated with Level 2: Robust. Table 3.16 describes the levels achieved for each component of professional noticing skills.

Table 3.16 The components of professional noticing skills and attained levels for each component.

	Levels	Description
Attending	Level 0: Lack	-Evaluate the correctness of the result without considering the student's strategies. - Focus on the number in the students' answers and accept the results correctly, even if they do not include units of measure.
	Level 1: Limited	-Identify the correctness of students' answers correctly. -Examine the solution concerning mathematical appropriateness without addressing every detail in students' solution strategies, such as counting dots and lengths to find the perimeter of the given shape.
	Level 2: Robust	- Examine the mathematical and pedagogical appropriateness of the solution by mentioning every detail of the students' strategies and using mathematical language correctly.
Interpreting	Level 0: Lack	-Make general comments without considering students' thinking of the perimeter concept. -Misinterpret the student's thinking of the perimeter concept. -Describe students' strategy rather than interpreting their understanding of the perimeter concept.
	Level 1: Limited	-Make general comments on students' understanding of the perimeter concept by considering the students' strategy.
	Level 2: Robust	-Make interpretive inferences on students' understanding of the perimeter concept by correctly mentioning all concepts and using mathematical language.
Deciding	Level 0: Lack	-Provide general explanations/questions by ignoring students' thinking of the perimeter concept. -Direct the student to the correct answer.
	Level 1: Limited	-Make students realize their mistakes as to the solutions.
	Level 2: Robust	-Extend or advance the students' thinking of the perimeter concept with appropriate mathematical language.

3.5 Trustworthiness of the Study

Validity and reliability are critical concepts that should be considered at all stages of research, including study design, analysis, and discussion of findings (Merriam, 1998; Patton, 2002). While the concepts of validity and reliability in qualitative studies differ from those in quantitative studies, they are referred to differently in qualitative studies (Cresswell, 2007; Lincoln & Guba, 1985). Lincoln and Guba

(1985) mentioned that credibility, transferability, dependability, and confirmability are more appropriate in qualitative research than validity and reliability.

According to Lincoln and Guba (1985), credibility is crucial in determining trustworthiness. Merriam (2009) stated that credibility is concerned with “the topic of how study findings mirror reality. How congruent are the findings with reality?” (p.213). Merriam (1998) offered six essential strategies for increasing the internal validity of the research: Triangulation, multiple investigators or multiple methods, member checks, long-term observation, peer review, and participatory or collaborative modes of research and research bias. Most of these methods were used in this study to increase credibility.

Firstly, the study used the triangulation strategy to establish credibility, which involves using more than one data sources to increase the accuracy of the data. To this end, the current study used multiple data sources, including pre-and post-tests, individual responses to the given tasks, group discussions, planning materials, field notes, and video recordings of the entire intervention.

In addition to triangulation, raw data from written responses and transcriptions of video recordings of the model and interviews were shared with each study participant. In this way, the member check strategy was met. In addition, there was constant interaction with the participants throughout the two-and-a-half-month period, including the entire study process. In this way, prolonged engagement with the study participants was met to enhance credibility, which is also one of the strengths of design research studies (Gravemeijer & Cobb, 2006).

The second one is transferability, which is concerned with “how we can generalize the results from these specific contexts to be useful for other contexts” (Bakker & van Eerde, 2015; p.444). In a design research study, the findings from the research study can be used to elucidate and explain specific cases (Gravemeijer & Cobb, 2006). In addition, design research results can influence future research and instructional design. To this end, "it is the responsibility of the investigator to ensure

that sufficient contextual information about the fieldwork sites is provided to enable the reader to make such transfer" (Shenton, 2004, p. 69–70). This study provided a thick description of the context and participants of the study, the purpose and context of the data collection instruments, and the period of the study to help other researchers understand the process and transfer it to other settings. In addition, the transcripts of the classroom discussions were provided to allow readers to understand the context of the study (Bakker & van Eerde, 2015).

The third one, dependability, refers to the replicability of research findings and the consistency between the data and the findings (Lincoln & Guba, 1985; Meriam, 1998; 2009). As Gravemeijer and Cobb (2006) stated, replicability is not mentioned in design research due to the variable structure of participants and research settings (Simon, 1995). Instead, ecological validity concerns a) how the study results could be generalized to real-life settings and b) using the same principles in design research (McKenney & Reeves, 2012). This way, it is assumed that the necessary adaptations will be considered for another research. In this regard, the research procedure of the current study, such as the assumptions, the theoretical details in the design of the model, the data collection process, the structure of the data collection tools, and the data analysis processes, were explained in detail. At the same time, the data were triangulated by using multiple data sources such as pre-posttests, classroom activities, and teaching experiences. In addition, the second coder participated in the data analysis process to reduce the researcher's bias and to reach a consensus on the study's findings. Thus, the second coder's presence increased the research quality.

Finally, confirmability is related to the objectivity of the study. In other words, it was related to “the extent to which the characteristics of the data, as posited by the researcher, can be confirmed by others who read or review the research results” (Bradley, 1993, p.437). It concerns “how independently of the researcher the data are collected and analyzed” (Bakker & van Eerde, 2015, p. 445).

The role and position of the researcher play an essential role in confirmability. I participated in the current study process as an instructor and researcher. As the

instructor, I introduced the activities and facilitated class discussions. As a researcher, I collected data from the pre-service teachers who participated in the current study. I was involved in the entire research process. First, I informed the participants about the purpose of the study. Then, I explained that their participation in the intervention would not affect their grades. Then, I emphasized using pseudonyms rather than their real names. I also emphasized that I would not share the raw data with anyone, making them comfortable. When I asked the pre-service teachers if they were willing to meet the requirements of the model, such as mandatory attendance, completing the tasks, and attending the performances as an observer, five of the course participants were willing to meet the requirements of the study. The entire research process was recorded using the Zoom program to ensure confirmability. I used verbatim transcription, and when I showed them the transcription of the model and their responses to the pre/post-test, post-interview, and tasks, they confirmed the transcriptions and responses. At the same time, the transcriptions were shared with the research team when necessary. In addition, direct quotes from their responses and explanations and dialogues without adding comments were included in the study's findings. In addition, credibility was sought through detailed methodological information, multiple types of data collection methods, and the presence of the second coder.

CHAPTER 4

FINDINGS

This study aims to support the improvement of prospective middle school mathematics teachers' mathematical knowledge and professional noticing skills on the concept of perimeter through the development of a model based on the cycles of enactment and investigation within the framework of design-based research. In light of this purpose, this chapter summarized the current study's findings in two main sections and related subsections. Remarkably, in the first section, the development of the prospective middle school mathematics teacher's mathematical knowledge regarding the perimeter concept was analyzed through the use of predetermined codes (*the overt subject knowledge, using terminology, identifying error,* and *theoretical underpinning of pedagogy* under the Foundation Dimension of the KQ Model) considering their responses in the pre and post-test and the post-interview. Consequently, the findings that reveal the development of her mathematical knowledge regarding the concept of the perimeter are presented under the sections: Overt subject knowledge, using terminology, identifying error, and theoretical underpinning of pedagogy. Then, evidence for developing each knowledge throughout her involvement in the study is presented in the following subsections. Some examples from the individual responses to the classroom activities, the video transcripts from the nodes of the model, the lesson plans, and the interview are presented to show the development of the prospective mathematics teacher's mathematical knowledge of the perimeter concept during the present study. Table 4.1 summarizes her development of mathematical knowledge in the perimeter concept during the present study by referring to the nodes of the model designed in light of the cycles of enactment and investigation.

Table 4.1 Summary of the preservice teacher's (Maya's) mathematical knowledge regarding the perimeter.

Knowledge types	Nodes of the Model					
	Pre-test	Node 1: Pre-Preparation	Node 2: Preparation	Node 3: Implementation	Node 4: Post-Preparation	Post-test
Overt subject knowledge	Lack of mathematical knowledge	x*	x	x	x	Robust mathematical knowledge
Using terminology	Lack of mathematical language and terminology	x	x	x	x	Robust mathematical language and terminology
Identifying Error	Lack of knowledge of students' difficulties and ways to overcome them	x	x	x	x	Robust knowledge of students' difficulties and ways to overcome them
The theoretical underpinning of the pedagogy	Lack of knowledge of the teaching strategies and materials	x	x	x	x	Robust knowledge of the teaching strategies and materials

** The "x" sign indicates the node where the exemplary behavior(s) related to the relevant code is observed in the model.*

As shown in Table 4.1, after attending the intervention, the prospective mathematics teacher's knowledge of the *overt subject knowledge*, *using terminology*, *identifying the error*, and *theoretical underpinning of the pedagogy* was enhanced from the pre-assessment to the post-assessment. In the most general sense, the development of such knowledge was observed during all nodes of the model. The details of such improvements are presented below. Then, the information on developing preservice teachers' professional noticing skills on the perimeter concept will be shared in detail.

4.1 Development of Mathematical Knowledge regarding the Perimeter Concept

As stated above, the purpose of the current study was to support the improvement of prospective middle school mathematics teachers' mathematical knowledge and professional noticing skills on the concept of perimeter through the development of the model based on the cycles of enactment and investigation within the framework of design-based research. This section will share the analysis of improving the prospective mathematics teacher's (Maya) mathematical knowledge regarding the perimeter concept.

4.1.1 Mathematical Knowledge: *Overt Subject Knowledge*

This section examined how Maya defined the perimeter concept and explained the underlying reasons for the perimeter calculation, considering her pre- and post-test responses and the post-interview explanations. As Table 4.2 shows, Maya developed *overt subject knowledge* during the current study. Furthermore, the following table summarizes the findings regarding Maya's current *overt subject knowledge* and shows the development in Maya's knowledge from the pre-assessment to the post-assessment.

Table 4.2 Development of the *overt subject knowledge*.

Pre-intervention findings	Post-intervention findings
-She based the perimeter definition on addition operation without mentioning critical terms for the perimeter concept.	-She based the definition of the perimeter concept on its meaning by mentioning critical terms for the perimeter concept.
-She had misinformation about the perimeter concept.	- She put her knowledge of length measurement and the perimeter concept on a conceptual basis.
-She memorized the rules on the perimeter calculation without querying its meaning.	-She explained the underlying reason for perimeter calculation.

As shown in Table 4.2, Maya's responses in the pre-assessment indicated that she had difficulty defining the concept of perimeter. Notably, she was unaware of the

meaning of perimeter and did not use critical terms such as closed shape and boundary in her definition. For example, when asked to define the concept of perimeter in the pretest, she responded, "*The length that covers a shape or object*." As can be seen from her statements, she also misused the term "*covering*" in her definition because it refers to area concepts, not perimeter. In addition, since she used the terms "*shape or object*" in her perimeter definition, she was asked during the post-interview why she used such terms in her definition, and she stated that

> The object focuses on geometric objects, such as squares and rectangles, because the basic shapes that come to my mind are a square and a rectangle, both geometric objects and geometric shapes. I was not thinking in three dimensions when I used the word object.

In her explanation of Maya, she focused on geometric shapes such as squares and rectangles rather than three-dimensional shapes. In contrast, when she drew figures that represented the concept of perimeter and when she drew figures that did not represent the concept of perimeter during the present study, it was seen that she preferred to draw irregular shapes, as shown in Figure 4.1.

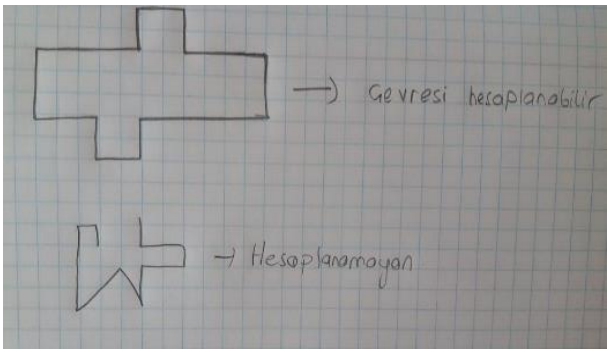


Figure 4.1. Maya's drawings for the perimeter and non-perimeter *(overt subject knowledge).*

There seemed to be an inconsistency between her statements and her drawings since she did not explain the concept correctly; she correctly drew the shapes to define the

perimeter. To explore the reasons for this situation, Maya was asked during the interview why she drew such figures. She explained, "*I saw them somewhere*." Her explanation indicated that she did not question the meaning of the perimeter while drawing them. She memorized the shapes of the figures regarding the perimeter concept.

Furthermore, it was observed that she had difficulty explaining the underlying reasons for the measurement processes. In particular, when she was asked to describe the perimeter calculation in detail in the pretest, it was observed that she based her explanations on addition operations. She did not emphasize whether the added lengths were in or around the shape or mention the measures of length in her answers. She explained the perimeter calculation: "*We can calculate the perimeter by adding the lengths that cover a shape or object."* Her explanation indicated that she again misused the terms "*cover*" and "*object*" and emphasized the addition operation in defining perimeter calculation. In addition, when asked in the pretest what she needed to know to calculate the perimeter of a shape, she gave superficial answers, listing only "*length*" as the required information. Considering her answers, she knew the perimeter as a length, but she did not question its meaning from a conceptual point of view. Her solution to the given problem in Figure 4.2 confirmed such an inference that she did not put the concept of perimeter into a conceptual base.

ABCD is a square, and ADE is a right triangle. Since the perimeter of triangle ADE is 48 cm, IECI=28 cm, and IABI=12, calculate the perimeter of trapezoid ABCE as much as possible ways.

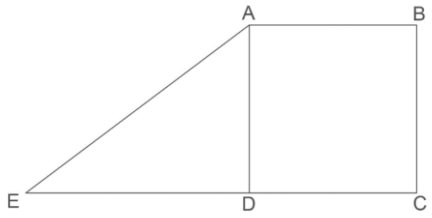


Figure 4.2. The question in the pre-test.

Although she could have solved the problem as much as possible solution ways, she produced one solution way that depended on using the properties of quadrilaterals and addition operations, as presented in Figure 4.3.

The perimeter ADE triangle is given 48 cm. Since AD is on one side of the square, it is 1two cm. The side EC is 28 cm, and the side DC is also one side of the square. It is 12 cm. Therefore, the side ED is 16 cm. Since we know the two sides of the triangle, we need to find the 3rd side. Since the perimeter of the triangle is given, we find 20 cm from 48-28. We can add the side EA + EC + AB + BC to calculate the perimeter to get two cm.

Figure 4.3. Evidence of *overt subject knowledge* from Maya's solution to the problem in the pre-test.

Figure 4.3 shows that she did not generate multiple solutions to the given problem, even though there are multiple ways to solve such a problem. To illustrate, she could have placed the side lengths in the shape's boundary on the ruler and calculated the total length. Alternatively, she could have used the Pythagorean Theorem to find the hypotenuse and then calculated the perimeter of the trapezoid ABCE. However, Maya chose to solve the question in the usual and expected way. On the other hand, producing different solutions indicated that she had well-developed domain knowledge (Rowland et al., 2005) and vice versa.

In addition, it was observed that she did not have comprehensive knowledge about the importance of using length measures in her answer. Specifically, when asked in the pretest to identify the student's solution path in Figure 4.4 and evaluate whether it was correct, it was observed that she accepted it as correct even though the student did not use the length measure in her response.

While solving this question, your student took the ruler on the board and made some measurements. Meanwhile, the following solution appeared on the board, and she told the class: "I found it in a very different way. My result is 72."

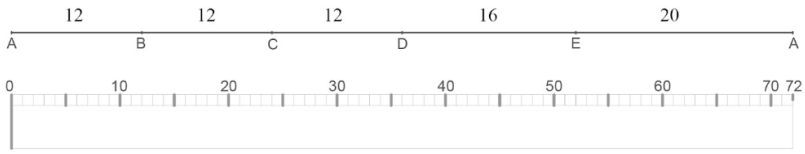


Figure 4.4. Excerpt from the pretest about the student's solution.

When examining such a strategy, Maya said, "*I think the student's solution is correct because she made the side lengths of the given shape a straight line*." Her response implied that she was not focused on expressing the unit results. Moreover, she was asked to interpret the given student's mathematical understanding of the concept of the perimeter to gain in-depth information about how she internalized the concept given the student's strategy. In this process, she focused on the addition operation and said, "*The student understood that calculating the perimeter of a shape means adding the side lengths.*" As shown in her explanations, she based the student's understanding of perimeter on the addition operation rather than its meaning.

In summary, Maya did not correctly define the concept of perimeter and based her explanation on the addition operation. She gave incorrect information in defining the concept and did not question its meaning from a conceptual point of view. Maya memorized the definition. All in all, she lacked *overt subject knowledge* of the perimeter concept before participating in the intervention of the current study.

After participating in the intervention, she was observed to base her knowledge of perimeter on a conceptual basis. She provided the correct definition regarding the perimeter concept by examining its meaning from a conceptual perspective. In addition, since she began to use critical terms for the perimeter concept in her definition, it could be said that she was questioning its meaning rather than simply memorizing its definition after participating in the intervention. For example, when

asked to define the perimeter concept in the post-test, she responded: "*When we start from the point of a closed shape and go around the shape when we come back to the starting point, we get the total length, called the perimeter*." Her explanations indicated that she included some critical terms such as "*closed shape*," "*go around*," and "*total length*" in her definition of perimeter.

In addition, she was able to explain the underlying reasons for the perimeter measurement processes correctly. To illustrate, when she was asked to describe the perimeter calculation in detail in the post-test, it was seen that she based her explanations on the meaning of the concept. She responded: "*We can find the perimeter of a shape by adding the outside lengths or measuring the total length of the shape's boundary*." As could be understood from her statements, she focused on the part and whole relationship in finding the perimeter by expressing "*adding outside lengths*" and "*total length of the boundary of the shape*." She also pointed out that the lengths around the figure should be summed to find its perimeter. Her response implied that she was aware of a measurement process in calculating perimeter. Her response in the post-test, when asked to explain what one needs to know to calculate the perimeter of a shape, also confirmed such an inference. It was seen that she emphasized the relationships between the concept of perimeter and length measurement and focused on units of length measurement in her explanations. She responded: "*The shape must first be closed, and the length measures of the shape's sides must be known to calculate the perimeter*." As her response shows, she explained the concept of perimeter more precisely than in her pretest response.

In addition, as shown in Figure 4.5, she tended to produce more than one solution to the given problem after her involvement in the study. Also, her solutions were more comprehensive than procedural understanding by following the addition operations or using the properties of quadrilaterals. She also mentioned the Pythagorean Theorem in her answers. These could be taken as an indicator that she understood the underlying reasons for calculating perimeter.

First, if IECI = 28 cm and IABI = 12 cm, since it is a square, IDCI = 12 cm. Then IEDI = 16 cm. Here, from the Pythagorean theorem, IAEI = 20 cm. The total side length of squares around this shape is 12 x 3 = 36 cm, IEDI = 16 cm, and if we add IAEI = 20 cm, we get 72 cm.
In another way, the perimeter of the triangle is already given to us without using Pythagoras, so we have IEDI = 16 cm. If we add up the lengths from the beginning and reach this point again, it will be 72 cm.

Figure 4.5. Evidence of *overt subject knowledge* from Maya's solution to the problem in the post-test.

Her statements during the post-interview also confirmed such inferences. Specifically, she stated:

> Before the class, I had never considered a circle or a closed curve as an example of a perimeter. This is because I did not question the concept of the perimeter before. I went by the definition in the textbook that I saw in the fifth-grade book, so I drew it for the perimeter, remembering the example in the book. I would have drawn basic shapes like squares and rectangles if I had not seen them. The circle was not a very popular figure for me to use to define the perimeter. I had more prototypical definitions in mind. You asked us to draw shapes... Also, before I focused on formulas to find perimeter... I ignored units. However, I learned that saying the number makes no sense without its units.

As can be seen from her statements, she did not examine the concept of perimeter conceptually before participating in the current study. She memorized its definition from the textbooks. She focused on the concept from a procedural aspect. However, she had the opportunity to co-construct her knowledge of the perimeter concept. She based her explanations on the meaning of the concepts rather than on the definitions, operations, or formulas in the textbooks after participating in the intervention. In addition, she was seen to question the reasoning behind the perimeter calculation,

which may have allowed her to generate more than one solution strategy for the problem. Since Maya correctly defined the perimeter concept by examining the meaning of the concept and mentioning the underlying reasons behind the perimeter calculation from a conceptual point of view, her *overt subject knowledge* regarding the perimeter concept was developed during the current study. Put differently, she made remarkable progress in her overt subject knowledge of the perimeter concept during her involvement in the study. Evidence for improving *overt subject knowledge* during the current study is presented below.

4.1.1.1 Evidence of Improvement in *Overt Subject Knowledge* during the Model

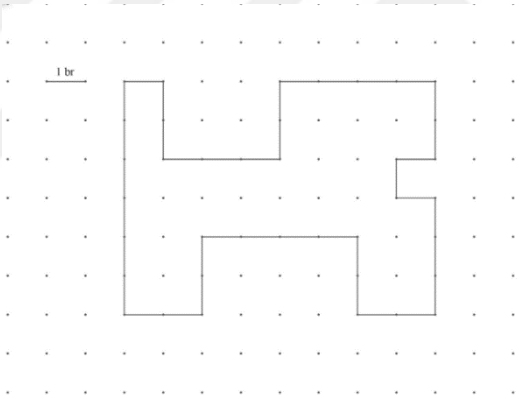
As noted above, Maya's *overt subject knowledge* about the concept of perimeter developed as she participated in the nodes of the model. In this section, I will share the evidence that she developed this knowledge during her participation in the current study's model, paying attention to the sequence of activities in each node of the model, as well as the sequence of the nodes of the model, which were referred to as *Pre-Preparation*, *Preparation*, *Implementation*, and *Post-Preparation*, respectively. The excerpts from the video transcriptions of the current intervention and the individual responses to the classroom activities were used to describe Maya's progress in *overt subject knowledge* during the intervention.

Table 4.1 shows evidence that Maya's *overt subject knowledge* development began to be observed in the first node of the model, *Pre-Preparation*. In this node, the preservice teachers were asked to examine Activity 1: *What is the Perimeter? Let Us Look at the Concept Through Your and the Student's Eyes*, and Activity 2: *Let Us Examine the Dialogue*. The preservice teachers were asked to answer all the questions individually in the first-class period. In the second lesson on implementing the first node, they were asked to explain their reasoning. In the first activity, there were questions about drawing some shapes that represent the perimeter concept and not the perimeter concept and solving the given question as much as possible in

examining the concept's meaning. The first three questions in this activity are shown in the following table.

These questions allowed preservice teachers to co-construct their mathematical knowledge about the concept of perimeter and to learn to justify their reasoning using correct mathematical expressions before responding to students' thoughts in simulated and natural classroom settings. Maya's drawings of these first two questions are shown in Figure 4.6.

Table 4.3 The first three questions in Activity 1.

Activity 1: *What is the Perimeter? Let us Look at the Concept through your and the Student's Eyes.*
1. Draw shapes that represent the perimeter concept. Explain your reasoning in detail.
2. Draw shapes that do not represent the perimeter concept. Explain your reasoning in detail.

3. Calculate the perimeter of the given shape as much as possible. Please indicate your solutions ways.

As shown in Figure 4.1, she attempted to explain the concept of the perimeter by focusing on whether the perimeter of a shape can be calculated. Although she correctly drew examples for the given questions, she had difficulty explaining the underlying reasons for such drawings in her responses. Her answers are as follows: "*I drew a figure for the perimeter to be calculated using unit squares. Since the other one does not have a closed shape, I thought its perimeter could not be calculated*". Her explanation included the term "*unit squares*," which is not correctly used in measuring length. It is used in calculating area.

Similarly, she used the term "*cover"* to define the perimeter in the pretest. Although Maya correctly drew examples that did and did not represent the perimeter, she had difficulty explaining why she drew such figures. Considering this, it could be said that there was a contradiction between her drawing of the perimeter and her explanations. She could memorize some pictures and definitions without questioning them. This situation implied that at the beginning of the study, she lacked *overt subject knowledge* about the concept of perimeter.

The preservice teachers were asked to share their reasoning about these questions at the beginning of the second-class period of the first node implementation. In this regard, the teacher initiated the discussion by asking them to explain a) their reasoning about why they drew such shapes to represent the perimeter and b) not the perimeter concept, and c) which of their friends' drawings caught their attention. It was then seen that Preservice Teacher 5 shared her drawings and explained her reasoning. Meanwhile, Maya responded to her drawings and explanations as follows:

> **Preservice Teacher 5:** Teacher, I think these figures represent the perimeter and non-perimeter of the shape. For example, the perimeter of the square is 20 cm. Students often include this edge to calculate the perimeter (She drew the shapes in Figure 4.6).

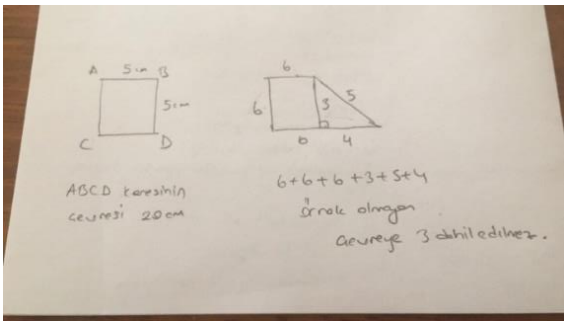


Figure 4.6. Evidence of *overt subject knowledge* from Preservice 5's drawings of the perimeter and non-perimeter shapes.

Maya: Actually, it is true; when several shapes come together, we only need to get the outside line segments, not the line segments in the middle or inside. In other words, instead of finding and adding all side lengths of the shapes, we focus on the shape's boundary. You are correct. Not all line segments. That is right, I never thought of it that way, but I like it.

In Preservice Teacher 5's explanation and drawing in Figure 4.7, it was seen that she focused on the drawings based on her personal experiences to define the concept and supported her claim with mathematical equations. As understood from Maya's statement, Preservice Teacher 5's explanations opened a new window for Maya to question the position of the shape's edges, such as the inside and outside of the shape, in defining the perimeter. Maya's explanations in the dialog concluded that she had not previously questioned the shape's boundary in terms of the perimeter, even though she had emphasized it in her drawings. The fact that she did not specify which edge should be added to the perimeter calculation in her pretest response also supported her explanation. Based on Preservice Teacher 5's assertion, it was assumed that Maya was beginning to investigate the properties of the perimeter in such a way that the sides existed as the boundary of the shape rather than all the sides. The class discussion continued as follows:

Preservice Teacher 2: Sir, can we say that if the shape is closed, no matter how the sides of it, whether linear or non-linear shapes, it represents the perimeter? Several closed regions with different structures could represent the perimeter examples, and concave shapes, while line segments do not represent the perimeter (She drew the shapes in Figure 4.7).

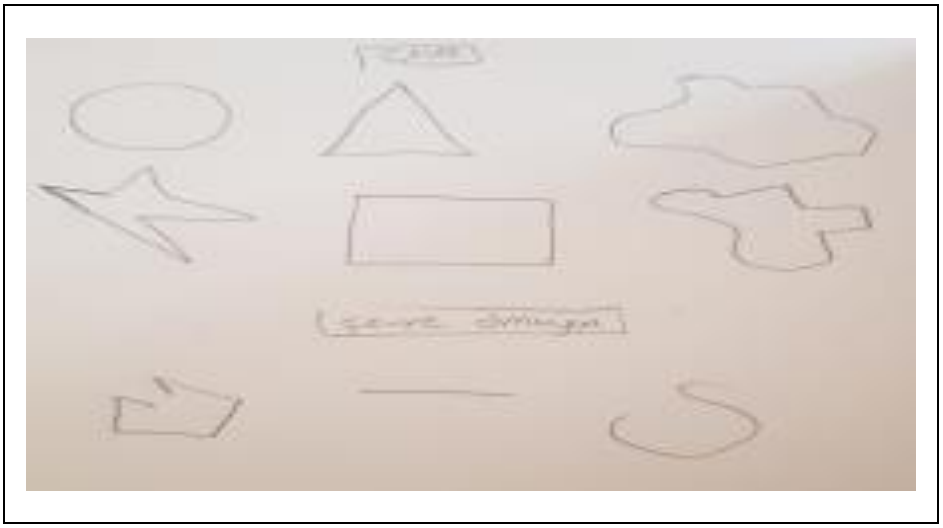


Figure 4.7. Evidence of *overt subject knowledge* from Preservice 2 drawings of perimeter and non-perimeter shapes.

> **Researcher:** A different perspective has come. Non-linear, said your friend; what do you think?

For the first time, the importance of being closed came to the fore to define the perimeter concept. In addition, Preservice Teacher 2 did not use the perimeter calculation or formula to explain the perimeter concept. Instead, she focused on the concept's meaning from a conceptual point of view. In response to Preservice Teacher 2's assertion, the teacher initiated the discussion by asking about the role of nonlinear terms in the definition to allow the preservice teachers to co-construct their mathematical knowledge about the perimeter concept. In this way, they actually questioned one of the properties of measurement called conservation, which is related to being aware that the length of an object does not change when it is moved or opened, specific to this situation. The discussion continued when Maya spoke as follows:

> **Maya:** I did not think it should be linear. It can be a circle if it does not have to be linear.
>
> **Researcher:** Your friend said circle; what do you think?

The above excerpts show that Maya questioned whether the shape's edges could be linear or nonlinear to represent the perimeter. Earlier, she had noticed that one of the essential properties of the perimeter was that it was closed. These inferences led Maya to conclude that the circle also had a circumference as a perimeter. She had probably restricted the concept of the perimeter to polygons before the intervention. Now, however, it was observed that she noticed and reasoned about the circle's circumference as a kind of perimeter with the help of such a collective analysis. Her explanations during the post-interview also confirmed this inference. She stated:

> I did not think a shape could or should be closed to represent the perimeter. I did not question it before. Besides, we taught and talked about the circle's circumference, but I never connected it with the concept of perimeter...During the course and with the help of discussions and examples from my friends, I realized that an irregular shape, including a curve, can also represent the perimeter if it is closed.

From what she said, she had never studied the perimeter conceptually before. She stated that she had memorized its definition without exploring its meaning. After Maya's reasoning about the circle's circumference as the perimeter, the teacher invited the others to share their ideas about her reasoning. The discussion then continued among the preservice teachers, Maya, and the researcher as follows:

> **Preservice Teacher 2:** I will come back to my definition. My teacher, I started from one point and returned to the same point, meaning the shape was closed. When we opened the circle, it became a length. I think it happens; we can mention the linear and non-linear ones in the perimeter definition.
>
> **Preservice Teacher 3:** How accurate is the sum of the side lengths starting from a point and reaching to close the shape? Should we tell the total length measurement around the shape in the definition?
>
> **Researcher:** Great, let us define the perimeter.

Maya: The perimeter is the total length measurement around the closed shape.

As seen in the dialogue above, Preservice Teacher 2 focused on the relationships between linear measurement and the properties of the perimeter, noting that the perimeter is the form of length measurement. She actually noticed that the perimeter involves determining the total number of unit lengths that fit end to end around the shape without overlaps or gaps. State it differently, she noticed that the perimeter is a property of an object and can be found by quantifying the distance between the object's endpoints. The preservice teachers then collectively examined the properties of the perimeter and constructed shared meanings relevant to the definition of the concept within the discussion. As a result, Maya defined perimeter as "*the total length measurement around the closed shape*." Through such a fruitful discussion, it could be suggested that Maya had the opportunity to co-construct her mathematical knowledge of perimeter and demonstrated exemplary behavior in developing her *overt subject knowledge.*

In addition to these questions in the first activity, since the solutions to the problem provided information about her *overt subject knowledge,* the third question in the same activity, as shown in Table 4.3, was directed to the preservice teachers to solve as much as possible, as mentioned earlier. Maya's response before the discussion showed that she was inclined to produce a one-way solution to the question that did not involve a high level of thinking, such as "*I counted units and found 38 is the perimeter of the shape*". As can be seen from her answer, she did not clearly explain her solution path or provide any length measures. For these reasons, her solution was not accepted as mathematically correct. During the second lesson on implementing the first node, it was observed that the preservice teachers did not tend to produce multiple solutions to the given problem. However, discussing solutions in mathematically correct terms could support the development of their mathematical knowledge of the perimeter concept. To illustrate,

Preservice Teacher 2: We need to count units to solve the problems.

Preservice Teacher 3: We need to count units around the shape.

Maya: I could solve the question by counting the units between two points around the shape and found that the shape's perimeter was 38 units (Evidence of *overt subject knowledge).*

As seen in the above dialog, Maya began to explain her solutions using the mathematically correct terms, such as "*around the shape*," in the explanations of Preservice Teachers 2 and 3. Maya showed exemplary behavior regarding *overt subject knowledge* in this activity since she replaced the critical terms and emphasized the length measures in her explanation during the discussion.

In addition, as noted above, the development of her *overt subject knowledge* continued to be monitored. After completing Activity 1 in the first-class period of the first node implementation, the preservice teachers were asked to examine the questions in Activity 2: *Let Us Examine the Dialogue* as a part of the representation of practice. In this regard, the preservice teachers were asked to examine the following dialogue, shown in Table 4.4, and respond individually to the prompts. In the shared dialogue, the student calculated the shape's perimeter by counting points rather than line segments. She then returned to a point before the starting point. As a result, the student arrived at the correct result. Although the student got the correct result, her thinking was wrong. When asked to identify the mathematical details of the student's strategy, Maya was observed to respond: "*The student counted the points*." As can be seen from her response, she was not asking for the details of the student's strategy. She mentioned what the student did as the solution to the problem.

Then, immediately following the discussion of Activity 1, they were asked to discuss their responses to Activity 2 in the second-class period of the *Pre-Preparation* node of the model. Although the primary goal of Activity 2 was to support the development of the preservice teachers' professional noticing skills regarding the perimeter concept, it was found that Maya also co-constructed her mathematical knowledge regarding the perimeter concept during the discussions. In other words,

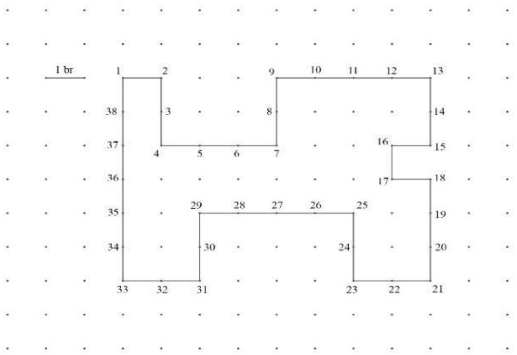
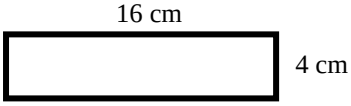
when the preservice teachers examined the given dialogue collectively, it was seen that Maya questioned the mathematical meaning of the perimeter concept in light of the given student strategy. More specifically, Maya based her mathematical knowledge of the perimeter concept on the conceptual base. For example,

> **Maya:** Wrong, my teacher.
>
> **Researcher:** Why?
>
> **Maya:** She did not count the length between two points but counted the points (Evidence of *overt subject knowledge*)
>
> **Preservice Teacher 3**: I disagree with you. The solution was right because when we count the points of a closed shape like this, we calculate its perimeter since the number of points equals the number of line segments.
>
> **Researcher:** Do you agree with Ali's opinion?
>
> **Maya:** I do not know, but for example, when calculating the perimeter, we say the sum of the sides. So, we need to know the distance. So, length. So, we do not need to know the sum of the lengths of the sides for the perimeter. We need to know the distance between two points, that is, how much so we can find the perimeter. We need the distance between the points, not the points. So, we need to have the length (Evidence of *overt subject knowledge*)
>
> **Preservice Teacher 3:** Since dots are not units.
>
> **Maya:** I do not think counting the point were not used in perimeter calculation.

As shown in the dialog above, Maya claimed that the student made an error in calculating the perimeter of the given shape. Meanwhile, she explained her reasoning by saying the student should have counted the distances to calculate the shape's perimeter. However, since the student got the correct answer to the given problem,

Preservice Teacher 3 thought that counting all the points once around the shape could be used to calculate the perimeter. The researcher then invited the preservice teachers to share their ideas about Preservice Teacher 3's assertion.

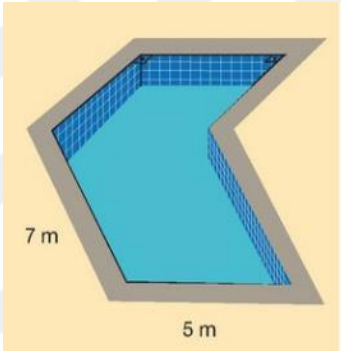
Table 4.4 Activity 2: *Let Us Examine the Dialogue.*

Representation of Practice	Prompts regarding the dialogue
Student: Teacher, the answer is 38. *Teacher:* How did you calculate Aleyna? Can you explain? *Student:* Teacher, I chose a reference point for myself, so I started *from one point and reached that point while passing 38 points.* *Teacher:* Can you show your solution on the board? *Student:* Of course, my teacher. I counted the points to calculate the shape's perimeter and got 38.	
	a. What do you think about the student's first strategy? Do you think it is true? b. What do you think about the student's second strategy? Do you think it is true? c. What do you think about the students' mathematical understanding of the perimeter concepts? D. Why do you think the teacher asked the student the second question? e. If you were the student's teacher, how would you continue the lesson? Which questions could you ask the student to advance her learning of the concept?
Teacher: How do we calculate the perimeter of the shape, Aleyna? *Student:* We count the dots around the shape. Specifically, we select a starting point around the shape and count the points until we reach the same point, teacher. *Teacher:* So, do we count the points when calculating the perimeter, Aleyna? *Student*: Yes, teacher. *Teacher:* How would you calculate if I did not place the dotted paper under the shape and asked you this question?	
	
Student: (16 + 4) x 2 = 40 cm teacher. *Teacher:* How did you calculate? *Student:* This shape is rectangular. To calculate the perimeter of the rectangle, I use the formula (a + b) x 2.	

In response to the researcher's invitation, the preservice teachers questioned the meaning of the concept of perimeter by considering the student's solution path. Specifically, Preservice Teacher 4 claimed that the student's solution could be correct if the number of dots were counted once. However, at the beginning of her explanation, Maya seemed hesitant about her idea that the distance should be counted to calculate the perimeter. However, she made sure of her idea while explaining her reasoning. In Maya's explanation, Preservice Teacher 3 seemed convinced that counting points would not be used to calculate the shape's perimeter. Maya concluded the discussion by saying that dots are not used in the perimeter calculation and focused on linear units to calculate the perimeterComparing Maya's response to such a question mentioned above, where Maya did not evaluate the student's strategy, and her explanations in the discussions, it could be said that Maya had a chance to put her understanding of the concept of the perimeter on a conceptual basis with the help of the collective analysis of the practice. Considering all the evidence, it could be said that she had a chance to develop her mathematical knowledge of the perimeter concept while engaging in the activities in the *Pre-Preparation* node of the model organized to improve preservice teachers' mathematical knowledge and professional noticing skills related to the perimeter concept. The following table summarizes Maya's developmental process regarding her *overt subject knowledge* during this node.

At the end of this node, Maya volunteered to practice teaching perimeter in the simulated classroom environment. After that, she was asked to prepare a lesson plan to teach the concept of perimeter for the next lesson by setting up dialogues between the teacher and the fifth-grade students. The second node of the model, *Preparation*, was mainly designed to prepare preservice teachers for teaching perimeter and making instructional decisions. In this regard, during the first class period of the implementation of this node, the preservice teachers collectively reviewed Maya's lesson plan to improve the quality of her teaching as part of Activity 1*: Prepare for*

Rehearsal. In this regard, Maya prepared the lesson plan individually by breaking down the exercises into pieces. Then, Maya and the others reviewed her lesson plan by focusing on the questions and dialogues between Maya and the fictional students in the second hour of implementing the second node. This allowed them to examine how Maya defined or explained the concept mathematically by decomposing practice into pieces. As Maya constructed the dialogues in her lesson plan, her explanation of the perimeter concept sometimes caught the preservice teachers' attention. Figure 4.8 shows a sample excerpt from Maya's lesson plan that she prepared as part of the first activity in this node.



Question: The pool consists of a parallelogram whose side lengths are 7 m and 5 m, respectively and the square. Please calculate the perimeter of the given pool.

Maya (teacher): Could you explain the perimeter of the given shape?

Student (fictitious student): The perimeter of the parallelogram is (7+5) x 2 =24 cm, the perimeter of the square is 5 x 4 from 20 cm, and the perimeter of these two shapes obtained is 44 cm.

Maya (teacher): What is the perimeter? It is the sum of the side lengths of the shape. It's like you went inside of the shape like this twice (Evidence of *overt subject knowledge*)

Student (fictitious student): Oh yes, teacher, it would be 34 cm.

Maya (teacher): Well done.

Figure 4.8. Evidence of *overt subject knowledge* in Maya's initial lesson plan.

Table 4.5 Sample evidence of the development of Maya's *overt subject knowledge* during the Pre-Preparation node of the model.

Activity Name	Excerpts from Maya's responses or explanations	Inferences about Maya's *overt subject knowledge*
Activity 1: *What is the Perimeter? Let Us Look at the Concept Through Your and the Student's Eyes (her response)*	I drew a figure for the perimeter to be calculated with the help of unit squares.	She provided misinformation about defining the perimeter concept using improper terms and unit squares.
Activity 2: *Let Us Examine the Dialogue (Her response)*	The student counted the points.	She did not provide any information about the concept's meaning.
Activity 1: *What is the Perimeter? Let Us Look at the Concept Through Your and the Student's Eyes* (*Her explanation during the discussion*)	"The perimeter is the total length measurement around the closed shape."	She used mathematically correct terms in defining the concept.
Activity 2: *Let Us Examine the Dialogue* (*Her explanation during the discussion*)	"I do not know, but for example, when calculating the perimeter, we say the sum of the sides. So, we need to know the distance. So, length. So, we do not need to know the sum of the lengths of the sides for the perimeter. We need to know the distance between two points, that is, how much so we can find the perimeter. We need the distance between the points, not the points. So, we need to have the length."	She made correct inferences about the concept by questioning its meaning from a mathematical aspect.

As shown in Figure 4.8, Maya created a dialog between herself and the fictitious student that might occur when the student calculates the perimeter of the shape given in the question. In this regard, Maya assumed that the student made a mistake by adding the side lengths of the parallelogram and square inside the pool to calculate its perimeter. However, to make the student realize her mistake, Maya defined the perimeter concept without specifying which edges of the figure should be added. When she shared her lesson plan with her friends, she received the following feedback from her friends about this issue:

> **Preservice Teacher 2:** You should correct your definition. It is not the correct definition to say the sum of the side lengths.
>
> **Preservice Teacher 3:** I agree with Preservice Teacher 2. In the perimeter definition, you should emphasize adding the sides as the shape's boundary.
>
> **Maya:** You are right. I will pay attention to these issues in the next lesson.

As understood from the above dialogue, Preservice Teachers 2 and 3 cautioned Maya to provide a conceptual basis for her definition by emphasizing the addition of the sides' lengths at the shape's boundary. In addition, the preservice teachers claimed that this question could lead to students' misconceptions about the learning environment and suggested that this question be removed from the lesson plan. Considering this suggestion, Maya removed such questions and dialogues from the lesson plan. Similarly, during the second and third lessons where the *Preparation* node was applied, it was observed that Maya paid attention to this feedback while teaching in a simulated environment as part of Activity 2: *Rehearse the Perimeter Teaching*. To illustrate:

> **Preservice Teacher 3 (in the role of the student):** I will multiply three by five to find the perimeter of the rectangular shape.

Maya (in the teacher role): But if you do not find the sum of the side lengths of the shape. Who wants to remind you of the definition of what was the perimeter?

Preservice Teacher 2 (in the role of the student): We start from a point and go until we reach that point again. I started from A point and returned to that point again to calculate its perimeter.

Maya (in the teacher role): You had to go around the shape or on the boundary of the shape. Then, you add the side lengths of the shape that you went around to find its perimeter (Evidence of *overt subject knowledge*).

As shown in the dialogue above, Maya rehearsed the perimeter teaching as part of the approximation of the practice, and Preservice Teachers 2 and 3 behaved, questioned, and produced strategies like fifth-grade students in this activity. Maya corrected Preservice Teachers 3 and 2's explanations as a rehearsal teacher by emphasizing the shape's boundary. In doing so, she exhibited exemplary behavior concerning the *overt subject knowledge* of the perimeter concept in her instruction.

Considering these, it could be said that taking feedback from her friends before teaching contributed to developing her *overt subject knowledge* regarding the perimeter concept. Still, it was seen that she did not always provide the mathematically correct definition of the perimeter during her teaching, especially when she faced the students' difficulties with the perimeter. For example, she used the following definition during the rehearsal when one of the preservice teachers who played the role of the student confused the perimeter and area formulas.

Maya (in the role of the teacher): The rectangle in the Figure on the board comprises congruent squares. Calculate the perimeter of the painted square since the longest side of the rectangle is 15 cm. Let us calculate the shape's perimeter.

Preservice Teacher 3 (in the role of the student): My teacher, I divide 15 by five and get three. Then, I multiply three by three. I found 9.

Maya (in the role of the teacher): You find the area of the shape. How did we calculate the perimeter? We were adding the side lengths, right? Now, you should multiply three by four and obtain 12 cm (Evidence of *overt subject knowledge*).

As shown in Maya's explanation, she did not emphasize which sides of the shape should be added to find its perimeter. However, while focusing on Activity 3, *Let Us Examine the Rehearsal,* the preservice teachers emphasized these situations when they encountered Maya's lack of a perimeter definition, as in the example above, as a part of the pedagogies of reflection. This activity involved the examination of the video recordings of the rehearsal, aiming to improve Maya's teaching as part of pedagogies of reflection, and was conducted during the fourth and fifth lesson hours of the implementation of the same node. For example,

Preservice Teacher 3: You directed the student to the solution way. However, you did not show or emphasize which sides of the shape should be added in your explanation.

Preservice Teacher 2: You should be more careful about the terms used to describe the perimeter calculation.

Maya: Okay.

As can be seen from the dialogue above, the pre-service teachers gave Maya feedback about being careful with her preferred terms for defining the concept. Namely, Maya was reminded to emphasize the shape's boundary when describing the perimeter. This kind of analysis was based on the analysis of the reflective perspective because it depended on Maya's teaching action and her reflection on it. Considering this feedback, Maya was asked to revise her lesson plan as homework in Activity 4: *Prepare for Enactment,* located at the same intervention node. In

reviewing her revised lesson plan as part of the decomposing and approximating practice, Maya emphasized the mathematically correct definition of the concept. She demonstrated exemplary behaviors in developing her *overt subject knowledge.* To illustrate, Maya defined the perimeter concept in her initial lesson plan as follows: "*What is the perimeter? It is the sum of the side lengths of the shape. It's like you went inside of the shape like this twice,"* as shown in Figure 4.9. As stated before, this definition was mathematically incomplete because it did not specify the sum of the edges around the shape. When Maya's final lesson plan prepared under the second node of the model was examined, it was seen that she tended to use mathematically essential terms for the perimeter definition, such as the boundary of the shape. The following table summarizes Maya's developmental process regarding her *overt subject knowledge* during the second node of the model.

In the first and second lesson hours of implementing the third node of the model called Implementation, Maya had an opportunity to teach the perimeter concept in one of the fifth-grade classrooms. She taught in the practice lesson hour of the Teaching Practice 2 course as part of pedagogies of enactment. This activity, called Activity 1: *Enact the Perimeter Teaching*, was carried out to enable preservice teachers to make instructional decisions and conduct perimeter teaching in an actual classroom environment as a part of the *Implementation* node of the model. Meanwhile, the other preservice teachers, the instructor, and the cooperating teacher attended Maya's lesson as the observer. During her teaching, it was seen that she mentioned some critical terms, such as the boundary of the shape and being closed in her perimeter definition. For example,

> **Student 5 (in the actual classroom):** The square's side length is five cm. When I multiply five by four, I obtain 20 cm.
>
> **Maya (teacher):** You are right. Let us remind you of the perimeter definition. The perimeter measures the length around the closed shape, right? Your friend added the lengths of the sides around the square's shape (Evidence of *overt subject knowledge*).

Table 4.6 Sample evidence of the development of Maya's *overt subject knowledge* during the Preparation node of the model.

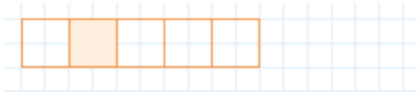
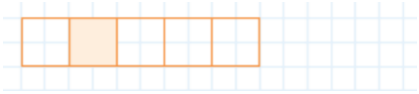
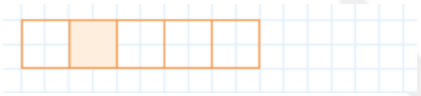
Activity Name	Excerpts from Maya's responses or explanations	Inferences about Maya's *overt subject knowledge*
Activity 1: *Prepare for the Rehearsal*	 Maya (teacher): The rectangle in the Figure on the board comprises congruent squares. Calculate the perimeter of the painted square since the longest side of the rectangle is 15 cm. Who wants to solve the question? Student (fictitious student): My teacher. May I solve it? Maya (teacher): Sure. Student (fictitious student): Divide 15 by five. We get three cm. If the side of the square is three cm, I find the perimeter of 12 cm. Maya (teacher): Well done.	She confirmed the student's mathematically correct solution.
Activity 2: *Rehearse the Perimeter Teaching!*	 Maya (in the role of teacher): The rectangle in the Figure on the board comprises congruent squares. Calculate the perimeter of the painted square since the longest side of the rectangle is 15 cm. Let us calculate the shape's perimeter. Preservice Teacher 3 (in the role of the student): My teacher, I divide 15 by five and get three. Then, I multiply three by three. I found 9. Maya (in the role of the teacher): You find the area of the shape. How did we calculate the perimeter? We *were adding the side lengths*, right? Now, you should multiply three by four and obtain 12 cm.	She did not emphasize the boundary of the shape.

Table 4.6 (Continued).

Activity Name	Excerpts from Maya's responses or explanations	Inferences about Maya's *overt subject knowledge*
Activity 3: *Let us Examine the Rehearsal*	Preservice Teacher 3: You directed the student to the solution way. However, you did not show or emphasize which sides of the shape should be added in your explanation. Preservice Teacher 2: You should be more careful about the terms you use to describe the perimeter calculation. Maya: Okay.	She was encouraged to use essential terms to define the concept.
Activity 4: *Prepare for the Enactment*	 Maya (teacher): The rectangle in the Figure on the board comprises congruent squares. Calculate the perimeter of the painted square since the longest side of the rectangle is 15 cm. Who wants to solve the question? How do we calculate the shape's perimeter? Student (fictitious student): By adding all the side lengths, the shape. Maya (teacher): *All sides of the shape? Can I add these sides exist inside the shape?* Student (fictitious student): We can add these shapes existing around the shape. Maya (teacher): Well done. *We should add the sides' lengths around the shape.* So, how do we calculate the shaded shape's perimeter? Student (fictitious student): Divide 15 by five. We get three cm. If the side of the square is three cm, I find the perimeter of 12 cm. Maya (teacher): Well done.	She reminded the student of the mathematically correct way to define the perimeter.

As shown in the dialogue above, Maya tended to remember the concept definition and procedures while explaining how Student 5 calculated the shape's perimeter. She displayed exemplary behavior regarding her overt subject knowledge since she defined the concept correctly during her teaching. Unfortunately, during her teaching, it was seen that she sometimes failed to define the concept, especially when she faced students' difficulties with the perimeter concept. A similar situation was observed in the rehearsal. The following excerpt from her perimeter teaching in the actual classroom was presented as an example of such a situation.

> **…**
>
> **Student 6 (in the actual classroom):** We multiply two sides to find the perimeter of the shapes.
>
> **Maya (teacher):** How could we find the perimeter? Could we multiply the sides?
>
> **Student 3 (in the actual classroom):** We add the sides and multiply by two.
>
> **Maya (teacher):** What else?
>
> **Student 2 (in the actual classroom):** We add the sides of the shapes.
>
> **Maya (teacher):** Yes, thank you. We can find the perimeter by adding all the side lengths. To summarize, when calculating the perimeter of the shapes, we add all the side lengths. What we did with the square's side length was equal to four times the side length of a square. So, we should multiply three by four to calculate the shape's perimeter (Evidence of *overt subject knowledge*).

As seen in the dialogue above, Maya, the classroom teacher, described the perimeter based on addition operation without mentioning critical terms such as the shape's boundary. Moreover, she misinformed the students by saying, "*adding all sides of the shapes*." In Activity 1: *Let Us Examine the Enactment,* they analyzed the video

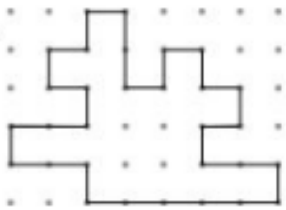
recordings of Maya's perimeter teaching in the first and second lesson hours of implementing the fourth node of the model, called *post-Preparation,* as a part of the pedagogies of reflection. In this regard, the preservice teachers suggested that Maya improve her discourse. For example, they told Maya to be careful with her perimeter definition as follows:

> **Preservice Teacher 3:** You should be careful how you define the concept. It would be best if you emphasized which edges would be added.
>
> **…**
>
> **Maya:** I know, but I was pretty excited about the enactment. Good thing I rehearsed; otherwise, I do not know what to do. I said whatever came to my mind when faced with unexpected situations.

The dialogue above indicated that Preservice Teacher 3 warned Maya to describe the concept correctly and use critical terms in her definition. Meanwhile, Maya stated she was excited and did not know what to do during her teaching, especially when faced with unexpected thoughts. It could be said that such suggestions helped her review her mathematical knowledge regarding the perimeter concept since it was observed that she gave place to some dialogues where students had misconceptions about the perimeter in Activity 2: *Prepare for Further Teaching*. This activity was assigned as homework and located at the same node. For example, she added the following question into her lesson plan, considering her experiences so far, and created the following dialogue between herself and a fictitious student, as shown in Figure 4.9.

Example excerpts above indicated that she focused on defining the concept using mathematically correct terms, which was an exemplary behavior regarding the *overt subject knowledge* observed in the fourth node of the model called *post-preparation.* The following table summarizes Maya's developmental process regarding her *overt*

subject knowledge during the third and fourth nodes of the model, called *enactment* and *post-preparation*, respectively.



Question: The distance between both dots is 3 cm on the above-dotted paper. What is the perimeter of this shape, then?

Maya (teacher): What is the perimeter of the given shape?

Student 1 (fictitious student): 93

Maya (teacher): How did you calculate it?

Student 1 (fictitious student): I multiply three by 31.

Maya (teacher): Why?

Student 1 (fictitious student): I counted the points and obtained 31. Then, I multiplied 31 by three and obtained 93.

Maya (teacher): Are there any solutions?

Student 2 (fictitious student): Teacher, I counted the distance between two points and found 30. I know that the distance between two points is 3 cm, and the perimeter of the given shape is 90 cm.

Maya (teacher): Yes, but it is not 90, right? It is 90 cm. *We need to emphasize the units in expressing the result* (Evidence of *overt subject knowledge*).

Student 2 (fictitious student): You are right, teacher.

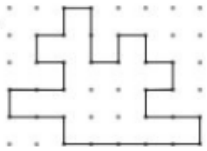
Maya (teacher): What did your friend do? *He added the lengths of the sides around the shape to find its perimeter* (Evidence of *overt subject knowledge*).

Figure 4.9. Evidence of *overt subject knowledge* from Maya's final lesson plan.

Table 4.7 Sample evidence of the development of Maya's *overt subject knowledge* during Enactment and Post-Preparation nodes of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *overt subject knowledge*
Activity 1: *Enact the Perimeter Teaching!* (*In the third node called Implementation*).	Student 6 (in the actual classroom): We multiply two sides to find the perimeter of the shapes. Maya (teacher): How could we find the perimeter? Could we multiply the sides? Student 2 (in the actual classroom): No. We add the sides of the shapes. Maya (teacher): Yes, thank you. We can find the perimeter by adding all the side lengths. To summarize, when calculating the perimeter of the shapes, we *add all the side lengths*. What we did with the square's side length was equal to four times the side length of a square. So, we should multiply three by four to calculate the shape's perimeter.	She misinformed the student about the perimeter by saying, "Adding all the side lengths."
Activity 1: *Enact the Perimeter Teaching!* (*In the third node called Implementation*).	Student 5 (in the actual classroom): The square's side length is five cm. When I multiply five by four, I obtain 20 cm. Maya (teacher): You are right. Let us remember the perimeter definition. *The perimeter measures the length around the closed shape, right? Your friend added the lengths of the side around the square's shape.*	She defined the concept by giving place to critical terms in her definition.
Activity 1: *Let us Examine the Enactment* *(In the fourth node called post-preparation).*	*(When Maya said "adding all sides of the shapes" to define the perimeter calculation in her teaching mentioned above, the preservice teachers commented.).* Preservice Teacher 3: You should be careful how you define the concept. It would be best if you emphasized which edges would be added. … Maya: I know, but I was pretty excited about the enactment. Good thing I rehearsed; otherwise, I do not know what to do. I said whatever came to my mind when faced with unexpected situations.	She was encouraged to define the concept correctly.

Table 4.7 (Continued).

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *overt subject knowledge*
Activity 2: *Prepare for Further Teaching (In the fourth node called post-preparation*).	 … Maya (teacher): What did your friend do? *He added the lengths of the sides around the shape to find its perimeter.*	She reminded the students of the perimeter calculation by using mathematically correct terms.

In addition to developing her mathematical knowledge of the perimeter concept, improving her knowledge of the trapezoid was also observed when the preservice teachers were engaging in Activity 1: *Let us Examine the Enactment* in the first and second lesson hours of implementing the *Post-Preparation* node as part of the decomposition and approximation of the practice. To be detailed, Maya defined a trapezoid with different sides in her perimeter teaching as follows:

> **Student 1 (the actual student):** Okay. Then, I drew my trapezoid. One side is 6 cm because we will divide by 4, and we will divide by 24. I drew an even trapezoid.
>
> *(She made the following explanations on the board shown in Figure 4.10).*



Figure 4.10. Student 1's explanations.

> **Maya (teacher):** It would not be a trapezoid, would it? When you divide it by 4, a rhombus is formed; when you divide it by four, a square is formed, right? What happens when the perimeter of 24 cm is divided by 4?
>
> **Student 2 (the actual student):** My teacher made the long side ten and the others 2, and their total is 24. I made the long side ten and the others 2, which total 24 cm.
>
> **Maya:** What did we create? We created a square, a rectangle, and an equilateral one, and let us create a trapezoid, shall we? I also created it for

you. Let me show you. So, I drew a trapezoid with all sides different in the perimeter of 24 cm (Evidence of *overt subject knowledge*).

In this dialogue, Student 2 tried to draw the trapezoids with different side lengths, as shown in the dialogue. However, Maya insisted that the length of each side was not equal, precisely six centimeters; thus, the side lengths of Student 2's drawings were incorrect. This dialogue caught the attention of the preservice teachers during the analysis of the video recording of the enactment. Then, they highlighted that the side lengths of the trapezoid did not have to be different as follows:

Preservice Teacher 3: Do you think a trapezoid means all edges differ? Would not we mention isosceles trapezoid?

Preservice Teacher 2: Or is the square not a kind of trapezoid?

Maya: Actually, you are right. We could mention isosceles trapezoid. I did not know why I said that.

As shown in the dialogue above, Preservice Teachers 2 and 3 directed questions to make Maya notice her mistake on the trapezoids. Maya also stated the following about this issue during the post-interview:

I did not notice I gave misinformation about the quadrilaterals. I just focused on how I could handle the student's saying since I did not expect such a student's solution. We did not discuss such kind of solution before stages. I guess I had a lack of information about the quadrilaterals. I noticed this situation in my friends' explanations.

As understood from Maya's explanations, the preservice teachers' explanation allowed her also to re-construct or review her mathematical knowledge regarding the trapezoids, which contributed to developing her *overt subject knowledge* regarding the trapezoids and, consequently, the perimeter concept.

In summary, Maya had the opportunity to develop her mathematical knowledge of the perimeter concept during the model of the current study, which consisted of four nodes: *Pre-Preparation, Preparation, Implementation*, and *Post-Preparation*. More specifically, it was found that Maya questioned the meaning of the perimeter concept and developed her knowledge from misinformation about the concept to conceptual knowledge as she engaged in productive discussions during the model. In other words, the discussions helped Maya to revise or co-construct her mathematical knowledge about the perimeter concept. In addition, she created dialogues between Maya and fictional students while preparing lesson plans related to perimeter concepts. Maya had the opportunity to develop her discourse about the definition or explanation of the concept with the help of the preservice teachers' explanations. In addition, she had the opportunity to conduct perimeter instruction in simulated and actual classroom environments. When there were shortcomings in her definition or explanation of the perimeter, the other preservice teachers alerted her to correct her explanations by reviewing the video recordings of her teaching, which helped her to reconstruct her mathematical knowledge of the perimeter concept. In addition, Maya had the opportunity to co-construct her knowledge of the trapezium while watching the video recordings of her perimeter lessons, which indirectly helped her develop her knowledge of the perimeter concept. Since all contributed to her defining the concept of the perimeter from conceptual aspects and explaining the underlying reasons for perimeter calculations, it could be said that her *overt subject knowledge* of the concept of the perimeter was enhanced when she was involved in the activities in the nodes of the model. The following section provides detailed information on her development *using terminology* knowledge of the perimeter concept during the intervention.

4.1.2 Mathematical Knowledge: *Using Terminology*

This section investigated how Maya used mathematical language and terminology regarding the perimeter, considering her pre- and post-test responses and post-interview

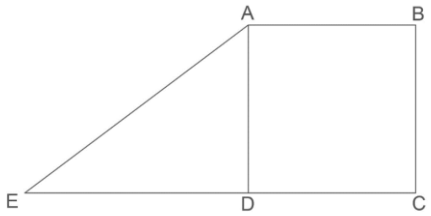
explanations. As Table 4.1 shows, she developed her knowledge of *using terminology* as she was involved in the nodes of the model. Furthermore, the following table summarizes the findings regarding Maya's current *using terminology* knowledge and shows the development in Maya's knowledge from the pre-assessment to the post-assessment.

Table 4.8 Development of Maya's knowledge of *using terminology*.

Pre-intervention based findings	Post-intervention-based findings
-She misused mathematical language on the perimeter concept.	-She used mathematical language on the perimeter concept correctly.
-She ignored using units of length and expressions.	-She paid attention to the use of units of lengths and expressions.

As shown in Table 4.8, Maya's responses in the pre-assessment indicated that she did not use mathematically correct language on the perimeter concept. Specifically, she misused mathematical language in defining the perimeter concept by saying, "*The length that covers a shape or object*." It was seen that Maya misused mathematical language for the perimeter concept by giving place to the "covering" term in her definition. Similarly, when explaining the underlying reasons for the perimeter measurement processes in the pre-test, it was observed that she described the perimeter calculation as follows: "*We can calculate the perimeter by summing the lengths that cover a shape or object.*" Her explanations showed that she again misused the mathematical language on the perimeter concept by saying the "cover" term. In addition to mathematical language, it was seen that she ignored using mathematical expressions correctly in her response. For example, when asked to solve the question in Figure 4.11, it was observed that Maya did not use units of length and expressions in her answer, as shown in Figure 4.11.

The Question: ABCD is a square, and ADE is a right triangle. Since the perimeter of triangle ADE is 48 cm, IECI = 28 cm, and IABI = 12, calculate the perimeter of trapezoid ABCE as much as possible ways.



Maya's response to it in the pre-test: The perimeter ADE triangle is given 48 cm. Since AD is on one side of the square, it is 1two cm. The side EC is 28 cm, and the side DC is also one side of the square. It is 12 cm. Therefore, the side ED is 16 cm. Since we know the two sides of the triangle, we need to find the 3rd side. Since the perimeter of the triangle is given, we find 20 cm from 48-28. We can add the side EA + EC + AB + BC to calculate the perimeter to get two cm.

Figure 4.11. The problem and evidence of *using terminology* knowledge from Maya's solution to it in the pretest.

As shown in her response, Maya did not use mathematical expressions such as equal signs and absolute value before attending the intervention. She explained her solution through verbal expressions rather than mathematical symbols. Besides, it was seen that Maya did not calculate the shape's perimeter or reach the result. She just mentioned how to calculate the perimeter of the shape. In addition, it was seen that Maya did not always pay attention to using the measures of the lengths in the expressions. To illustrate, when asked to evaluate the student's strategy mathematically, as shown in Figure 4.12, it was seen that she ignored missing the units in the student's answer and accepted the student's strategy as correct. More specifically, Maya responded: " *I think the student's solution is correct because she made the side lengths of the given shape a straight line."* Her response implied she ignored expressing the results with units. Besides, she again misused the mathematical language and said straight line rather than line segment in her response.

To sum up, Maya misused the mathematical language regarding the perimeter concept. She ignored using mathematical expressions and measures of length correctly before attending the intervention of the current study. For these reasons, it could be said that she lacked *using terminology* knowledge of the perimeter concept.

However, it was seen that she paid attention to using mathematically correct language regarding the perimeter concept in her responses after her involvement in the intervention. For example, while defining the perimeter concept in the post-test, she responded: "*When we start from the point of a closed shape and go around the shape when we reach the starting point again, we get the total length called the perimeter."* As shown from her response, she used mathematically correct terms such as "closed shape," "total lengths," and "go around the shape" in defining the perimeter concept. Similarly, she used mathematically correct language in determining the perimeter calculation process. She described it in the post-test: "*We can find the perimeter of a shape by adding outside lengths or measuring the total length of the boundary of the shape*." She again used terms such as "adding outside lengths or total lengths of the boundary of the shape," which belongs to the perimeter measurement process. Moreover, it was seen that she tended to pay attention to the mathematical expressions in her solution to the problem, as indicated in Figure 4.12

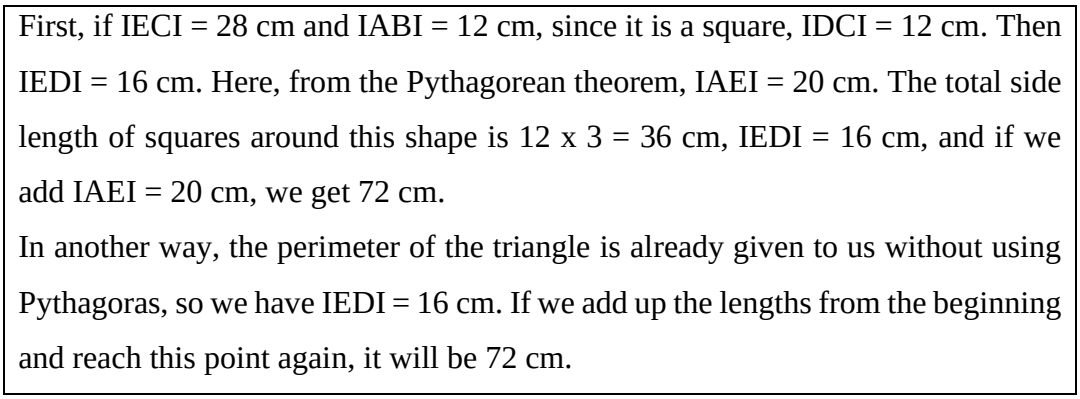

First, if IECI = 28 cm and IABI = 12 cm, since it is a square, IDCI = 12 cm. Then IEDI = 16 cm. Here, from the Pythagorean theorem, IAEI = 20 cm. The total side length of squares around this shape is 12 x 3 = 36 cm, IEDI = 16 cm, and if we add IAEI = 20 cm, we get 72 cm.
In another way, the perimeter of the triangle is already given to us without using Pythagoras, so we have IEDI = 16 cm. If we add up the lengths from the beginning and reach this point again, it will be 72 cm.

Figure 4.12. Evidence of *using terminology* knowledge from Maya's solution to the problem in the post-test.

As shown in her response, Maya tended to use mathematical expressions such as equal signs and absolute values in her response while she was calculating the perimeter of the given shape. She was inclined to explain her solution by using mathematical symbols. However, before attending to the intervention, it was seen that she did not have calculated the shape's perimeter. She just specified how to calculate the perimeter of the given shape.

Moreover, as seen in Figure 4.12, she paid attention to using units in her response. In addition, it was seen that she noticed the given students' answers (see Figure 4.4) was missing due to not expressing the measures of the length. Her response to the post-test was as follows:

> The student's strategy is correct, but her expression of the result is wrong. She found the perimeter of the figure by calculating the total length formed by adding the lengths of the sides around the shape end to end through the ruler. She may have forgotten to mention units of measure out of excitement. This solution method can be used in lessons.

As understood from her response, she did not accept the given student's answer as correct. She emphasized that the student should have used the units in her response. Since she used mathematical language correctly on the perimeter concept and paid attention to mathematical expressions correctly, her *using terminology* knowledge of the perimeter concept was robust when she was involved in the intervention. Her explanations during the interview also confirmed such inferences. Specifically, she stated, "*…(Before) I did not pay attention to units. However, I learned that saying the number does not make sense without its units*". As understood from her statements, she did not grasp the importance of using units to calculate the shape's perimeter before attending the intervention. She just focused on reaching the number as the solution to the problem. However, during the intervention, she had a chance to re-revise her mathematical language on the perimeter concept. She paid attention to using mathematical language correctly and used units and mathematical expressions in her responses. Besides, she stated during the interview that her

mathematical language improved as she conceptually understood the perimeter's meaning. Her explanation during the post-interview was as follows:

> We examined the meaning of the perimeter concept first. Indeed, we discussed what the perimeter means, which property it has, etc. This way, I noticed that the perimeter is a length and belongs to a closed shape. Later, when I noticed it was a kind of length, I started to use or pay attention to using units in my response. Besides, as I grasped the meaning of the perimeter concept, I used the correct terms to define the concept.

Her explanations implied that the development of her mathematical knowledge influenced her mathematical language positively. Overall, since Maya used mathematical language and terminology correctly for the perimeter concept when examining her responses in the post-test and explanations during the interview, it could be said that she had a robust knowledge of *using terminology* regarding the perimeter concept after participating in the intervention. In other words, it could be said that she had made remarkable progress in her knowledge of *using terminology* regarding the perimeter concept during her involvement in the intervention of the current study. Below, evidence for improving *using terminology* knowledge during the intervention is presented.

4.1.2.1 Evidence for Improvement in *Using Terminology* Knowledge during the Model

As stated above, Maya's knowledge of the *using terminology* regarding the perimeter concept was developed as she was involved in the nodes of the model. In this section, the evidence that she developed this knowledge during her participation in the intervention of the current study is shared, paying attention to the sequence of activities in each node of the model, as well as the sequence of the nodes of the model, which are called *Pre-Preparation, Preparation, Implementation,* and *Post-Preparation,* respectively. The excerpts from the current intervention's video

transcriptions and individual responses to classroom activities were shared to describe Maya's progress in *using terminology* knowledge during the intervention.

As displayed in Table 4.1, evidence for the development of Maya's knowledge of the *using terminology* started to be observed in the first node of the *Pre-Preparation*. In this node, the preservice teachers were asked to examine the activities called Activity 1: *What is the Perimeter? Let Us Look at the Concept Through Your and the Student's Eyes*, and Activity 2: *Let Us Examine the Dialogue* individually and collectively. Indeed, the preservice teachers were asked to respond to all questions individually in the first lesson hour and then explain their reasoning in the second lesson hour of implementing the *Pre-Preparation* node of the model. In the first activity, there were questions about drawing several shapes that represent the perimeter concept and not the perimeter concept and solving the given question as much as possible in the examining concept meaning activity. More specifically, there were several questions in the first activity to examine and support the development of the preservice teachers' mathematical language and terminology on the perimeter concept. Within this context, the preservice teachers were asked to draw the shapes to present the perimeter, not the perimeter, explaining their reasoning as stated in Table 4.3. Moreover, they were asked to solve the given question, aiming to examine and support the development of her mathematical terminology. Specifically, the first three questions in the first activity, called examining the concept used in this node of the model to extend their knowledge of *using terminology*, were presented in Table 4.3.

In the first hour of implementing the *Pre-Preparation* node, the preservice teachers individually responded to the questions in the first and second activities. Her response explained the underlying reasons for such drawings: "*I drew a figure for the perimeter to be calculated with the help of unit squares. Since the other one has no closed shape, I thought its perimeter could not be calculated*." As shown in her explanations, she used "*unit squares*" in her perimeter definition. However, it was an

inappropriate unit of measure to describe the perimeter. Rather than focusing on the square units, she could have focused on the linear units in her explanations.

On the one hand, it was seen that she started to use mathematically correct language by questioning the place of "side lengths around the shape," "boundary," "curve," "being a circle," "being closed," and "total length around the closed shape" terms when explaining the reasoning about such drawings in the second lesson hour of the implementation of this node of the model. Below are exemplary excerpts from the discussion held within the scope of these questions. Meanwhile, from now on, some words in Maya's explanations in the exemplary excerpts were italicized to emphasize the evolution of her mathematical language on the perimeter concept.

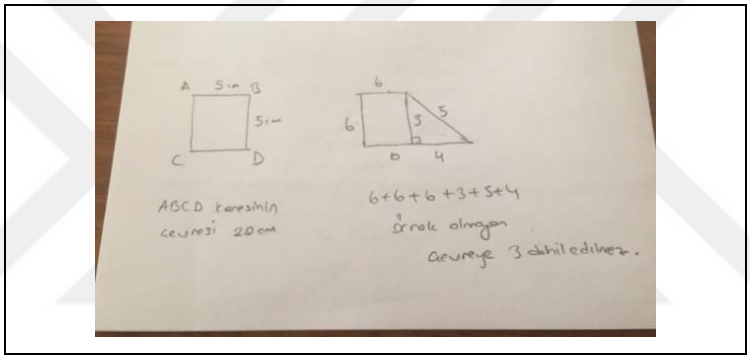


Figure 4.13. Evidence of *using terminology* knowledge from Preservice 5's drawing of the perimeter and not-perimeter.

Preservice Teacher 5: Teacher, I think these figures represent the perimeter and non-perimeter of the shape. For example, the perimeter of the square is 20 cm. Students often include this edge to calculate the perimeter (Her drawings are presented in Figure 4.13).

Maya: Actually, it is true; when several shapes come together, we only need to get the *outside line segments*, not the line segments in the middle or inside. In other words, instead of finding and adding all side lengths of the shapes, we focus on the shape's *boundary*. You are correct. Not all line segments.

That is right; I never thought of it that way, but I like it (Evidence of *using terminology* knowledge).

Preservice Teacher 2: Sir, can we say that if the shape is closed, no matter how the sides of it, whether linear or non-linear shapes, it represents the perimeter? Several closed regions with different structures could represent the perimeter examples, and concave shapes, including line segments, do not represent the perimeter (Her drawings are presented in Figure 4.14).

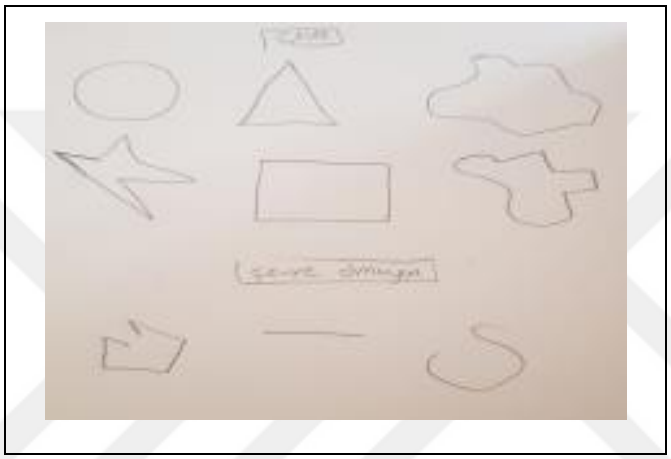


Figure 4.14. Evidence of *using terminology* knowledge from Preservice 2's drawing of the perimeter and not the perimeter.

Researcher: A different perspective has come. Non-linear, said your friend; what do you think?

Maya: I did not think that it should be linear. It can be a *circle* if it does *not have to be linear* (Evidence of *using terminology* knowledge).

Researcher: Your friend said circle; what do you think?

Preservice Teacher 2: I will come back to my definition. My teacher started from one point and returned to the same point, meaning the shape was closed. When we opened the circle, it became a length. I think it happens; there are linear and nonlinear ones.

Preservice Teacher 3: How accurate would it be to say the sum of the side lengths starting from a point and reaching that point around the shape, sir? Should we tell the measurement of the total length around the shape in the definition?

Researcher: Great, let us define the perimeter.

Maya: The perimeter is the *total length measurement around the closed shape* (Evidence of *using terminology* knowledge).

As seen in the excerpts above, she had the opportunity to improve her mathematical language during the discussion. Indeed, before the discussions, she misused the terms in her responses when she defined the concept by saying, "*…with the help of unit squares."* As shown in the dialogue above, she emphasized: *"outside line segments" (line segments) not be linear" and "total length measurement around the closed shape"* terms during the discussions, respectively. Considering these, it could be said that she noticed that the perimeter is a kind of linear measurement. Moreover, it was seen that as she put her mathematical knowledge of the perimeter concept on a conceptual basis, it was seen that she developed her mathematical language. In other words, a reflection of the development of her mathematical knowledge was observed in her mathematical language.

In addition to these questions in the first activity, since the solutions to the problem gave information about her *using terminology* knowledge, the third question in the same activity, as shown in Table 4.3, was directed to the preservice teachers to solve it with as much as possible as stated before. Maya responded: "*I counted units and found 38 on the perimeter of the shape*". As shown from her response, she did not express the length measures; consequently, her solution was not accepted as mathematically correct. During the second hour of the lesson in the first week of the model, discussing solutions using mathematical expressions might help Maya explain her reasoning using mathematically correct terms for the perimeter concept. For example,

Preservice Teacher 2: We need to count units to solve the problems.

Preservice Teacher 3: We need to count units around the shape.

Maya: I could solve the question by counting the units between two points around the shape and found the perimeter of the shape 38 units (Evidence of *using terminology* knowledge).

As understood from the dialogue above, Maya used mathematically correct terms in her definition, such as "around the shape," and gave place to the measure of length in her explanations. Since she gave place to critical terms and emphasized the measures of the lengths in her explanation during the discussion, it could be said that Maya displayed exemplary behavior regarding the *using terminology* knowledge under this activity.

Furthermore, as stated above, the development of her *using terminology* knowledge was observed while the preservice teachers were asked to examine Activity 2: *Let us Examine the Dialogue*. In this regard, the preservice teachers were asked to examine the dialogue shown in Table 4.4 as part of the representation of the practice and respond to the prompts regarding this dialogue individually right after completing Activity 1 in the first lesson hour of the first week of the implementation of the *Pre-Preparation* node. In this way, while doing Activity 2, Maya had the opportunity to reflect on the development in her mathematical knowledge of the concept of the perimeter in the mathematical language she used. For instance:

Researcher: What do you think about the student's strategy? Do you think it's true?

Maya: Wrong, my teacher.

Researcher: Why?

Maya: She did *not count the length between* two points but *counted the points* (Evidence of *using terminology* knowledge).

Preservice Teacher 3: I disagree with you. The solution was right because when we count the points of a closed shape like this, we calculate its perimeter since the number of points equals the number of line segments.

Researcher*:* Do you agree with Ali's opinion?

Preservice Teacher 4*:* I was very conflicted, but I thought the student's solution was wrong while answering. However, I tried what PST1 said in other shapes, which could be true.

Maya: I do not know, but for example, when calculating the perimeter, we say the sum of the sides. So, *we need to know the distance*. So, *length.* So, we do not need to know the *sum of the lengths of the sides* for the perimeter. We need to know the distance between two points, that is, how much so we can find the perimeter. We need *the distance between the points, not the points*. So, we need to have the *length (*Evidence of *using terminology* knowledge).

Preservice Teacher 2: Since dots are not units.

Maya: I do not think *counting the point is used in perimeter calculation*. In addition, if we accept the student's solution method is correct like Preservice Teacher 3 says, the students and we can generalize it to other situations. For example, let us image a rectangle. They could say that the rectangle's perimeter is four because it has four corners. So, I think this solution method was wrong, too (Evidence of *using terminology* knowledge).

Preservice Teacher 5*:* You are right.

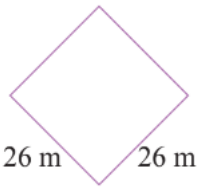
As understood from the dialogue above, Maya displayed exemplary behavior *using terminology* knowledge by using correct expressions such as "*the distance"* while explaining her reasoning for the perimeter. Considering the evidence, it could be said that Maya developed her mathematical language while engaging in the activities in the first node of the model, called *pre-preparatio*n. The following table summarizes

Maya's developmental process regarding *using terminology* knowledge during this node.

Table 4.9 Sample evidence of the development of Maya *using terminology* knowledge during the Pre-Preparation node of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *using terminology* knowledge
Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes (Her response).*	I drew a figure for the *perimeter* to be calculated with the help *of unit squares*.	She misused mathematical language in defining the perimeter by saying "unit square."
Activity 2: *Let Us Examine the Dialogue (Her response).*	The student counted the points.	She did not provide any information about her mathematical language regarding the perimeter.
Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes* (*Her explanation during the discussion*).	*"…outside line segments,"* "…a *circle* if it does *not have to be linear*." "*The perimeter is the total length measurement around the closed shape*."	She used mathematically correct language in the perimeter definition.
Activity 2: *Let Us Examine the Dialogue* (*Her explanation during the discussion*).	"*…counting the point is not used in perimeter calculation*." "*We need to know the distance."*	She used mathematically correct terms regarding the perimeter in her explanations.

At the end of this node, Maya was asked to prepare a lesson plan to teach the perimeter concept to fifth-grade students by creating dialogues between the teacher (herself) and fictitious students as part of Activity 1: *Prepare for the Rehearsal*. This activity under the second node of the model, called *Preparation*, was mainly designed to prepare preservice teachers for perimeter teaching and making instructional decisions in light of the decomposition and approximation of the practice. During this activity, she sometimes displayed exemplary behavior regarding *using terminology* knowledge by using mathematical language correctly. The following excerpt was taken from Maya's initial lesson plan.



Maya (teacher): Let us calculate the perimeter of the rhombus-shaped garden above.

Student (fictitious): My teacher, I multiply 26 m by four. I obtained 104 m.

Maya (teacher): You *added the lengths of the sides around the shape* to find the perimeter (Evidence of *using terminology* knowledge).

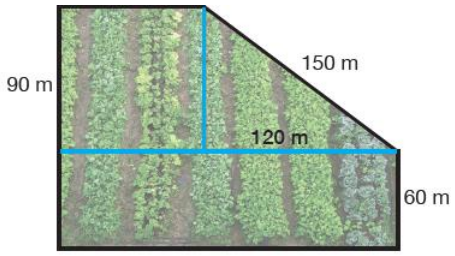
Figure 4.15. Evidence of *using terminology* knowledge from Maya's initial lesson plan.

As shown in the lesson plan, she used the mathematical language correctly by saying, "…*adding sides lengths around the shape*". During the first lesson hour of the implementation of the second node of the model called *Preparation,* the preservice teachers examined Maya's lesson plan collectively, aiming to improve the quality of her teaching. In doing so, they focused on how Maya used mathematical language and terminology in her lesson plan and gave her feedback if necessary. To illustrate, when Maya constructed the dialogues in her lesson plan shown in Figure 4.16, her language regarding the perimeter concept attracted the preservice teachers' attention. Indeed, Maya gave place to the following dialogue in her lesson plan.

As shown in Figure 4.16, Maya ignored using length measures in the dialogue. The preservice teachers noticed such situations while examining her lesson plan and provided the following recommendations concerning the above dialogue:

> **Preservice Teacher 2:** In the student's answer, it was seen that she said 120, 210, and 60 in solving the question. It would help if you were careful to use measures of length always.
>
> **Preservice Teacher 4:** Units are essential.

Maya: You are right.



Maya (teacher): On the right, some side lengths of a garden consisting of square, triangular, and rectangular sections are given. How many meters of wire would it take to run two rows around this garden?

Student (fictitious student): Teacher, I found it. There's a square there. It goes down to 120. The bottom is 210. The Opposite is 60. Now 60+90+90+150+60+210=660 (Evidence of *using terminology* knowledge).

Maya (teacher): Yes, well done

Figure 4.16. Evidence of *using terminology* knowledge from Maya's initial lesson plan.

As shown in the dialogue above, the preservice teachers highlighted the importance of the lengths of measures in perimeter teaching. This kind of feedback helped Maya carefully use units in her explanations and contributed to developing her knowledge of *using terminology*. At the same, this suggestion could contribute to the constructing knowledge regarding the perimeter as a kind of measurement and involves assigning a numeric value to the total length of a closed shape. Meanwhile, while examining her lesson plan, the preservice teachers encouraged Maya to add some activities to her lesson plans. To illustrate, they suggested that Maya add the following activity to her lesson plan.

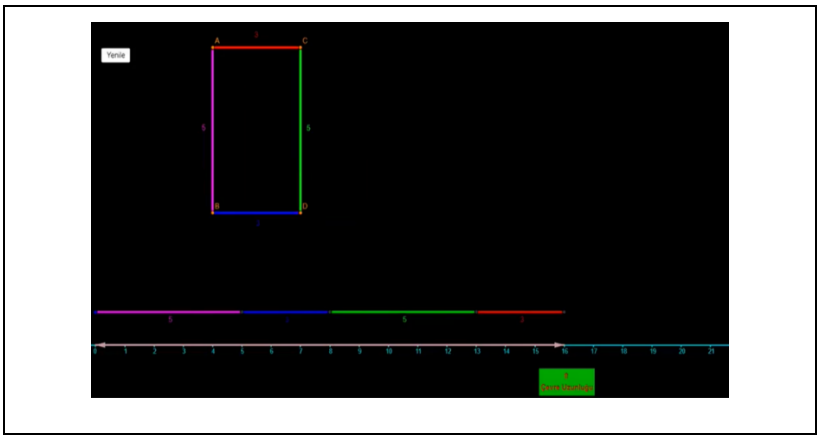


Figure 4.17. GeoGebra Activity suggested by the preservice teacher.

In this activity, a rectangle with each side of a different color existed, as shown in Figure 4.17. The students can put these side lengths on the number line and measure the perimeter of the rectangle. They can create various rectangles with various side lengths with the help of GeoGebra. However, the side lengths of rectangles in GeoGebra did not include the lengths of measure. The preservice teachers and Maya overlooked such mistakes at this stage.

Consequently, Maya gave place to this activity in her perimeter teaching as part of Activity 2: *Rehearse the Perimeter Teaching* conducted in the second and third lesson hours of node implementation. In this activity, Maya had the role of teacher, and other preservice teachers were the role of students as part of the approximation of the practice. During this activity, some of the preservice teachers in the role of students noticed that the side lengths of the figures did not include the length measures. Then, they tried to make Maya actualize such a mistake during the rehearsal as follows:

> **Preservice teacher 2 (in the role of the fifth-grade student):** The teacher, the length between AC is 5.
>
> **Maya (in the role of the teacher):** Yes, 5.

Preservice teacher 3 (in the role of the fifth-grade student): Is the distance between two points possible to be five?

Maya (in the role of the teacher): Yes, why not? (Evidence of *using terminology knowledge*).

Even though Preservice Teachers 2 and 3 implied that Maya's discourse was improper, she did not notice her mistake, as shown in the above excerpt. While examining the video recordings of the rehearsal as part of pedagogies of reflection through Activity 3, called *Let Us Examine the Rehearsal*, which was applied in the fourth and fifth lesson hours of the same node, the preservice teachers emphasized the following points:

Preservice teacher 2: We discuss length in expressing the distance between two points. I told you to highlight the unit by asking such a question, but you did not notice.

Maya: Actually, I did not notice.

Preservice teacher 4: I found such activity on the internet within the scope of another course. I encouraged Maya to use it in the rehearsal.

Preservice Teacher 3: I encouraged Maya to use it, too. I believe this activity could enable students to explore the concept's meaning. However, I overlooked such a situation until Preservice Teacher 2 asked the question.

Preservice teacher 4: Me, too.

Maya: The activity did not include units. Although I overlooked such a situation, I learned that this is an essential property of perimeter measurement (Evidence of *using terminology knowledge).* However, in the enactment of the actual classroom environment, we should be careful of the appropriateness of the activity found on the internet.

As understood from the dialogue above, Maya and most did not notice such a situation until this fruitful discussion. Gaining awareness of the importance of using the lengths of measure might support the development of her *using terminology* knowledge. In addition, within the scope of the same node of the model, it was observed that while Maya was conducting perimeter teaching in a simulated classroom environment, she could not express the measurement units sometimes. A similar emphasis from the preservice teachers mentioned above was shared when the preservice teachers were engaging in the video recordings of the rehearsal. For example,

Excerpt 1

Preservice Teacher 3: You accepted the answer as correct when I told 32. Did you notice?

Maya: Yes, it was.

Preservice Teacher 2: Does using 32 in the perimeter calculation make sense?

Maya: I should have emphasized units. You are right. I overlooked the units when Preservice Teacher 3 gave the correct answer.

Excerpt 2

Preservice Teacher 3: You said 24, but 24 did make any sense. Do you agree?

Maya: Yeah, you are right. I forgot to say out of excitement.

Preservice Teacher 4: You should be more careful in the next lesson.

As understood from the excerpts above, Maya did not use the length of measures while conducting perimeter teaching in the simulated classroom environment. As understood from her explanations, she focused on the number when the student

reached the correct number. However, during the examination of the rehearsal video recordings, Preservice Teachers 2 and 3 encouraged Maya to carefully use the length measures in her perimeter teaching, as shown in the first excerpt. Similarly, as implied in the second excerpt, it was seen that Maya forgot the units of expression in her rehearsal teaching. The preservice teachers again warned Maya to be careful in using measures. In addition, it was seen that Maya, as a teacher, sometimes confused the meter and centimeter in her rehearsal teaching. To illustrate,

> **Maya (in the role of the teacher):** The long side is 5 cm, and the opposite side is 5 cm; since this is two cm, the opposite side becomes two cm again (shows a rectangle-shaped pool). (Evidence of *using terminology* knowledge).
>
> **Preservice Teacher 2 (in the role of the fifth-grade student):** Teacher, is not two cm too little, or could it be two cm off the long side of a pool?
>
> **Maya (in the role of the teacher):** I wrote two cm more. There is more.
>
> **Preservice Teacher 2 (in the role of the fifth-grade student):** Should it not be necessary to write meters, teacher?
>
> **Maya (in the role of the teacher):** Sorry, did I say cm? Sorry, meters.

The excerpt above showed that Preservice Teacher 2, acting like a student, noticed Maya's mistake in specifying the length measures in calculating the pool's perimeter during the rehearsal. For this purpose, Preservice Teacher 2 asked, "Is not two cm too little, or could it be two cm of the long side of a pool?" to make Maya actualize her mistake in the moment of instruction. After such emphasis, it was observed that Maya also paid more attention to using length measures correctly during the rehearsal. For example,

> **Maya (in the role of the teacher):** Is there any other way to solve it?

Preservice Teacher 2 (in the role of the fifth-grade student): Teacher, I solved it, counted the points, and found 33.

Maya (in the role of the teacher): So why did you count the dots?

Preservice Teacher 2 (the fifth-grade student): I counted the dots.

Maya (in the role of the teacher): The distance between two points is one unit. Right? So, we need to choose how we count units. We need to select one starting point. For example, one unit, two units, three units… 32 units, so we return to the starting point and find the perimeter (Evidence of *using terminology* knowledge).

As shown in the dialogue above, Maya paid attention to using appropriate units, linear units, while explaining the problem's solution during the rehearsal. Consequently, she displayed exemplary behavior regarding *using terminology* knowledge in her rehearsal teaching. In addition, when she was asked for homework Activity 4, called *Prepare for Enactment*, at the end of this node, she showed exemplary behavior about *using terminology* knowledge by emphasizing the length of the measures and using the mathematically correct language Table 4.10. This activity enabled her to decompose and approximate the practice through lesson planning. The following table summarizes Maya's developmental process using terminology knowledge during the second node of the model, *Preparation.*

Then, Maya conducted perimeter teaching in one of the fifth-grade classrooms as a part of the pedagogies of enactment. This teaching was conducted in the first and second lesson hours of implementation of the third node of the model, called *Implementation*, within the context of a practical hour of the model. During her teaching, it was seen that Maya frequently emphasized the length of measures and paid attention to using mathematically correct language. For example,

…

Maya (teacher): What is the length of this side?

Student 7(in the actual classroom): 8

Maya (teacher): 8 cm, right? It would be best if you emphasized units. Then? (Evidence of *using terminology* knowledge).

Student 7 (in the actual classroom): I added sorry 8 cm, 8 cm, 8 cm, 8 cm and obtained 32 cm.

Maya (teacher): What did your friend do? Add the lengths of the sides around the square's shape (Evidence of *using terminology* knowledge).

Table 4.10 Sample evidence of the development of Maya *using terminology* knowledge during the Preparation node of the model.

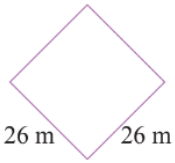
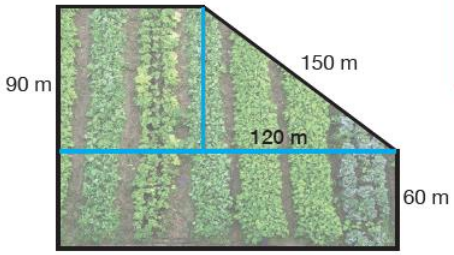
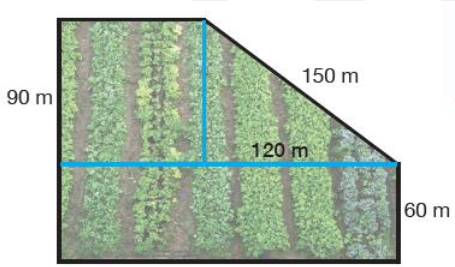
Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *using terminology* knowledge
Activity 1: *Prepare for the Rehearsal*	 Maya (teacher): Let us calculate the perimeter of the rhombus-shaped garden above. Student (fictitious): My teacher, I multiply 26 m by four. I obtained 104 m. Maya (teacher): You *added the lengths of the sides around the shape* to find the perimeter.	She used mathematically correct language and the units of length in her explanation.
Activity 2: *Rehearse the Perimeter Teaching!*	 Maya (teacher): On the right, some side lengths of a garden consisting of square, triangular, and rectangular sections are given. How many meters of wire would it take to run two rows around this garden? Preservice teacher 2 (in the role of the fifth-grade student): Teacher, I found it. There's a square there. It goes down to 120. The bottom is 210. The Opposite is 60. Now 60+90+90+150+60+210=*660.* Maya (teacher): Well done.	She did emphasize the units of length in her explanation.

Table 4.10 (Continued).

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *using terminology* knowledge
Activity 3: *Let us Examine the Rehearsal*	Preservice Teacher 2: In the student's answer, it was seen that she said 120, 210, and 60 in solving the question. It would help if you were careful to use measures of length always. Preservice Teacher 4: Units are essential. Maya: You are right.	She was encouraged to use the units of length in her responses.
Activity 4: *Prepare for the Enactment*	 Maya (teacher): On the right, some side lengths of a garden consisting of square, triangular, and rectangular sections are given. How many meters of wire would it take to run two rows around this garden? Student (fictitious): Teacher, I found it. There's a square there. It goes down, and its *side lengths* are 120 *cm*. The bottom is 210 *cm*. The opposite *side length* is 60 cm. Then, I added 60 cm +90 cm +90 cm +150 cm +60 cm +210 cm =*660* cm. Maya (teacher): Well done.	She used the units of length and mathematically correct terminology in her responses.

As seen in the dialogue above, she expressed the unit when Student 7 forgot to say it in her explanation. Besides, Maya summarized what the student did by using mathematically correct language. Thus, it could be said that she displayed exemplary behavior regarding *using terminology* knowledge in her perimeter teaching. Still, Maya sometimes failed to use mathematically correct language and terminology in her teaching regarding the perimeter. The sample snapshot regarding this issue is shown in Figure 4.18.

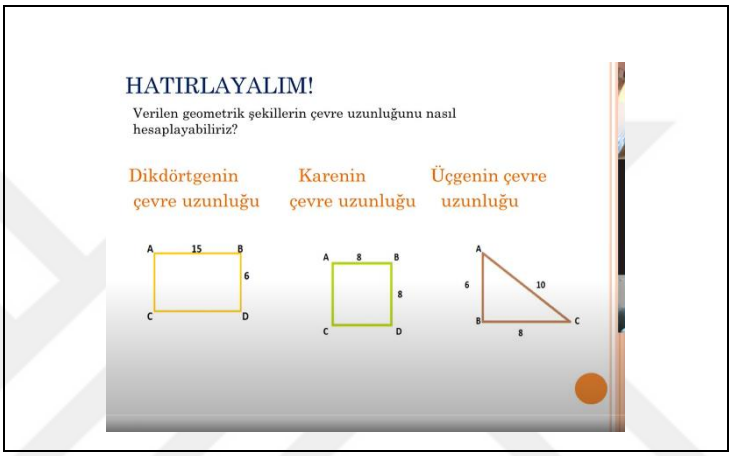


Figure 4.18. Snapshot from the actual classroom regarding evidence of *using terminology* knowledge.

As shown in Figure 4.18, Maya, the lesson's teacher, started the lesson by calculating the perimeter of the most used shapes, such as rectangles, squares, and triangles. Although she used centimeters in her verbal expressions while introducing the side lengths of these shapes in her teaching, she did not add the measure of the side lengths of the given shapes visually. Other preservice teachers recognized such situations and shared their suggestions while analyzing the video recording of Maya's perimeter teaching in Activity 1: *Let Us Examine the Enactment*, which took place during the first and second lesson hours of implementing the fourth node of the model called *post-Preparation.* They emphasized the importance of Maya's use of length measurements in her teaching as follows:

Preservice teacher 2: I noticed that the lengths of sides were not shown on such drawings. You assigned cm to the side lengths of the shape while calculating the perimeter of such shapes.

Preservice teacher 3: You should be careful in this issue. Otherwise, you prepared the ground for the misconception to occur. In this way, students may not associate the concept of the perimeter with length.

Maya: You are right. I missed such an issue. However, I noticed this situation and said cm in my teaching. But still, I promised I would be more careful in other lessons.

From the above dialogue, Maya was warned by others to be careful about showing the units of side length of the shapes on the visuals. In response, Maya said she noticed her deficiency in showing the length of measures in her lessons. When Maya was asked to complete Activity 2, *Prepare for Further Teaching*, as a homework assignment within the same node of the model, it was observed that Maya exhibited exemplary behaviors regarding the use of terminology knowledge in her lesson plan. For example, as shown in Figure 4.19, she correctly used mathematical terminology and the length of measures while calculating the perimeter of the shapes.

Her examples in the lesson plan showed that Maya became more sensitive to using units to construct the concept of the perimeter as a length in the students' minds. Overall, the following table summarizes Maya's developmental process regarding *using terminology* knowledge during the third and fourth nodes of the model, called *Implementation* and *Post-Preparation*, respectively.

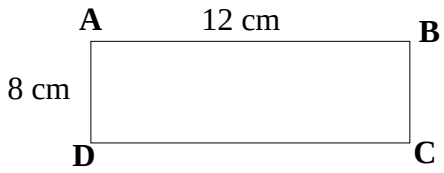
In addition to this evidence, she emphasized that she would be more careful in her future teaching while using length measures in perimeter teaching. She expressed her views during the post-interview as follows:

Besides, I felt the effect of the model while I conducted the perimeter teaching in a real classroom. For example, I needed to internalize the concept

> of unit expressions. I must internalize their importance to transfer them to the students. Before, for example, I would describe the perimeter definition as in the textbook (in my private teaching). I would not emphasize the length and the unit there. However, thanks to these experiences, I focused on the critical points…For example, the student said 24 to the class to answer the problem, and I accepted 24 as the correct answer in the rehearsal. But what is 24? Unless the unit is used, the answer is incomplete… (Before). I ignored units. However, I learned during the intervention that saying the number does not make sense without its units.

As shown in her explanations, she gained awareness of the importance of using mathematical language, the lengths of measures, linear measurement, and expressions correctly after her involvement in the intervention. To sum up, it could be said Maya had a chance to develop her mathematical language and paid attention to the use of length of measures regarding the perimeter concept during the activities in the nodes of the current study's intervention, which consisted of four nodes: *Pre-Preparation, Preparation, Implementation*, and *Post-Preparation*. More specifically, it was seen that as Maya questioned the perimeter concept's meaning, she developed her mathematical language from misusing the terms about the concept to mathematically correct terms in her explanations. Indeed, the collective analysis helped Maya to re-revise her mathematical language regarding the perimeter concept. In addition, she created dialogues between Maya and fictitious students while preparing lesson plans for perimeter concepts. In these ways, if there were deficiencies in her language or a miss to express the units of measures about the perimeter, the other preservice teachers warned her to correct her deficiencies, which helped her to re-revise her mathematical language and terminology regarding the perimeter. All may have contributed to improving the knowledge of *using terminology*. The below section gave information on the development of Maya's *identifying errors* knowledge during the intervention.

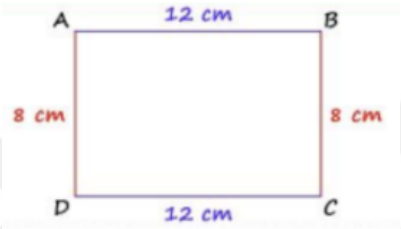
Teacher (Maya): What is the perimeter of the given shape?



Student 2 (fictitious student): Teacher, the answer is 40 cm.

Teacher (Maya): How did you find it?

Student 2 (fictitious student): I added 12 and 8 and found 20 cm. Then, I multiply it by two.

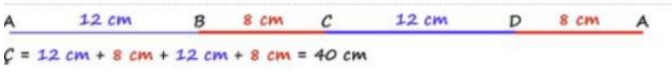


Teacher (Maya): Yes, well done. Are there any solution ways?

Student 1 (fictitious student): I added 12, 8, 12, and 8 and found a 40 cm teacher.

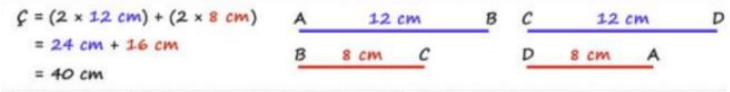
Teacher (Maya): You could also solve the question in these ways (I will open GeoGebra software).

First way:



The first way included adding the sides lengths of shape by adding end to end.

Second way:



The second way included adding together equal lengths.

Third way:



The third way included the following formula.

Figure 4.19. Evidence of *using terminology* knowledge from Maya's final lesson plan.

Table 4.11 Sample evidence of the development of Maya *using terminology* knowledge during the Implementation and Post-Preparation nodes of the model.

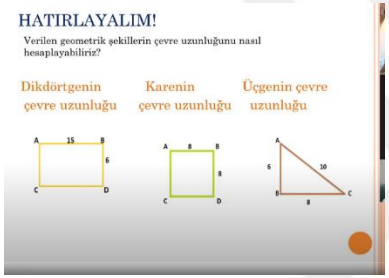
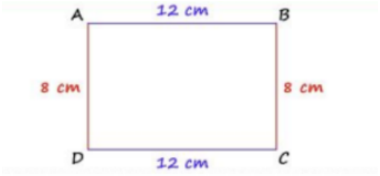
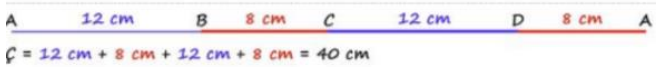
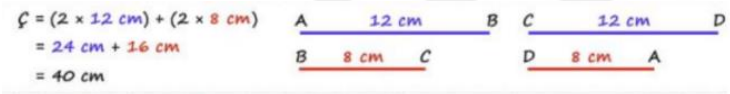
Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *using terminology* knowledge
Activity 1: *Enact the Perimeter Teaching!* (in the third node called enactment).	Maya (teacher): What is the length of this side? Student 7(in the actual classroom): 8 Maya (teacher): 8 cm, right? *It would be best if you emphasized units*. Then? Student 7 (in the actual classroom): I added eight, sorry eight cm, eight cm, eight cm, eight cm, and obtained eight cm. Maya (teacher): What did your friend do? *Add the lengths of the sides around the square's shape.*	She used mathematically correct language and units of measure in her explanations.
Activity 1: *Enact the Perimeter Teaching!* (in the third node called enactment).		She misused mathematical expressions since she forgot to add the units of measure visually.
Activity 1: *Let us Examine the Enactment* (in the fourth node called post-preparation).	Preservice teacher 2: I noticed that the lengths of sides were not shown on such drawings. You assigned cm to the side lengths of the shape while calculating the perimeter of such shapes. Preservice teacher 3: You should be careful in this issue. Otherwise, you prepared the ground for the misconception to occur. In this way, students may not associate the concept of the perimeter with length. Maya: You are right. I missed such an issue. However, I noticed this situation and said cm in my teaching. But still, I promised I would be more careful in other lessons.	She was encouraged to show units of length visually.

Table 4.11 (Continued).

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *using terminology* knowledge
Activity 2: *Prepare for Further Teaching* (in the fourth node called post-preparation).	….  Maya (Teacher): You could also solve the question in these ways. … First way:  The first way included adding the sides lengths of shape by adding end to end. Second way:  The second way included adding together equal lengths. Third way:  The third way included the following formula.	She paid attention to using mathematically correct language and using units of measure.

4.1.3 Mathematical Knowledge: *Identifying Error*

This section examined what Maya knew about the potential students' difficulties with the perimeter concept, their sources, and how she planned to overcome them, considering her responses in the pre-and post-test and the interview. As Table 4.1 shows, Maya's mathematical knowledge of *identifying errors* regarding the perimeter was enhanced during the intervention. The following table summarizes the findings regarding Maya's current knowledge of *identifying errors* and shows the development in Maya's knowledge from the pre-assessment to the post-assessment.

Table 4.12 Development of the *identifying error* knowledge.

Pre-intervention based findings	Post-intervention-based findings
-She had difficulty identifying errors in the student's answers and difficulties that students may have regarding the perimeter concept.	-She could identify errors in the student's answers and difficulties that students may have regarding the perimeter concept.
-She lacked knowledge of the possible sources of students' difficulties regarding the perimeter concept.	- She correctly reasoned the potential sources of students' difficulties regarding the perimeter concept.

As shown in Table 4.1, Maya's responses in the pre-assessment indicated that she lacked knowledge of fifth-grade students' potential difficulties with the perimeter concept. To be mentioned in detail, she said conceptual and computational errors regarding perimeter concepts that students might have superficially. When asked to describe the possible difficulties of fifth-grade students on the perimeter concept in the pre-test, she responded, "*When students are given complex shapes, they can add the sides twice due to forgetting their starting points. They can confuse the perimeter and area concepts."* As seen from her explanation, she mainly focused on computational errors, such as counting twice the side lengths of the shape, and common misconceptions, such as confusion about the perimeter and area concept, while explaining the possible students' difficulties with the perimeter concept.

Similarly, when asked to describe the possible sources of such difficulties, it was seen that she gave general information about them such that "*Having lack of previous*

knowledge may cause such difficulties. *For example, they may encounter these problems when they cannot grasp the properties of the square or rectangle*." Her response implied that she thought students might have such difficulty due to a lack of knowledge of the quadrilaterals' properties rather than the perimeter's meaning. Put differently, her explanations were not directly related to the sources of such difficulties with the concept's meaning. Meanwhile, it was observed that she had difficulty identifying the given student's error (mentioned in the previous paragraphs and shown in Figure 4.4) related to not expressing length units.

In addition, it was seen that she did not provide a clear teaching strategy about how she planned to overcome students with misconceptions regarding the perimeter concept when she encountered them in the instruction. Put differently, although she knew she should focus on students' prior knowledge to prevent such misconceptions, she did not know how. For instance, her response in the pre-test about how she planned to deal with such misconceptions was as follows: *"First, I include questions based on measuring students' prior knowledge. Later, I would mainly focus on concrete materials and associating the concept with daily life."* Her responses showed that Maya had no plan of action to prevent these misconceptions from occurring in the perimeter instruction. To sum up, it could be said that she did not provide transparent information about students' difficulties with the perimeter concept and their possible sources and did not have any strategy to prevent them from teaching. For these reasons, it could be said that she lacked knowledge of *identifying errors* regarding the perimeter concept before attending the intervention.

On the contrary, the analysis of the post-assessments showed that Maya tended to present more detailed explanations about possible fifth-grade students' difficulties with the perimeter, their sources, and how to deal with them. Specifically, she described the possible fifth-grade students' difficulties with the perimeter concept in the post-test, as presented in Figure 4.20.

Students can confuse the concepts of perimeter and area with each other. They can confuse their formulas and the units of measure with perimeter and area. They can confuse the relationship between perimeter and area concepts. They can also make an operational error by calculating the perimeter, such as counting the sides twice.

Figure 4.20. Evidence of *identifying error* knowledge in the pretest.

As understood from her response, she mainly mentioned conceptual difficulties that originated from a lack of understanding of the concepts. Besides, she pointed out computational errors in calculating the shape's perimeter, such as counting the sides twice. All misconceptions that she mentioned also existed in the literature on learning the perimeter concept (e.g., Reinke, 1997). Similarly, her response in the post-test showed that she noticed the given student's response in Figure 4.4 was missing due to not expressing units. In addition, it was seen that she approached possible sources of these difficulties from a conceptual aspect. For example, when asked to describe the potential sources of them in the post-test, she responded to the question as presented in Figure 4.21.

The source of this difficulty may be that the students do not understand the concept of length and perimeter. In addition, they may have problems because they do not internalize the measurement process and the units of measure.

Figure 4.21. First evidence of *identifying error* knowledge in the post-test.

Although she pointed out the possible sources of difficulties as a lack of understanding of the properties of squares and rectangles before participating in the intervention, she said that they might have been because students did not have enough knowledge about the measurement process, as shown in her response above. Besides, she emphasized the role of grasping the meaning of length measurement, the perimeter, and the length measures in understanding the perimeter concept. In addition to these, it was seen that she seemed to have action plans on how to

overcome such difficulties in teaching. To be detailed, her response in the post-test, when asked how she planned to overcome such misconceptions, was as presented in Figure 4.22.

First, their prior knowledge can be measured to overcome students' difficulties. For example, it can be observed whether there is a lack of previous knowledge about length measurement. It can be emphasized that the perimeter is a particular (closed) case of length measurement. Concrete materials can help students construct these concepts in their minds. Collaborative activities on perimeter calculation can be included. For example, each group can calculate the perimeter of a different concrete shape with rope and then measure on the ruler to calculate the total length.

Figure 4.22. Second evidence of *identifying error* knowledge in the post-test.

As understood from her response, she clarified what she planned to do when encountering some students' difficulties with the perimeter concept during the instruction. Furthermore, she said during the interview that she had some misconceptions regarding the perimeter concept before attending the intervention. She said as she developed her mathematical knowledge on the perimeter, she more easily identified students' difficulties with the concept. In particular, she said the following during the post-interview:

> When I looked at the misconceptions that the students had, there were misconceptions and misunderstandings that I had as well... For example, the relationship between perimeter and area. When we calculate the perimeter or area of a shape, I see that even if the shape changes, its area can remain constant. For example, I noticed that it depends on the shape instead of assuming a linear relationship between its perimeter and area. Now, I know that there is no clear relationship between perimeter and area, such that increasing the area of a shape does not mean increasing the perimeter. I saw that it depends on the shape. Students also have this misconception. So do I.

> However, if I had not observed it before graduation, I might have said the perimeter would increase if the area increased. I would not have emphasized the students' misconception because I had it, too. However, during the intervention, I saw that this was not the case, and even though the shape's perimeter remained the same, the shape's area could remain the same... So I enjoyed learning about it. Seeing all this contributed significantly to my development... Because I understood, it was more eventful to understand and make sense of the student's misconceptions.

Her explanation showed that she did not correctly grasp the relationship between the perimeter and the area of the shape before her involvement in the intervention. For this reason, she might not have given specific examples of possible students' difficulties with the perimeter concept. She stated she had a chance to improve her knowledge of the perimeter and area concepts during the intervention of the current study. Thus, after her intervention involvement, she might have explained the possible sources of such misconceptions in more detail. Put differently, as she was more aware of the mistakes and misconceptions that students may have about the perimeter concept, she reasoned more clearly about their possible sources.

To sum up, it could be said that she correctly and elaborately explained potential fifth-grade students' difficulties and described possible sources for them. She also explained her action plans and what if she faced such challenges in the instruction. For these reasons, it could be said that she had a robust knowledge of *identifying errors* regarding the perimeter concept after attending the present study. Thus, it could be said that she had made remarkable progress in *identifying errors in the* knowledge of the perimeter concept during her involvement in the activities in the nodes of the model of the current study. Below, evidence for improving *identifying error* knowledge during the model is presented.

4.1.3.1 Evidence for Improvement in *Identifying Error* Knowledge during the Model

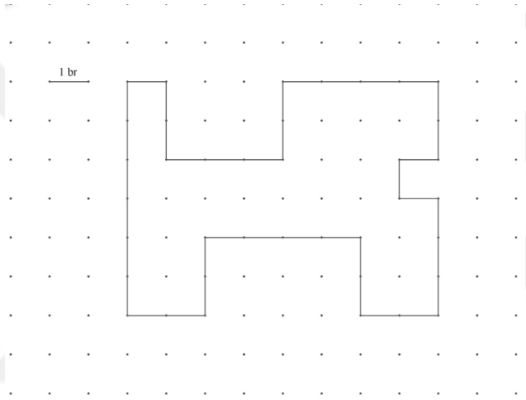
As stated above, Maya's knowledge of *identifying errors* regarding the perimeter concept was developed when she was involved in the nodes of the model. In this section, the evidence that she developed this knowledge during her participation in the current study is shared, paying attention to the sequence of activities in each node of the model, as well as the sequence of the nodes of the model, which are called *Pre-Preparation, Preparation, Implementation,* and *Post-Preparation*, respectively. The excerpts from the current intervention's video transcriptions and individual responses to classroom activities were shared to describe Maya's progress in *identifying error* knowledge during the intervention.

As displayed in Table 4.1, evidence for the development of Maya's knowledge of *identifying errors* started to be observed as the preservice teachers engaged in Activity 1: *What is the perimeter? Let us Look at the Concept through your and the Student's Eyes,* and Activity 2: *Let Us Examine the Dialogue* at the first node of the model, called *Pre-Preparation*. More specifically, there were several questions about identifying possible difficulties of fifth-grade students while calculating the perimeter of the given shape. In the first hour of implementing the *Pre-Preparation* node, the preservice teachers were asked to explain the students' difficulties in solving the third question in the same activity, identify their possible sources, and how they planned to deal with them individually. Then, they were asked to share their reasoning with the classroom in the second lesson hour of implementing the first node.

As shown in Table 4.13, the fourth question in the activity enabled the preservice teachers to reason misconceptions about the concept of perimeter, considering the third question in the same activity and the possible sources of them. In addition, they were also asked to identify possible difficulties with the concept of the perimeter by considering the third question in the activity since the second activity in the same

node included the dialogue between the teacher and students around this question. In this way, before discussing the student's strategy in the given dialogue as part of the second activity, the preservice teachers obtained a chance to reason about the possible students' misconceptions about the perimeter concept, considering the problem mentioned in the third question.

Table 4.13 Third, fourth, and fifth questions in the first activity in the Pre-Preparation node of the model.

Activity 1: What is the perimeter? Let us look at the concept through your and the student's eyes (continued).

3. Calculate the perimeter of the given shape as much as possible. Please indicate your solutions ways. Then, share your solution ways with your friends.
4. What mistakes do students make when solving the above question? What are the possible sources of them? Please think individually and then share your thoughts with your friends.
5. How do you handle such mistakes or prevent them from occurring such mistakes during the instruction? Please think individually and then share your thoughts with your friends.

Maya answered the fourth question in the first activity individually: "*Students count the dots twice and reach the incorrect result*." Her response implied that she was aware that counting dots around the shape was one of the mistakes in the perimeter calculation since the starting point was counted twice. But still, she failed to explain the possible source of such difficulty in her answer.

Furthermore, the fifth question in the first activity, as shown in Table 4.13, was related to explaining what the preservice teachers would do to prevent such mistakes from occurring or handling them. Such a question was asked to allow the preservice

teachers to reason about what they could do when they faced such situations before the perimeter teaching. Maya responded: "*I could remember its definition and rules to calculate perimeter*." As seen in her response, she tended to remember the procedures regarding perimeter calculation to overcome the mistake she mentioned in the fourth question. In the second lesson hour of the implementation of the first node, when the preservice teachers discussed their explanations about the fourth and fifth questions, it was seen that Maya tended to mention students' difficulties with the perimeter concept from a conceptual aspect. Meanwhile, she explained the possible underlying sources and how to deal with them in more detail. For example:

> **Preservice Teacher 2:** Students could count the dots of the shape by closing the shape. In such a situation, they do not understand the perimeter… I would solve more examples.
>
> **Preservice Teacher 4:** Students can be confused about where they started. Thus, I would tell them to mark the starting point. If the counted lines are indicated with a colored pencil, the calculated lines will not be counted again.
>
> **Maya:** Students could confuse the area and the perimeter if they do not internalize the formulas. They can count the squares of the units in this question. If I meet such situations, I could remind students about defining the perimeter and the area (Evidence of *identifying error* knowledge).

As seen in the dialogue above, the preservice teachers began to think about the students' potential difficulties with the perimeter concept given the given problem. While Preservice Teacher 4 mentioned the operational errors, such as miscounting, Preservice Teacher 2 and Maya expressed conceptual strategies that might stem from a lack of understanding of the concept, such as confusing the perimeter and area concepts. Meanwhile, the preservice teachers began to think about overcoming them in the discussion, even though they provided superficial explanations. However, it was found that at this stage, Maya and others tended to overcome such difficulties using procedural rather than conceptual strategies. Nevertheless, it was assumed that

Maya showed exemplary behavior in *identifying error* knowledge, as she became aware of both students' difficulties with the perimeter question, even though she did not address them conceptually.

Then, the preservice teachers passed to discuss the student's strategy in the given dialogue as part of the second activity mentioned above. As mentioned in Table 4.4, this activity included a dialogue between the fifth-grade student and the teacher on the perimeter concept as part of the representation of the practice. However, before, they were asked to individually respond to prompts in this activity in the first lesson hour of implementing the first node. Maya responded to the prompt regarding identifying the details of the student's strategy and evaluating its correctness: "*The student counted the points."* As seen in her response, she did not query the student's strategy from a mathematical aspect. She just mentioned what the student did as the solution to the problem. However, when the pre-service teachers were asked to examine the given dialogue collectively in the second lesson hour of the implementation of the first node, it was seen that Maya started to evaluate the correctness of the student's strategy on the researcher's question. The excerpt is presented below.

> **Researcher:** What do you think about the student's strategy? Do you think it's true?
>
> **Maya:** Wrong, my teacher (Evidence of *identifying error* knowledge).

Although she avoided commenting on whether the student's answer was correct in her response, at the beginning of the discussion, she evaluated the student's result correctly, as shown in the dialogue above. Consequently, it could be said that she successfully identified that the student made a mistake. The discussion about the mathematical details of the student's strategy continued with the following:

> **Researcher:** Why?

Maya: She did not count the length between two points but counted the points (Indicator of *identifying error* knowledge).

Preservice Teacher 3: I disagree with you. The solution was right because when we count the points of a closed shape like this, we calculate its perimeter since the number of points equals the number of line segments.

Researcher*:* Do you agree with Ali's opinion?

Preservice Teacher 4*:* I was very conflicted, but I thought the student's solution was wrong while answering. However, I tried what Preservice Teacher 3 said in other shapes, which could be true.

Maya: I do not know, but for example, when calculating the perimeter, we say the sum of the sides. So, we need to know the distance. So, length. So, we do not need to know the sum of the lengths of the sides for the perimeter. We need to know the distance between two points, that is, how much so we can find the perimeter. We need the distance between the points, not the points. So, we need to have the length (Evidence of *identifying error* knowledge).

As shown in the excerpt above, in the middle of the discussion, Maya hesitated about the correctness of her thought, considering the explanations of Preservice Teachers 3 and 4. Meanwhile, Maya questioned the mathematical meaning of the concept and concluded that the student had made a mistake. Maya then considered what would happen if the student's solution were accepted as correct:

Preservice Teacher 2: Since dots are not units.

Maya: I do not think counting the point is used in perimeter calculation. In addition, if we accept the student's solution method is correct like Preservice Teacher 3 says, the students and we can generalize it to other situations. For example, let us image a rectangle. They could say that the rectangle's

perimeter is four because it has four corners. So, I think this solution method was wrong, too (Evidence of *identifying error* knowledge).

Preservice Teacher 5*:* You are right.

As shown in the dialogue above, when Preservice Teacher 2's claim supported Maya's thought, Maya handled Preservice Teacher 3's claim from a pedagogical aspect. She concluded that accepting such a strategy as the correct one might lead to misconceptions in the perimeter calculation. Later, Maya ensured that counting dots were not used in the perimeter calculation. In this way, she did not accept this answer as correct and took precautions against possible misconceptions in her further perimeter teaching. Hence, it could be said that Maya displayed exemplary behavior regarding *identifying error* knowledge.

A similar situation was observed when, as part of the approximation of practice, the fourth prompt of the second activity, the preservice teachers discussed how they planned to continue the lesson in light of this student's strategy, as shown in Table 4.4. Indeed, Maya's response was as follows: "*I could ask similar questions as a drill."* As shown in her response, she had no plans to handle such a mistake. In contrast, while the preservice teachers were discussing their explanations about this prompt, it was seen that she started to display actions to prevent such mistakes. To illustrate, she stated:

> I want the student to think that between two points, each point is a city, and there are two cities. When calculating the distance between these two cities, we do not take the distance as two by counting the cities. We calculate the distance between two cities, for example, how many kilometers the distance between two cities is. It also has value. In other words, we can think that one started to travel to Istanbul and returned to Istanbul by making a circular route to other cities by moving from Istanbul, Ankara, and Bursa. We calculate that distance in this pathway by looking at the total distance among cities. If we

look at the student's method, we might say three as an answer (Evidence of *identifying error* knowledge).

As shown above, she said she would display some actions to overcome the student's mistakes during her teaching. Since she planned to prevent such mistakes in her teaching, it could be said that she displayed exemplary behavior regarding *identifying error* knowledge at this stage of the model. Moreover, she said the following about this situation during the post-interview:

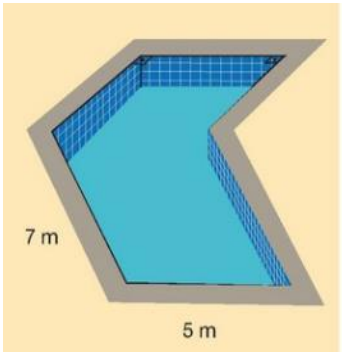
> We saw that counting the distances between two points, the lengths, helps us reach the shape's perimeter. First, the discussion on the perimeter created a lot of confusion for us. I wonder if the method the student followed was correct. Instead of counting the points, calculating the perimeter would be more accurate by adding the side lengths between two points around the shape. The more I understand the concept's meaning, the better I can handle the problems related to the student's thinking. I felt more confident.

As understood from her explanation, when she put the concept on conceptual foundations, she produced ways to cope with difficulties faced by students. The following table summarizes Maya's developmental process of *identifying error* knowledge during the *Pre-Preparation* node of the model.

At the end of this node, Maya volunteered to prepare a lesson plan to teach the concept of perimeter by setting up dialogues between herself as a teacher and fifth-grade students. While creating such dialogue as part of the decomposition and approximation of the practice, she produced the student's strategy for the problem, including correct solutions and misconceptions. When examining her lesson plan, it was seen that Maya tended to produce correct solution strategies for the given problem. However, she sometimes produced incorrect solution strategies in the lesson plan. A sample excerpt from her initial lesson plan is presented in Figure 4.23.

Table 4.14 Sample evidence of the development of Maya's *identifying error* knowledge during the first Pre-Preparation of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *identifying error* knowledge
Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes (Her response to 4th question).*	Students count the dots twice and reach the incorrect result.	She had difficulty expressing the underlying reasons for students' possible difficulties with the perimeter.
Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes (Her explanation of 4th question during the discussion).*	"Students could confuse the area and the perimeter if they do not internalize the formulas. They can count the squares of the units in this question. If I meet such situations, I could remind students about defining the perimeter and the area".	She identified the underlying reasons for students' possible difficulties with the perimeter and how she could deal with them.
Activity 2: *Let Us Examine the Dialogue (Her response to the fourth prompt).*	I could ask similar questions as a drill.	She did not act to prevent or overcome the student's mistake.
Activity 2: *Let Us Examine the Dialogue* (*Her explanation during the discussion of the fourth prompt*).	I want the student to think that between two points, each point is a city, and there are two cities. When calculating the distance between these two cities, we do not take the distance as two by counting the cities. We calculate the distance between two cities, for example, how many kilometers the distance between two cities is. It also has value. In other words, we can think that one started to travel to Istanbul and returned to Istanbul by making a circular route to other cities by moving from Istanbul, Ankara, and Bursa. We calculate that distance in this pathway by looking at the total distance among cities. If we look at the student's method, we might say three as an answer.	She took action to realize her mistake to the student.



Question: The pool consists of a parallelogram whose side lengths are 7 m and 5 m, respectively and the square. Please calculate the perimeter of the given pool.

Maya (teacher): Could you explain the perimeter of the given shape?

Student (fictitious student): The perimeter of the parallelogram is (7+5). x 2 =24 cm, the perimeter of the square is 5 x 4 from 20 cm, and the perimeter of these two shapes obtained is 44 cm (Evidence of *identifying error* knowledge).

Maya (teacher): What is the perimeter? It is the sum of the side lengths of the shape. It's like you went inside of the shape like this twice (Evidence of *identifying error* knowledge).

Student (fictitious student): Oh yes, teacher, it would be 34 cm.

Maya (teacher): Well done.

Figure 4.23. Evidence of *identifying error* knowledge from Maya's initial lesson plan.

As seen in the dialogue in Figure 4.23, Maya claimed that the student added the insides of the shape when calculating its perimeter. In this way, she identified one of the students' mistakes about the perimeter. Meanwhile, it was seen that even if she tended to display procedural strategies to overcome such students' mistakes as the teacher, she gave place on how she planned to overcome such mistakes in her instruction. When she shared her lesson plan with the preservice teachers in the first lesson hour of the implementation of the second node, which was called *Preparation*, she got the following feedback about the dialogue mentioned in Figure 4.20:

Preservice Teacher 2: You should remove such an example from the lesson plan since students might not focus on the perimeter. If they have limited knowledge of quadrilaterals, they cannot calculate the perimeter of the given shape.

Preservice Teacher 3: Also, it could cause misconception. The visual is not proper.

Maya: You are right. I will pay attention to these issues.

As understood from the dialogue above, Preservice Teachers 2 and 3 warned Maya to remove such a question from her lesson plan since they thought this question might lead to misconceptions in perimeter teaching due to the ambiguity in the visual. Considering these suggestions, Maya removed this question and dialogue from her lesson plan. Considering Maya's behavior, it could be said that since Maya identified students' difficulties with the perimeter concept and took necessary precautions to prevent students' mistakes regarding such problems by removing this question from her lesson plan, she displayed exemplary behavior regarding *identifying error* knowledge. In the second and third lesson hours of the same node, Maya conducted the perimeter teaching in the simulated classroom environment as part of the approximation of the practice, where she behaved as a teacher in Activity 2, *Rehearse the Perimeter Teaching*! More specifically, the other preservice teachers produced solution strategies and ways of thinking about the problems the teacher, Maya, posed during the rehearsal. At the same time, Maya had a chance to meet the students' difficulties and produce ways to overcome them during the instruction conducted in a sheltered environment. The following excerpt was taken from the rehearsal to show this issue about the question shown in Figure 4.24.

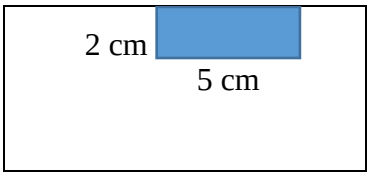


Figure 4.24. The question used in the rehearsal.

Preservice Teacher 3 (in the role of the fifth-grade student): No. We multiply the long side with the short side, do we not? We are calculating the perimeter in that way.

Maya (in the role of the teacher): How do we calculate the perimeter? When we add the rectangle's side lengths, we can calculate the shape's perimeter by adding two, two, five, and five and getting 14 cm (Evidence of *identifying error* knowledge).

In the dialogue above, Preservice Teacher 3 introduced a solution strategy that included common misconceptions, such as confusion about perimeter and area formulas. Such thinking allowed Maya to meet the potential fifth-grade students' conceptions regarding the perimeter at the moment of the instruction. At the same time, she had a chance to respond to such thinking to avoid students' misconceptions. However, as shown in the dialogue above, Maya, as the teacher, was unsuccessful in overcoming such misconceptions. Concerning the above dialogue, it could be said that she tended to remember the formula of the perimeter calculation for parallelograms and then directly explain the problem's solution. Indeed, she focused on procedures of perimeter calculation. While engaging in Activity 3, called *Let Us Examine the Rehearsal*, in the fourth and fifth hours of implementing the second node, it was seen that the preservice teachers collectively reasoned how to overcome such difficulty from a conceptual aspect as part of the pedagogies of the reflection. In this activity, they examined the video recordings of the perimeter teaching rehearsal, aiming to improve the quality of her perimeter teaching. In connection with the above dialogue, they made the following comments to overcome Preservice Teacher 3's difficulty and confusion about the perimeter and area formulas.

Preservice Teacher 4: You can ask students what the area means and how to calculate the area of the given shape.

Preservice Teacher 3: You can ask for length and area measurement units.

Maya: You are right. I would ask such questions when I met such misconception.

Preservice teacher 2: You can start the lesson with the example where students construct the concept's meaning as the kind of length to prevent such misconceptions.

As in the dialogues above, the preservice teachers discussed what questions could be asked to students who confused the area and the perimeter confusion. Meanwhile, Preservice Teachers 3 and 4 suggested focusing on understanding students' thinking on the concept. Moreover, Preservice Teacher 5 suggested providing concrete experiences for students to support their understanding. At the same time, as mentioned before, since the units of the side lengths were not specified and thinking that this may lead to misconceptions, Maya removed the activity, shown in Figure 4.9, from the lesson plan in Activity 4: *Prepare for Classroom Enactment*, which was located in the same node of the model. This activity was designed in light of the decomposition and approximation of the practices. Overall, it could be said that such kind of feedback enabled her to enhance her knowledge of *identifying errors*. In the post-interview, Maya emphasized her role in teaching the activities in the second node as follows:

> Creating dialogues in the planning process, rehearsing the perimeter teaching, and getting feedback from my friends allowed me to see how students can think. For example, we have seen that you can confuse area and perimeter formulas. The mistake was one of the most common mistakes. Therefore, the formula should not be given as a formula but to the student after concrete experiences. It was very effective for me to talk about student difficulties before the enactment to take precautions against them.

As understood from her statements, she learned to look at the problems/concepts from students' perspectives and mistakes/misunderstandings/misconceptions she might encounter in her classroom as she was involved in the activities. Still, it was

observed that she had difficulty producing conceptual strategies to overcome students' difficulties, although she tried to give students the properties of the perimeter, such as length. The following table summarizes Maya's developmental process of *identifying error* knowledge during the intervention's *Preparation* node.

Table 4.15 Sample evidence of the development of Maya's *identifying error* knowledge during the Preparation node of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *identifying error* knowledge
Activity 1: *Prepare for the Rehearsal*	Remove the question shown in Figure 4.8.	She removed the activity from her lesson plan to avoid misconceptions in her teaching.
Activity 2: *Rehearse the Perimeter Teaching!*	Preservice Teacher 3 (in the role of the fifth-grade student): No. We multiply the long side with the short side, do we not? We are calculating the perimeter in that way. Maya (in the role of the teacher): How do we calculate the perimeter? When we add the rectangle's side lengths, we can calculate the shape's perimeter by adding two, two, five, and five and getting 14 cm (Evidence of *identifying error* knowledge).	She emphasized the meaning of the perimeter to overcome misconceptions.
Activity 3: *Let us Examine the Rehearsal*	Preservice Teacher 4: You can ask students what the area means and how to calculate the area of the given shape. Preservice Teacher 3: You can ask for length and area measurement units. … Maya: You are right. I would ask such questions when I met such misconception. Preservice teacher 2: You can start the lesson with the example where students construct the concept's meaning as the kind of length to prevent such misconceptions.	She was encouraged to connect the perimeter and length to avoid misconceptions.
Activity 4: *Prepare for the Enactment*	Remove the question shown in Figure 4.17.	She removed the activity from her lesson plan to avoid misconceptions in her teaching.

In the context of Activity 1, *Enact the Perimeter Teaching*, under the third node of the model called I*mplementation*, Maya demonstrated exemplary behavior in *identifying error* knowledge while conducting the perimeter teaching in the actual classroom environment as part of the implementation pedagogies. The following excerpt and Figure 4.25 are presented to illustrate one of these behaviors.

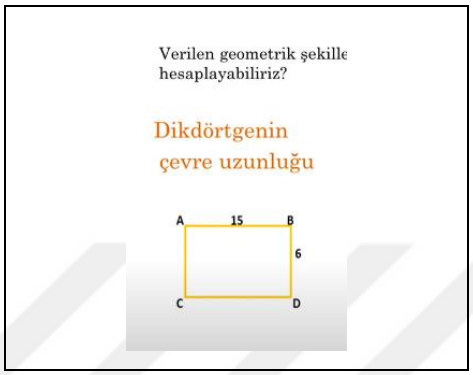


Figure 4.25. One of the questions used in the lesson.

Maya (teacher): What is the perimeter of this shape?

Student 3 (in the actual classroom): 90

Maya (teacher): How did you calculate it?

Student 3 (in the actual classroom): I multiplied 15 by nine and obtained 90.

Student 3 (in the actual classroom): Okay. Let us remind the perimeter. It is a kind of length. What is the area? Space inside of the shape, right? What do we do to calculate the perimeter of the given shape if the perimeter is a length? We add the lengths of the sides around the right shape. So? (Evidence of *identifying error* knowledge).

Student 3 (in the actual classroom): I added 15.

Maya (teacher): 15 cm, right? We should emphasize units. Then? (Evidence of *identifying error* knowledge).

Student 3 (in the actual classroom): I added 15 sorry 15 cm, 6 cm, 15 cm, 6 cm, and obtained 90 cm.

As seen in the dialogue above, Student 3 made mistakes, such as confusion about the area and perimeter formula in calculating the shape's perimeter. This mistake was similar to the mistake that Maya faced during the rehearsal mentioned above. Upon this mistake and Maya's actions towards students during the rehearsal, the preservice teachers suggested that Maya display actions to enable students to notice the perimeter is a kind of length and difference between perimeter and area during the analysis of the rehearsal. It was seen that Maya adapted these suggestions into her actions during her perimeter teaching, which was carried out in the actual classroom environment in the first and second lesson hours when the third node was implemented. As stated in the dialogue above, she tried to enable Student 3 to notice the perimeter is the kind of length. Moreover, she emphasized the difference between the perimeter and area concepts to handle Student 3's difficulty. Consequently, Maya identified Student 3's mistake and showed actions to overcome such mistake. Unfortunately, she made an error on this node that could lead to a misconception. Indeed, she did not put the measures of side lengths on the rectangles' visual, as shown in the dialogue above. This was essential to support students' understanding of the perimeter concept since seeing side lengths with measures enabled students to notice the perimeter is the kind of length. This situation attracted the attention of the preservice teachers in the first and second lessons of the *Post-Preparation* node while they were doing Activity 1, called *Let Us Examine the Enactment* in the scope of the pedagogy of reflection. To illustrate,

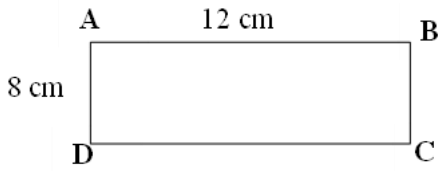
Preservice teacher 2: I noticed that the lengths of sides were not shown on such drawings. You assigned cm to the side lengths of the shape while calculating the perimeter of such shapes.

Preservice teacher 3: You should be careful in this issue. Otherwise, you prepared the ground for the misconception to occur. In this way, students may not associate the concept of the perimeter with length.

Maya: You are right. I missed such an issue. However, I noticed this situation and said cm in my teaching. But still, I promised I would be more careful in other lessons.

As understood from the dialogue above, Maya received a warning from others to pay attention to displaying the units of side lengths of the shapes on the visuals. Meanwhile, Maya also said she noticed her deficiency in showing the length of measures in her teaching. When Maya was asked to engage in Activity 2, which was called *Prepare for Further Teaching* and included reviewing the lesson plan last time as homework within the context of the same node of the model, it was seen that she displayed more exemplary behaviors regarding *identifying errors* in knowledge in her lesson plan. To illustrate, as shown in Figure 4.26, Maya avoided forming students' misconceptions about the perimeter concept by indicating cm, and the perimeter is the length. Consequently, she displayed exemplary behavior regarding *identifying error* knowledge in her final lesson plan.

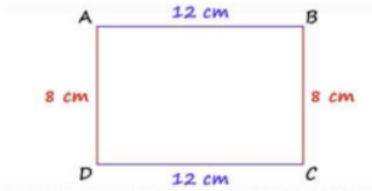
Teacher (Maya): What is the perimeter of the given shape?



Student 2 (fictitious student): Teacher, the answer is 40 cm.

Teacher (Maya): How did you find it?

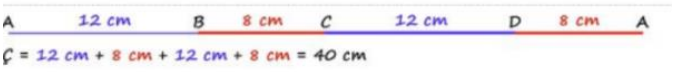
Student 2 (fictitious student): I added 12 and 8 and found 20 cm. Then, I multiply it by two.



Teacher (Maya): Yes, well done. Are there any solution ways?

Student 1 (fictitious student): I added 12, 8, 12, and 8 and found a 40 cm teacher.

Teacher (Maya): You could also solve the question in these ways (I will open GeoGebra software).



First way: The first way included adding the sides lengths of shape by adding end to end.



Second way: The second way included adding together equal lengths



.

Third way: The third way included the following formula.

Figure 4.26. Evidence of *identifying error* knowledge from Maya's latest lesson plan.

Meanwhile, while conducting her perimeter teaching in an actual classroom, Maya encountered students' thoughts she had not encountered before. The following dialogue and Figures 4.27, Figure 4.28, and Figure 4.29 are presented to unpack such situations.

Student 5 (in the actual classroom): My teacher, I found another method.

Maya (teacher): I am listening to you.

Student 5 (in the actual classroom): I added 15 and 15 and found 30. These are adding the long sides. There are six squares here (see Figure 4.27). Thus, I divided 30 by five and obtained six. Then, I added six with six and obtained 12.

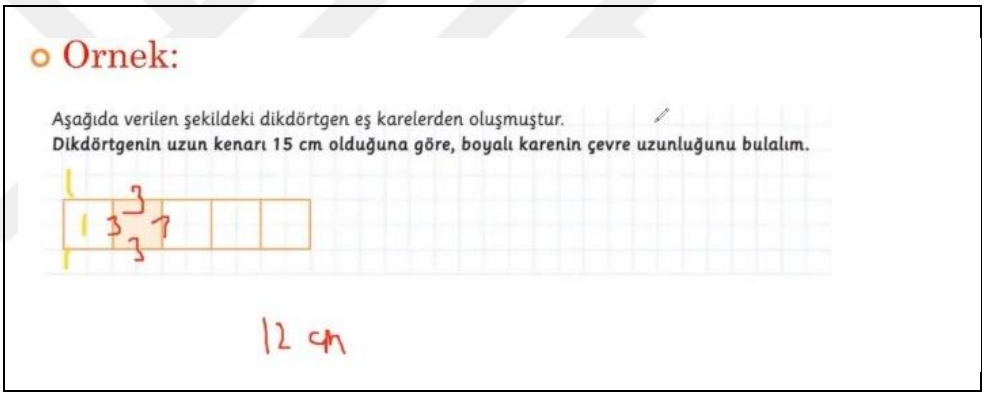


Figure 4.27. The first attempt Student 5's to solve the question.

Maya (teacher): Look, the length of this long side is 15 cm. How many congruent squares are there? Five. What do I get if I divide 15 cm by five? So here three, here three, here and here three. Now, what happened is a square with a side of three cm. How do I find the phase of it? (She drew the following figures at the same time.).

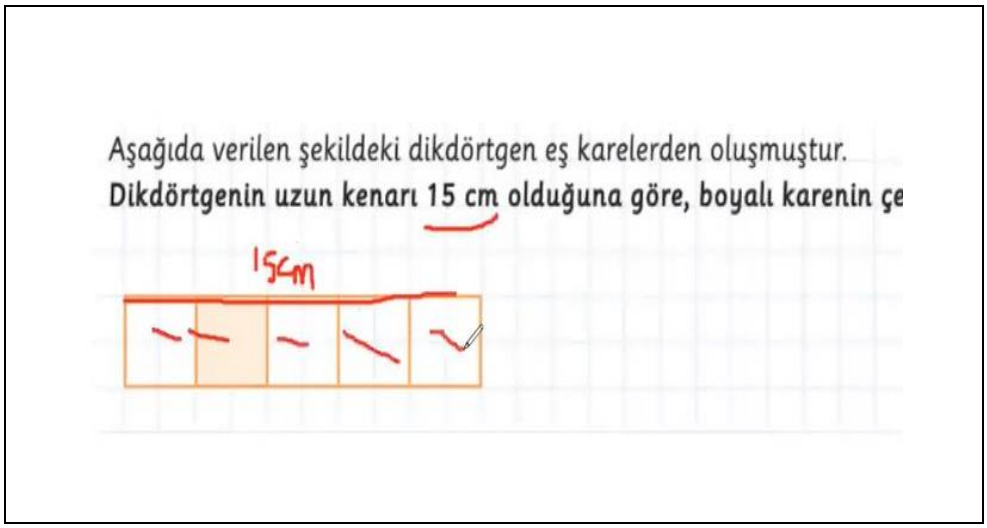


Figure 4.28. The second attempt Student 5's to solve the question.

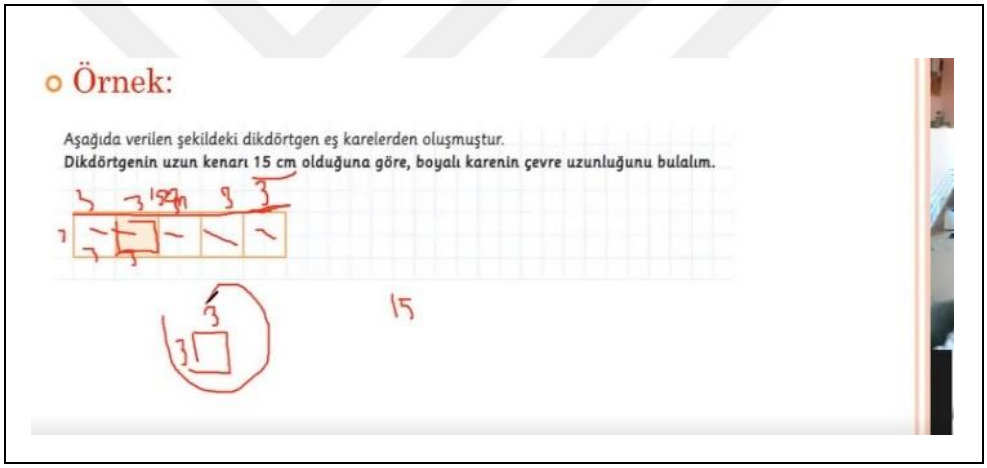


Figure 4.29. The third attempt Student 5's to solve the question

Student 5 (in the actual classroom): I multiply three by four.

Maya (teacher): So?

Student 5 (in the actual classroom): I found a 12 cm teacher.

As shown in the dialogue and figures above, Student 5 found the correct number, but she followed the wrong solution way. Indeed, she added to the long sides of the rectangle at first. Then, she divided the obtained number by five since the rectangle had five big squares. Then, Maya drew the following figures for Student 5 to enable

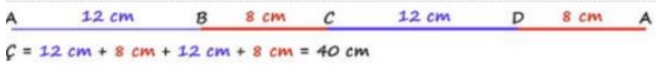
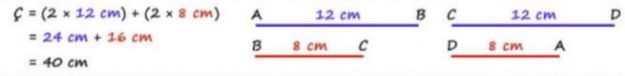
her to understand the problem more efficiently. Since she handled the student's mistake during the instruction, she displayed exemplary behavior regarding *identifying error* knowledge during her instruction. The following table summarizes Maya's developmental process regarding *identifying error* knowledge during the model's *Implementation* and *Post-Preparation* nodes, respectively.

Overall, it could be said that Maya made progress in *identifying errors* in her knowledge of the perimeter concept during the intervention. Maya increased her knowledge of students' difficulties with the perimeter concept and reflected on how to deal with them during the intervention activities. In addition, as a classroom teacher, she had the opportunity to confront potential and actual student difficulties during lesson planning, rehearsal, and authentic perimeter instruction. During these processes, Maya displayed exemplary behaviors in identifying error knowledge by attending to students' difficulties and taking precautions to prevent them. However, when she did not successfully avoid misconceptions in her perimeter teaching, she received feedback from her peers on overcoming these misconceptions during lesson plan preparation, rehearsal, and enactment analysis. All in all, it could be said that her knowledge of *identifying errors* in the perimeter concept developed as she engaged in the intervention activities. The following provided information on the development of Maya's *theoretical underpinnings of pedagogy* knowledge as she was involved in the current study's model.

Table 4.16 Sample evidence of the development of Maya's *identifying error* knowledge during the Implementation and Post-Preparation nodes of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *identifying errors* knowledge
Activity 1: *Enact the Perimeter Teaching!* (in the third node called enactment).		She forgot to express the units of the side lengths, which can lead to misconceptions.
Activity 1: *Let us Examine the Enactment* (in the fourth node called post-preparation).	(Preservice teacher 2: I noticed that the lengths of sides were not shown on such drawings. You assigned cm to the side lengths of the shape while calculating the perimeter of such shapes. Preservice teacher 3: You should be careful in this issue. Otherwise, you prepared the ground for the misconception to occur. In this way, students may not associate the concept of the perimeter with length. Maya: You are right. I missed such an issue. However, I noticed this situation and said cm in my teaching. But still, I promised I would be more careful in other lessons when faced with unexpected situations.	Maya was warned not to cause possible misconceptions with her discourses.

Table 4.16 (Continued).

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *identifying errors* knowledge
Activity 2: *Prepare for Further Teaching* (in the fourth node called post-preparation).	… Teacher (Maya): Yes, well done. Are there any solution ways? Student 1 (fictitious student): I added 12, 8, 12, and 8 and found a 40 cm teacher. Teacher (Maya): You could also solve the question in these ways (I will open GeoGebra software).  First way: The first way included adding the sides lengths of shape by adding end to end.  Second way: The second way included adding together equal lengths.  Third way: The third way included the following formula.	She used mathematically correct expressions to avoid misconceptions.

4.1.4 Mathematical Knowledge: *Theoretical Underpinning of the Pedagogy*

In this section, what kind of materials and teaching strategies Maya planned to use in the perimeter instruction, considering her responses in the pre-and post-test and post-interview, were explored within the context of the *theoretical underpinning of the pedagogy*. As shown in Table 4.1, Maya' developed her knowledge of the *theoretical underpinning of the pedagogy* during the model. Furthermore, the following table summarizes the findings regarding Maya's existing knowledge of the *theoretical underpinnings of pedagogy* to show her knowledge's development from the pre-assessment to the post-assessment.

Table 4.17 Development of the *theoretical underpinning of the pedagogy* knowledge.

Pre-intervention based findings	Post-intervention-based findings
-She provided a lack of information about planning to use representations and materials in teaching the perimeter concept. -She did not plan to provide space for students to construct knowledge regarding the concept. - She focused on getting students procedural fluency by giving direct instruction.	-She provided comprehensive or obvious explanations on how to plan to use representations and materials in teaching the perimeter concept. -She planned to provide space for students to construct knowledge regarding the concept. -She tended to use conceptual teaching strategies by displaying a student-centered approach.

As shown in Table 4.1, before attending to the intervention, she did not provide a clear teaching strategy to overcome students with misconceptions regarding the perimeter concept when she encountered them. Put differently, although she knew she should focus on students' prior knowledge to prevent such misconceptions, she did not know how. For instance, her response in the pre-test about how she planned to deal with such misconceptions was as follows: *"First, I include questions based on measuring students' prior knowledge. Later, I would mainly focus on concrete materials and associating the concept with daily life."* Her responses showed that she tended to use concrete manipulatives to teach the concept, but she did not know

how to use them to enhance the quality of her teachings. She presented simple explanations about overcoming such misconceptions without giving any details. Besides, when asked to define the perimeter concept for middle school students, she offered a similar approach. Her response is presented in Figure 4.30:

> I would mention the concept going through an example. I ask the definitions, such as how we measure the perimeter of a football field. Then, I say that the perimeter is calculated around this field. The definition does not change according to the class level.

Figure 4.30. First evidence of the *theoretical underpinning of the pedagogy* knowledge from the pretest.

Her response displayed that she defined the measurement concept with insufficient information. Although she presented examples from daily life, such as the football field, to describe the perimeter concept, she did not clearly emphasize what the perimeter means. She did not mention how students can measure the perimeter of the football field; however, she assumed that students would understand the concept by such an example. Besides, she stated she would use the same definitions or examples regardless of their grade levels. She did not elaborate on her definition based on students' grade levels.

Moreover, as understood from her statements, she tended to use direct instruction to teach the perimeter. She did not plan to provide students with a space to construct their knowledge of the perimeter concept. Specifically, when asked what kind of representations she planned to use in teaching the concept of perimeter, she said she planned to use them without providing any conceptual basis and an opportunity for students to explore the concept's meaning. Her response in the pre-test for such a question is presented in Figure 4.31.

I mainly give examples from actual-life situations, such as a school football field, to explain the concept of perimeter. At the same time, I can provide an example of a board we see around us. Then, I will give a word problem to teach the concepts that could be related to daily life examples. Besides, I would use GeoGebra and the Word Wall site, which includes game-style math activities. I preferred to use GeoGebra to calculate the perimeter of the square and rectangle. I like to use the Word Wall site more to reinforce the subject.

Figure 4.31. Second evidence of the *theoretical underpinning of the pedagogy* knowledge from the pretest.

As understood from Figure 4.31, she planned to use several representations, such as virtual manipulative, including GeoGebra and Word Wall, and daily life examples in teaching the perimeter concept. However, she did not establish connections between multiple representations and the idea of the perimeter. She mainly preferred such representations to strengthen students' procedural skills through drills. When asked how she planned to continue the lesson, her answer in the pre-test, considering the given student's strategy in Figure 4.4, also confirmed this inference. Her response in the post-test, was presented in Figure 4.32.

I would have included a question similar to this example. Because, as in this example, the edge common to both ways may cause misconceptions for students. If the student cannot understand the concept of perimeter, she tries to reach the result by including this edge.

Figure 4.32. First evidence of the *theoretical underpinning of the pedagogy* knowledge from the posttest.

As understood from Figure 4.32, she ignored the student's way of solution. She used drills to reinforce the student's understanding of the perimeter concept. She tended to use general statements without considering the student's understanding. Besides,

she did not prefer to use the materials in a way that served the purpose of the subject and did not use conceptual strategies to extend the student's understanding. Besides, her explanations implied that she planned to structure the lesson as teacher-centered. Considering all, it could be said that she lacked knowledge of the teaching strategies and teaching materials to teach the perimeter concept before attending to the intervention, which gave information about her knowledge of the *theoretical underpinning of the pedagogy*.

After attending the intervention, it was seen that Maya used conceptual teaching strategies to enable students to construct their knowledge of the concept. Besides, she presented more detailed information about overcoming students' difficulties through manipulative and tended to design her lesson by focusing on collaborative activities. Her response in the post-test, when asked how she planned to overcome such misconceptions, is presented in Figure 4.33.

> First, their prior knowledge can be measured to overcome students' difficulties. For example, it can be observed whether there is a lack of previous knowledge about length measurement. It can be emphasized that the perimeter is a particular (closed) case of length measurement. Concrete materials can help students construct these concepts in their minds. Collaborative activities on perimeter calculation can be included. For example, each group can calculate the perimeter of a different concrete shape with rope and then measure on the ruler to calculate the total length.

Figure 4.33. Second evidence of the *theoretical underpinning of the pedagogy* knowledge from the posttest.

As understood from Figure 4.33, she mentioned the collaborative activities to enable students to construct the perimeter's meaning. In this way, she planned to design lessons as student-centered. She stated that students could explore its meaning as a particular form of length measurement in this way. She planned to allow students to make sense of the concept of perimeter through concrete experiences. Meanwhile,

her response indicated that she was inclined to use concrete manipulatives to teach the concept with a purpose. Besides, she used various representations by putting them on a conceptual basis. Her response in the post-test, when asked what kind of representations she planned to use in teaching the perimeter concept, is presented in Figure 4.34.

> I could give examples that we can use the perimeter of in daily life. To measure the length, I also use a rope whose length is predetermined. I can give several shapes whose perimeter is known, but students are expected to guess the perimeter of the given shapes using the rope. For example, to calculate the perimeter of a notebook, I take a certain length of string, wrap it around that notebook, and then take this length. Later, I would give an example of everyday life where different shapes are intertwined. The student observed that some edges of that shape could not be included in the perimeter. In this way, I follow how students make sense of the concept of perimeter. Besides, I use GeoGebra to allow them to explore the perimeter with various shapes. In this way, they explore that the perimeter is not limited to just a square and rectangle but that the perimeter can be calculated when the shape is closed.

Figure 4.34. Third evidence of the *theoretical underpinning of the pedagogy* knowledge from the posttest.

As understood from Figure 4.34, she planned to use manipulative and examples while explaining the reasons for uses. She tended to provide a chance for students to construct their knowledge of the concept with concrete experiences. Consequently, it could be said that she seemed to use conceptual strategies to teach the perimeter concept. When asked to define the perimeter concept for middle school students, she made the following explanations in the post-test, shown in Figure 4.35.

First, I define the perimeter similarly at each grade level, but there may be a difference according to the grade level. I define the sum of the lengths around the closed shape or the total length around the closed shape. I provide them with concrete experience to calculate the perimeter of several geometric shapes, such as a square, rectangular, and circle. Thus, my examples could change according to grade level. For 5th grade, I can use around the school's football field. In higher grades, I can use circles.

Figure 4.35. Fourth evidence of the *theoretical underpinning of the pedagogy*.

As understood from Figure 4.35, she focused on the conceptual strategies to describe the perimeter concept by providing concrete experiences. In this way, she planned to give a chance for the students to explore the meaning of the perimeter and establish relations between the length measurement and the perimeter. On the contrary, she had mentioned she would use representations without providing any information in the pre-assessments. When asked how to continue the lesson considering the given student's strategy shown in Figure 4.4, it was observed that she also had a similar attitude. Specifically, she displayed conceptual approaches to support students' understanding of the concept using concrete manipulative. Her response in the post-test is presented in Figure 4.36.

Then, I want her to calculate the circumference of the shape. In the meantime, I keep the rope in the classroom. From here, I move on to the circumference calculation of circle shapes. I will calculate the circle's circumference with a string and a ruler and give the circle's radius. My goal is to enable the student not to limit the perimeter with the quadrilaterals. Besides, I will allow her to notice units. Then, I asked if there was a relationship among the circles whose circumferences she calculated (to discover pi and radius).

Figure 4.36. Fifth evidence of the *theoretical underpinning of the pedagogy* knowledge from the posttest.

Her explanation implied that she planned to display actions to support the student's understanding. In addition, she said the following during the post-interview:

> …Now, for example, I would start with a more complex shape instead of a rectangle and square so that the student understands the meaning of the concept…However, it would not have occurred to me to start with complex shapes to teach the concept of the perimeter. I would reinforce the concept by using formulas of the perimeter over shapes such as squares and rectangles, which are mostly given side lengths. I would teach the concept through formulas. That is what I thought. But right now, I would use various representations such as GeoGebra to enable students to construct the knowledge by paying attention to the units and noticing that a perimeter is a particular form of the length measurement.

Considering her explanation, it could be said that she tended to present conceptual teaching strategies to support students' understanding of the perimeter by using representations and manipulatives within a purpose. Besides, after attending the intervention, she changed her teaching approach from teacher-centered to student-centered. Thus, it could be inferred that she had a robust knowledge of *the theoretical underpinning of the pedagogy*. Put differently, it could be said that she had made remarkable progress in her knowledge of *the theoretical underpinning of the pedagogy* regarding the perimeter concept during her involvement in the intervention of the current study. Below, evidence for improving the *theoretical underpinning of the pedagogy* knowledge during the intervention is presented.

4.1.4.1 Evidence for Improvement in *Theoretical Underpinning of Pedagogy* Knowledge during the Model.

As stated above, Maya improved her knowledge of the *theoretical underpinnings of the pedagogy* regarding the perimeter concept during the intervention. In this section, the evidence that she developed this knowledge during her participation in the

intervention of the current study is shared, paying attention to the sequence of activities in each node of the model, as well as the sequence of the nodes of the model, which are called *Pre-Preparation, Preparation, Implementation,* and *Post-Preparation*, respectively. The excerpts from the current intervention's video transcriptions and individual responses to classroom activities were shared to describe Maya's progress in the *theoretical underpinning of the pedagogy* knowledge during the intervention.

As displayed in Table 4.1, Evidence of the development of Maya's theoretical underpinnings of pedagogical knowledge began to be observed as the preservice teachers engaged in Activity 2: *Let Us Examine the Dialogue,* the first node of the model, called *Pre-Preparation.* Indeed, although the main aim of Activity 2 was to develop the preservice teachers' professional noticing skills on the perimeter concept by representing the practice, their responses to the prompt fifth in this activity gave information about their teaching strategies, which was designed in the light of the approximation of the practice. The prompt was, "If you were the student's teacher, how would you continue the lesson? Which questions could you ask the student to advance her learning of the concept?" In the first hour of the implementation of the first node, the preservice teachers were asked to respond to the prompt in this activity individually. Maya responded, "*I could ask similar questions as a drill."* As understood from her explanation, she focused on supporting the way students think from the procedural aspect. In other words, she planned to ask a similar question to enhance students' procedural fluency. In the second lesson hour of the implementation of the first node, the preservice teachers were asked to share their reasoning with the classroom regarding Activity 1, which was related to examining the concept's meaning from learners' and students' eyes, and then Activity 2. During the collective analysis, it was observed that they produced conceptual teaching strategies enriched with materials to support the student's understanding of the given dialogue in Activity 2. Meanwhile, Maya modeled the idea of the perimeter on daily life examples as follows:

> For example, I would like the student to think that between two points, each point is a city, and there are two cities. When calculating the distance between these two cities, we do not take the distance as two by counting the cities. We calculate the distance between two cities, for example, how many kilometers the distance between two cities is. It also has value. In other words, we can think that one started to travel to Istanbul and returned to Istanbul by making a circular route to other cities by moving from Istanbul, Ankara, and Bursa. We calculate that distance in this pathway by looking at the total distance among cities. If we look at the student's method, we might say three as an answer.

Maya's explanation showed that she tried to construct the meaning of the perimeter concept by modeling the distance among the cities. Notably, she modeled the length and the perimeter through daily life examples. Before explaining their reasoning about the prompt, joining the discussions about the concept's meaning and students' difficulties might help her successfully respond to the student's thinking on the perimeter concept using conceptual strategies. Maya's explanation during the post-interview confirmed such a conclusion. She said:

> When I understood the concept, I could explain it better to students…Rather than focusing on the formula, I tried to explain the concept. Besides, I enabled students to use concrete materials to grasp the concept's meaning.

As Maya stated, developing her mathematical knowledge of the perimeter concept contributed to developing her teaching strategies and using materials in the instruction. Rather than basing the teaching strategies on addition operations to explain the perimeter calculation and using drills to facilitate the student's understanding, Maya tended to explain the underlying reasons for the perimeter calculation by modeling the concept through daily life examples. Meanwhile, Maya tried to allow students to make sense of the concept's meaning using materials. Thus, she exhibited exemplary behavior regarding the *theoretical underpinning of the pedagogy* knowledge while engaging in the activities in the first node of the model.

The following table summarizes Maya's developmental process regarding *the theoretical underpinning of the pedagogy* knowledge during this node.

Table 4.18 Sample evidence of the development of Maya's *theoretical underpinning of the pedagogy* knowledge during the Pre-Preparation node of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *theoretical underpinning of the pedagogy* knowledge
Activity 2: *Let Us Examine the Dialogue (Her response).*	I could ask similar questions as a drill.	She planned to use procedural strategies in her lessons.
Activity 2: *Let Us Examine the Dialogue* (Her explanation during the discussion).	"For example, I would like the student to think that between two points, each point is a city, and there are two cities. When calculating the distance between these two cities, we do not take the distance as two by counting the cities. We calculate the distance between two cities, for example, how many kilometers the distance between two cities is. It also has value. In other words, we can think that one started to travel to Istanbul and returned to Istanbul by making a circular route to other cities by moving from Istanbul, Ankara, and Bursa. We calculate that distance in this pathway by looking at the total distance among cities. If we look at the student's method, we might say three as an answer."	She planned to enable students to construct their knowledge regarding the perimeter.

In addition, the activities in other nodes also nurtured the improvement of Maya's knowledge of the *theoretical underpinning of the pedagogy*. For example, at the end of the first node, Maya volunteered to rehearse the perimeter teaching in the simulated classroom environment as part of the approximation of the practice. After that, she was asked to prepare a lesson plan to teach the concept of the perimeter for the next lesson by setting up dialogues between the teacher and fifth-grade students within the context of Activity 1, which was called *Prepare for the Rehearsal* under the second node of the model, called *preparation*. In this plan, Maya focused on

mainly procedural strategies, such as reminding the formula and giving place to the problems that could be solved through the formula while decomposing and approximating the practice, as shown in Figure 4.37. For example, she planned to start her teaching with the following question in her initial lesson plan after she defined the perimeter concept.

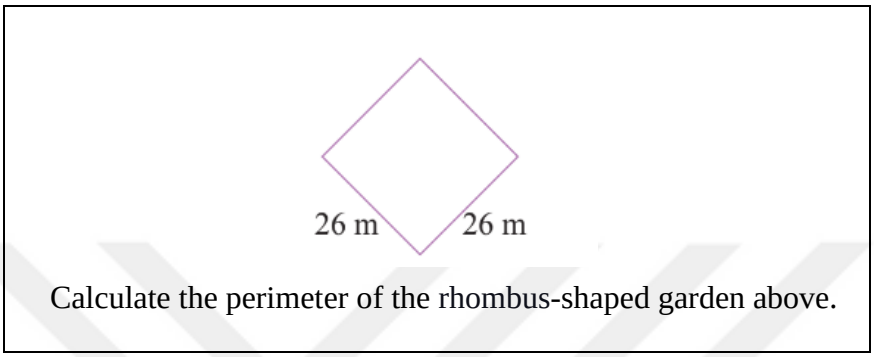


Figure 4.37. The question in the lesson plan.

The perimeter of the rhombus-shaped garden, shown in Figure 4.37, could be calculated by using the formula of the perimeter for parallelograms. At the same time, she did not mainly use manipulatives to construct the perimeter meaning. When examining her lesson plan collectively, the preservice teachers encouraged her to use conceptual strategies to teach the perimeter concept. Indeed, the preservice teachers stated their opinions about this question as follows:

> **Preservice Teacher 3**: If you start with this question, students cannot associate the perimeter with the length measurement.
>
> **Preservice Teacher 5:** They just followed the rules, most probably.
>
> **Maya:** I will consider your sayings.

As explained above, Preservice Teachers 3 and 5 claimed that students might have difficulty establishing relations between the length measurement and the perimeter concept since the question could be solved by following the formula. Later, Preservice Teacher 2 suggested using manipulatives in the lesson to visually

construct the meaning of the perimeter in the students' minds. Mainly, she offered to use the activity shown in Figure 4.38.

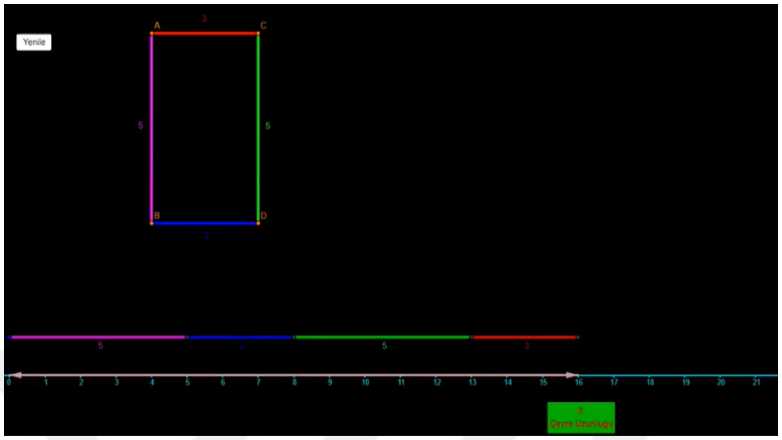


Figure 4.38. GeoGebra activity

In this activity, a rectangle with each side of a different color existed, as shown in Figure 4.28. Students were able to put these side lengths on the number line and measure the perimeter of the rectangle. Besides, they can create various rectangles with various side lengths with the help of GeoGebra. The preservice teachers thought she could use such an activity to allow students to explore the perimeter concept, a particular case of length measurement. In this way, Maya would have obtained an opportunity to explain the underlying reasons for the addition in perimeter calculation and used GeoGebra to facilitate students' understanding of the concept. Although the side lengths of rectangles in GeoGebra did not specify the length measures as stated above, it can be accepted as a suggestion that would strengthen Maya's perimeter teaching with manipulatives and enable Maya to establish relations between length and perimeter. Then, Maya added the following activity to her lesson plan. Considering this, it could be said that she displayed exemplary behavior regarding *the theoretical underpinning of the pedagogy* knowledge. In addition, during the interview, she pointed out the role of her friends in her lesson preparation process as follows:" …*After the discussion, I questioned*

what could be the example that students can make the most sense of the concept or that does not seem complicated at first glance?" Her explanations indicated that such discussion helped her to review her teaching on allowing students to construct their knowledge of the perimeter concept. Another contribution from the other preservice teachers to the improvement of her *theoretical underpinning of the pedagogy* knowledge was that although Maya focused on the activities from the leading textbook MoNE (2018) published in her lesson plan, the preservice teachers suggested using various resources, such as GeoGebra, as mentioned above.

Then, Maya conducted her perimeter rehearsal teaching in the second and third lesson hours of implementing the same node in the simulated classroom environment in Activity 2: *Rehearse the Perimeter teaching* as part of the approximation of the practice. Although Maya was encouraged to use conceptual strategies in the lesson planning process, during the rehearsal teaching, it was seen that Maya tended to use procedural strategies as follows:

> **Maya (in the teacher role):** The rectangle in the figure below comprises congruent squares. Since the long side of the rectangle is 15 cm, let us find the perimeter of the painted square. (The visual regarding the question is presented in Figure 4.39).

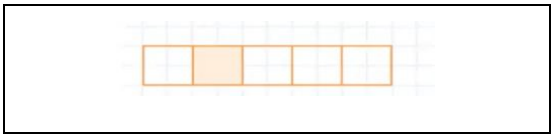


Figure 4.39. The visual regarding the question.

> **…**

> **Preservice Teacher 2 (in the role of a student):** My teacher asked about the perimeter of the painted square since the sides are short, two cm, and the long side is 10 cm. Then I add all the sides and get two, two, two, two, to eight since one side is two cm.

> **Maya (in the teacher role):** Could you reread the question? The question indicated 15 cm of the long side of the rectangle. So, it did not give information that each little square's side is one cm. Let us do it together. Since the shape consisted of five congruent squares, we should divide 15 by 5. Then, we get 3 cm, the small square's side length. Then, to find the perimeter of the painted shape, we add three, three, three, three, and get 1two cm. Do you understand now?

As shown in the dialogue above, Maya tended to explain the solution to the problem directly rather than giving place the student a place to construct her knowledge. She used direct instruction. While examining the rehearsal records of Activity 3: *Let us Examine the Rehearsal* within the scope of pedagogies of reflection in the fourth and fifth lesson hours of node application, the preservice teachers encouraged Maya to design her lesson as student-centered. For example,

> **Preservice Teacher 4:** You could have invited others to evaluate students' thinking.

> **Preservice Teacher 3:** You could have provided a space for the student to think about the solution to the problem after you said 15 cm. In the meantime, you can help the little one.

> **Maya:** Yes, I could have asked.

As understood in the dialogue above, Preservice Teacher 4 suggested that Maya encourages the other students to comment so that the student could build collaboratively on her knowledge of the concept of perimeter. Moreover, Preservice Teacher 3 suggested the student provide opportunities to construct her knowledge. Maya said the following about this issue during the post-interview:

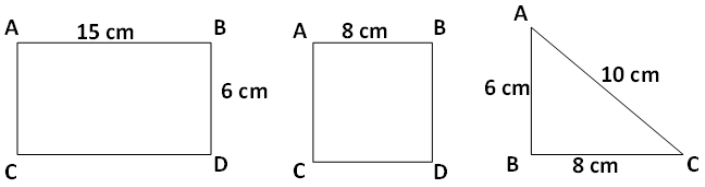
> Before, I did not see any problem in students' solving the problem regarding the perimeter concept with formula…After the rehearsal and the analysis of the rehearsal, I started to think the idea should be introduced by providing

> concrete experiences. Not on the formula…After internalizing the concept, the perimeter formula should be shared with them to improve their procedural fluency. Before, the space should be provided for students to explore the concept's meaning...In this way, I could face fewer difficulties understanding the concept and unexpected students' thinking, and gaining experience before the application helped me… to revise my teaching approach and the approach to the students.

As understood from her statements, she started to approach teaching strategies from a conceptual point of view after her involvement in activities in the second node of the model. She thought that after students gained concrete experience with the concept and operations, she preferred to give the formula to allow students to gain procedural fluency. Hence, such experiences enabled her to restructure her teaching through conceptual strategies and several models. For example, she added the following figures, shown in Figure 4.40, into her revised lesson plan prepared for Activity 4, which was called *Prepare for Enactment,* at the same node of the model, designed in light of the decomposition and approximation of the practice. This activity was assigned as homework.

As shown in Figure 4.40, rather than starting the lesson with a formula, as mentioned in Figure 4.37, she aimed to explain the underlying meaning of the concept through visuals. Overall, since she focused on the conceptual strategies to teach the concept and tended to use various sources and representations to teach the perimeter, her knowledge of the *theoretical underpinning of the pedagogy* was enhanced as she was involved in the activities in the second node of the model, *Preparation.* Table 4.19 is presented to summarize the sample evidence of the development of Maya's *theoretical underpinning of the pedagogy* knowledge during the Preparation node of the model.

Maya (Teacher): Let us find the following figures' perimeter without using the formula.



Student 1 (fictitious student): 15 plus 6 plus 15 plus 6 plus. I found 42 cm for the first one.

Student 2 (fictitious student): 8 plus 8 plus 8 plus 8 plus. I found 40 cm for the second one.

Student 3 (fictitious student): 6 plus 8 plus 10 plus. I found 24 cm for the second one.

Maya (Teacher): Well done.

Figure 4.40. Snapshot from Maya's revised lesson plan regarding evidence of *theoretical underpinning of the pedagogy*.

Then, Maya conducted perimeter teaching in one of the fifth-grade classrooms as part of the pedagogy of the enactment. This teaching was conducted as part of Activity 1, called *Enact the Perimeter Teaching*, in the first and second lesson hours of implementation of the third node, called *Implementation,* within the context of practical hours of the model. Meanwhile, the other preservice teachers, the instructor, and the cooperating teacher attended Maya's lesson as the observer. In her teaching, she mentioned some critical terms to enable students to make sense of the perimeter, such that "…*the perimeter is the measure of the length around the closed shape*," which could be accepted as exemplary behavior regarding her *theoretical underpinning of the pedagogy* knowledge. However, sometimes, and especially when she faced unexpected students' thinking or students' misconceptions about the perimeter concept, it was seen that she tended to base her definition on the formula or addition operation. The following excerpt was shared to display one of such situations.

Student 6 (in the actual classroom): We multiply two sides to find the perimeter of the shapes.

Maya (teacher): How could we find the perimeter? Could we multiply the sides?

Student 2 (in the actual classroom): No. We add the sides of the shapes.

Maya (teacher): Yes, thank you. We can find the perimeter by adding all the side lengths. To summarize, when calculating the perimeter of the shapes, we add all the side lengths. What we did with the square's side length was equal to four times the side length of a square. The perimeter of the rectangle was twice the sum of its short and long sides, or the perimeter of a rectangle was twice the sum of its short and long sides (Evidence of the *theoretical underpinning of the pedagogy* knowledge).

As seen in the dialogue above, Maya, as the classroom teacher, faced one of the typical student misconceptions such that they confused the formula of the area and the perimeter in her teaching. Upon this misconception, she tended to direct students to the correct answer and then defined the concept based on addition operation. Besides, Maya immediately mentioned the formula of perimeter calculation for rectangles and squares after explaining the concept.

Table 4.19 Sample evidence of the development of Maya's *theoretical underpinning of the pedagogy* knowledge during the Preparation node of the model.

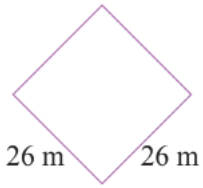
Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *theoretical underpinning of the pedagogy* knowledge
Activity 1: *Prepare for the Rehearsal*	 Maya (teacher): Let us calculate the perimeter of the rhombus-shaped garden above. Student (fictitious): My teacher, I multiply 26 m by four. I obtained 104 m. Maya (teacher): You *added the lengths of the sides around the shape* to find the perimeter.	She tended to use procedural strategies to teach the concept and misinformed students about the meaning of the concept.
Activity 2: *Rehearse the Perimeter Teaching!*	Use the GeoGebra Activity shown in Figure 4.28.	She used virtual manipulatives, aiming to improve the quality of her lessons.
Activity 3: *Let us Examine the Rehearsal*	Preservice Teacher 4: You could have invited others to evaluate students' thinking. Preservice Teacher 3: You could have provided a space for the student to think about the solution to the problem after you said 15 cm. In the meantime, you can help the little one. Maya: Yes, I could have asked.	She was encouraged to use different students' ideas in her lessons to enable students to construct their knowledge.

Table 4.19 (Continued).

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *theoretical underpinning of the pedagogy* knowledge
Activity 4: *Prepare for the Enactment*	Maya (Teacher): Let us find the following figures' perimeter without using the formula. …. Student 1 (fictitious student): 15 plus 6 plus 15 plus 6 plus. I found 42 cm for the first one. Student 2 (fictitious student): 8 plus 8 plus 8 plus 8 plus. I found 40 cm for the second one. Student 3 (fictitious student): 6 plus 8 plus 10 plus. I found 24 cm for the second one. Maya (Teacher): Well done.	She used drawings and asked students to solve questions without formulas to explore the concept's meaning.

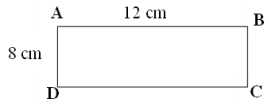
Her behavior implied that she might display procedural teaching strategies when she faced difficulties learning the concept. While examining the video recordings of her perimeter teaching as part of pedagogies of reflection through Activity 1: *Let Us Examine the Enactment* in the first and second lessons hours of the implementation of the fourth node, *post-preparation*, the preservice teachers told Maya to be careful in her language. They emphasized that adding the sides of the shape was not a proper definition for students as follows:

> **Preservice Teacher 3:** You should be careful how you define the concept. It would be best if you emphasized which edges would be added.
>
> **Preservice Teacher 2:** You could model the perimeter concept with concrete manipulatives rather than emphasizing the definition.
>
> **Maya:** I know, but I was pretty excited about the enactment. Good thing I rehearsed; otherwise, I do not know what to do. I said whatever came to my mind when faced with unexpected situations.

The dialogue above indicated that Preservice Teachers 2 and 3 encouraged Maya to use conceptual strategies reinforced with manipulative to improve the quality of her perimeter instruction. In response to these suggestions, Maya stated she was excited and did not know what to do, especially when faced with unexpected thoughts. However, these suggestions must have helped her develop her *theoretical underpinnings of pedagogy* knowledge since it was observed that Maya aimed to use conceptual strategies in her final lesson plan and to support her lesson with materials. In the same intervention node, she reviewed her lesson plan for the following perimeter teaching as part of Activity 2, called *Prepare for Further Teaching*, as part of the decomposition and approximation of the practice. She did this activity as homework. In this regard, the most notable change in her lesson plan was adding the following representations shown in Figure 4.41 to support her perimeter teaching.

Figure 4.41 shows that in subsequent lessons, she planned to have students observe the relationships between operations, formulas, length measurement, and perimeter concepts. In these ways, she planned to explain the underlying causes of the formulas. The following table summarizes Maya's developmental process and her *theoretical underpinning of the pedagogy* knowledge during the second node of the model, *Preparation.*

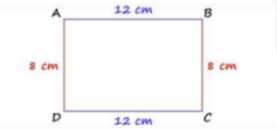
Teacher (Maya): What is the perimeter of the given shape?



Student 2 (fictitious student): Teacher, the answer is 40 cm.

Teacher (Maya): How did you find it?

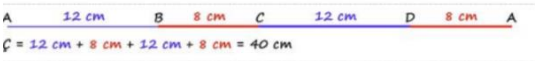
Student 2 (fictitious student): I added 12 and 8 and found 20 cm. Then, I multiply it by two.



Teacher (Maya): Yes, well done. Are there any solution ways?

Student 1 (fictitious student): I added 12, 8, 12, and 8 and found a 40 cm teacher.

Teacher (Maya): You could also solve the question in these ways (I will open GeoGebra software).

First way: 

The first way included adding the sides lengths of shape by adding end to end.

Second way: 

The second way included adding together equal lengths.

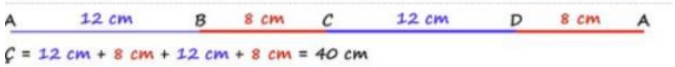
Third way: The third way included the following formula.

Figure 4.41. Evidence of *theoretical underpinning of the pedagogy* knowledge from Maya's re-revised lesson plan.

Table 4.20 Sample evidence of the development of Maya's theoretical underpinning of the pedagogy knowledge during the Preparation node of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *theoretical underpinning of the pedagogy* knowledge
Activity 1: *Enact the Perimeter Teaching!* (*In the third node called Implementation*).	*"…the perimeter is the measure of the length around the closed shape*."	She defined the concept in her teaching from a conceptual aspect.
Activity 1: *Enact the Perimeter Teaching!* (*In the third node called Implementation).*	Student 6 (in the actual classroom): We multiply two sides to find the perimeter of the shapes. Maya (teacher): How could we find the perimeter? Could we multiply the sides? Student 2 (in the actual classroom): No. We add the sides of the shapes. Maya (teacher): Yes, thank you. We can find the perimeter by adding all the side lengths. To summarize, when calculating the perimeter of the shapes, we add all the side lengths. What we did with the square's side length was equal to four times the side length of a square. The perimeter of the rectangle was twice the sum of its short and long sides, or the perimeter of a rectangle was twice the sum of its short and long sides.	She misinformed students about the meaning of the concept.
Activity 1: *Let us Examine the Enactment* (*In the fourth node called post-preparation).*	Preservice Teacher 3: You should be careful how you define the concept. It would be best if you emphasized which edges would be added. Preservice Teacher 2: You could model the perimeter concept with concrete manipulatives rather than emphasizing the definition. Maya: I know, but I was pretty excited about the enactment. Good thing I rehearsed; otherwise, I do not know what to do. I said whatever came to my mind when faced with unexpected situations.	She was encouraged to define the perimeter conceptually and allow students to share their ideas.

Table 4.20 (Continued).

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *theoretical underpinning of the pedagogy* knowledge
Activity 2: *Prepare for Further Teaching* (*in the fourth node called post-preparation*).	Teacher (Maya): Yes, well done. Are there any solution ways? Student 1 (fictitious student): I added 12, 8, 12, and 8 and found a 40 cm teacher. Teacher (Maya): You could also solve the question in these ways (I will open GeoGebra software). First way:  The first way included adding the sides lengths of shape by adding end to end. Second way:  The second way included adding together equal lengths. Third way:  The third way included the following formula.	She paid attention to using virtual manipulatives to increase the quality of her lesson.

Unfortunately, some technical problems during the perimeter may affect the lesson's teaching and teaching. Specifically, since the lesson was held via Zoom program with distance education due to the COVID-19 pandemic, Maya and the students had some difficulties during the lesson. For example, as the lesson teacher, Maya could not use concrete manipulatives such as geoboard and geometry strips and virtual manipulatives such as GeoGebra in her teaching. Still, she prepared a PowerPoint presentation for her lesson and gave place to the visual examples to teach the perimeter concept, as shown in Figure 4.42.

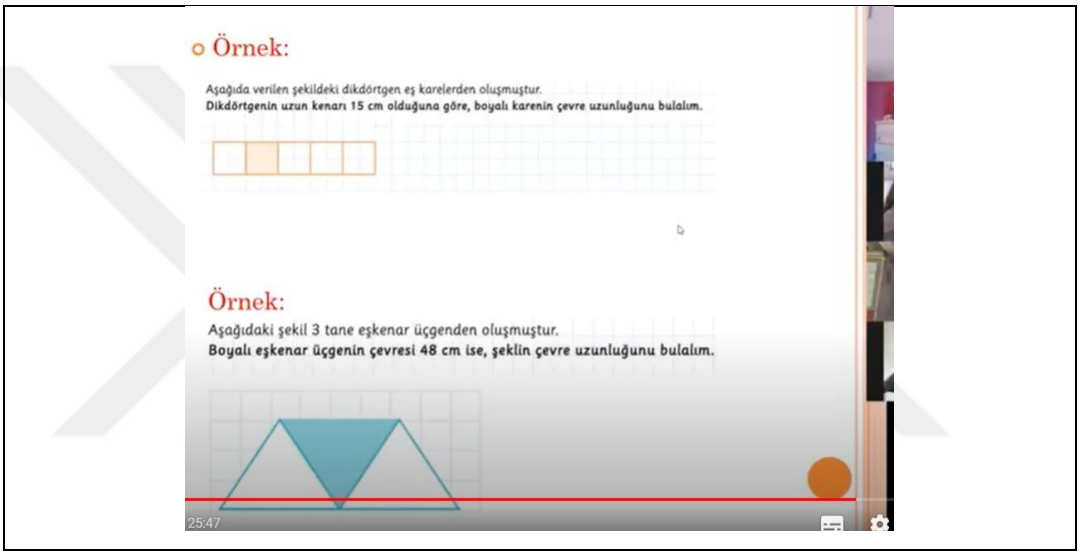


Figure 4.42. Sample examples used in Maya's perimeter teaching.

As shown in Figure 4.42, she selected examples whose drawings existed on dot paper so that students could easily calculate the shape's perimeter by using side their sides' lengths. In addition to the PowerPoint presentation, she wanted to use the EBA platform to support her lesson visually through the videos. Unfortunately, she faced a technical problem and could not open it during her teaching. However, she tried to open such a video for approximately three minutes. Then, she gave up opening it but lost minutes in her lesson. Figure 4.43 shows the screenshots were taken from her lessons to display her effort to open the video.

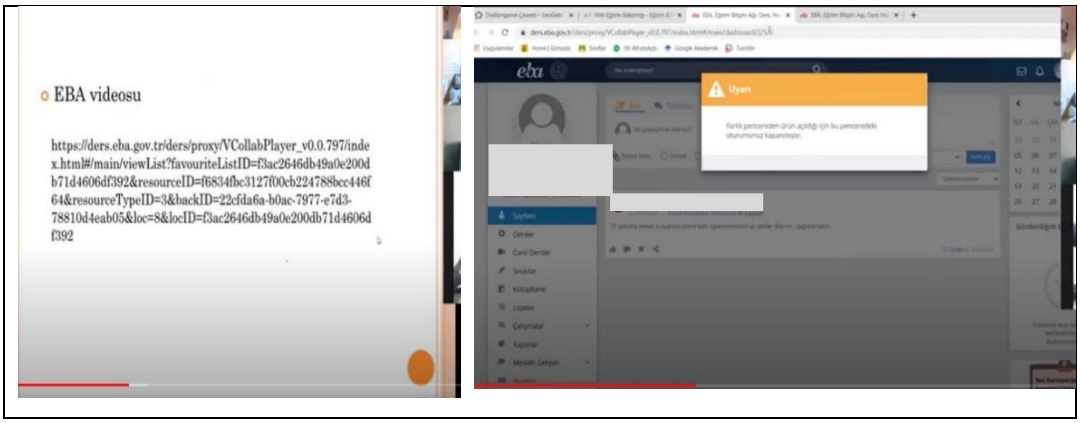


Figure 4.43. Screenshots were taken from Maya's perimeter teaching.

Furthermore, she sometimes took control and taught a teacher-centered lesson due to technical problems. For example, she shared the example with the students and gave time to them to think about it. Then, she selected one of the students to solve the problem. However, the selected student did not use the pencil on the whiteboard, which is the Zoom program's property, in solving the problem. Then Maya sharpened the pen, took control, and promised the student to explain her solution. Meanwhile, Maya started to model the students' words with the pencil on the whiteboard as follows:

Student 1 (in the actual classroom): Teacher, my answer is 32.

Maya (teacher): How did you find it?

Student 1 (in the actual classroom): I counted.

Maya (teacher): How did you count? Could you show it on the board?

Student 1 (in the actual classroom): (He tried to use a pencil on the Zoom Program but failed.).

Maya (teacher): Let us count together. We select a starting point, right? We need to close the shape by going around the shape. Let us count. 1,2 (She used the board, as shown in Figures 4.44 and 4.45).

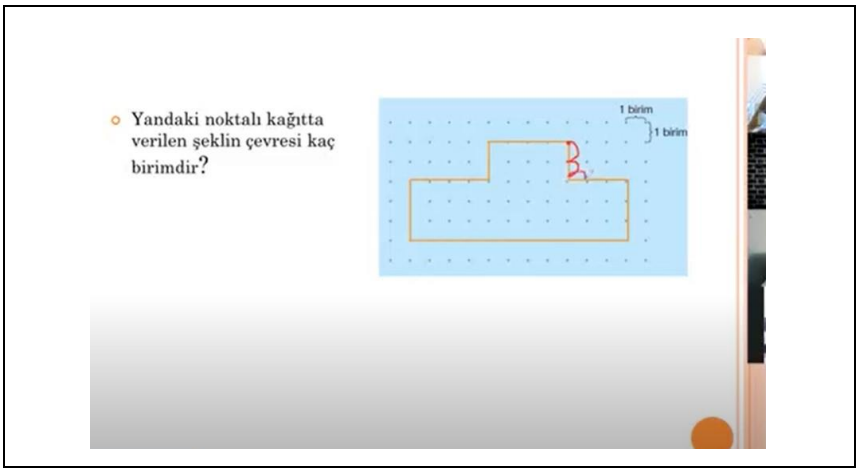


Figure 4.44. Student 1's first drawings.

Student 1 (in the actual classroom): 2,3.....30

Maya (teacher): 30,31

Student 1 (in the actual classroom):32.

Maya (teacher): 32, yes. We found 32 units and counted all units. Is it clear? (She drew the following shape.).

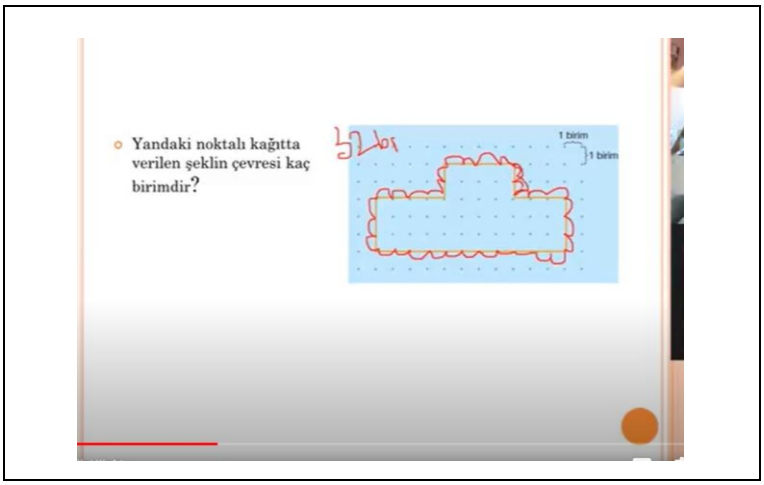


Figure 4.45. Student 1'ssecond drawings.

Student 1 (in the actual classroom): If my teacher asked about the perimeter, it would be 32 cm, right?

> **Maya (teacher):** Here, the distance is not given as cm. It is provided as a unit.

Maya interfered with Student 1's solution as she could not use the pen to explain Student 1's answer, as shown in the dialogue above. In fact, at the beginning of the lesson, Maya allowed the student to show and explain her answer. During the post-interview, she said to explain such a situation:

> In my teaching, I wanted to understand the students' solutions as much as possible. However, technical problems on the internet made me panic about the instructions because I was afraid of being unable to set the time.

As can be seen in her explanations, Maya's teaching through the Zoom program caused her not to be able to teach her lesson as she wanted due to technical problems. Considering all these, it can be said that if she had conducted the course face to face, she could have better reflected and observed the development of her knowledge of the *theoretical underpinnings of the pedagogy*.

In summary, Maya's *theoretical underpinnings of pedagogy* knowledge were strengthened during the intervention. Maya co-constructed her teaching strategies in the concept of perimeter and was encouraged to design her lesson as student-centered during the collective analysis. When preparing the lesson plan, she gave space to activities that allowed students to construct the concept's meaning. In addition, she demonstrated exemplary behavior regarding the *theoretical underpinnings of the pedagogy* knowledge in her perimeter lesson. In addition, the analysis of the perimeter lesson enabled her to revise her teaching strategies and encouraged her to use manipulatives and design student-centered lessons. Thus, her knowledge of the *theoretical underpinnings of the pedagogy* developed as she participated in the intervention. The following section provides information about developing her professional noticing skills of the perimeter concept during the intervention.

4.2 Professional Noticing Skills on the Perimeter Concept

As stated above, the current study aimed to support the improvement of preservice middle school mathematics teachers' mathematical knowledge and professional noticing skills related to the concept of perimeter through the development of a model based on the cycles of enactment and investigation within the framework of design-based research. In this section, the analysis of the improvement of the prospective mathematics teacher's professional noticing skills on the perimeter concept is presented under the following sections: Attending, interpreting, and deciding. As stated above, these are components of professional noticing skills proposed by Jacobs et al. (2010). The data mainly obtained from her pre-and post-test responses and her explanations during the post-interview were used to reveal to what extent the prospective mathematics teacher (Maya) improved her professional noticing skills on the perimeter concept. As Table 4.21 shows, Maya progressed her attending and deciding skills from lack to robust levels. In addition, her interpreting skills were enhanced from limited to robust levels from the pre-assessment to the post-assessment.

Table 4.21 The development of Maya's professional noticing skills during the model.

		Nodes of the model				
Components of Professional Noticing Skills	Pre-test	Node 1: Pre-Preparation	Node 2: Preparation	Node 3: Enactment	Node 4: Post-Preparation	Post-test
Attending	Lack of evidence	x*	x	x	x	Robust evidence
Interpreting	Limited evidence	x			x	Robust evidence
Deciding	Lack of evidence	x	x	x	x	Robust evidence

** The "x" sign indicates the node where the exemplary behavior(s) related to the relevant code is observed in the model.*

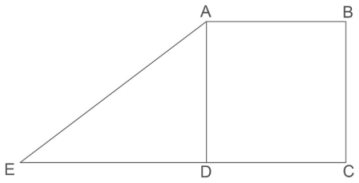
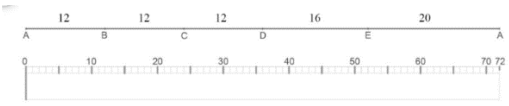
Since the improvement was observed in all components of the professional noticing skills from the pre-assessment to the post-assessment, it could be implied that the model nurtured Maya's professional noticing skills on the perimeter concept. Evidence of the development of each component of professional noticing skills during the intervention is provided in the following subsections.

4.2.1 Development of *Attending Skill*

In this section, how Maya identified mathematically important details in students' strategy, such as how they count numbers, find perimeter, and use manipulative, and whether she correctly evaluated the students' strategy, were examined within the context of *attending skills*. As Table 4.21 shows, Maya progressed her *attending skill* from lack to robust levels from the pre-assessment to the post-assessment.

To measure her professional noticing skills in the pre-and-post-tests, she was asked to examine the fifth-grade student's written solution strategy for the given problem related to finding the perimeter of the given shape. The problem and the strategy used by the student to solve the problem are shown in Table 4.22.

Table 4.22 The question and the student's solution strategy for the question.

Let us imagine that you are teaching the concept of the perimeter to 5th grades, and you started the lesson by asking the following question: ABCD is a square, and ADE is a right triangle. Since the perimeter of triangle ADE is 48 cm, IECI = 28 cm, and IABI = 12, calculate the perimeter of trapezoid ABCE.  While solving this question, your student took the ruler on the board and made some measurements. Meanwhile, the following solution appeared on the board, and she told the class: "I found it in a very different way. My result is 72." 

As shown in Table 4.22, while calculating the perimeter of the given figure, the student placed the side lengths outside the figure on the ruler, starting from the line segment AB, end to end, respectively. Thus, instead of reaching the result by adding the side lengths, she reached the total length on the ruler. However, she did not indicate the measures of length in her response. For this reason, her response was not mathematically correct.

To measure Maya's attending skills, the question "*Explain in detail how the student solved this problem (the strategy she used, etc.). Do you think the student's solution is correct? Why?"* was directed to her in the pre and post-test. Her response in the pre-test was coded at the lack of evidence level because Maya accepted the student's answer as correct even though the student did not express the units of length in her answer. Her response to such a question in the pre-test is as follows: "*I think the student's solution is correct because she made the side lengths of the given shape a straight line*." As shown in her explanations, she mentioned some details of the student's solution strategy, such as making the side lengths of the shape a straight line, even if she did not use the mathematical language correctly. However, she could have mentioned that the student had written the lengths of the sides around the shape on the ruler. Meanwhile, she would be expected to emphasize the lack of cm in the student's answer, and she should have said that the student's answer was incorrect for this reason. Maya did not make these points in her response, so her *attending s*kill was coded at the lack of evidence level.

Her response in the post-test indicated that she attended to the student's solution properly and expressed that the use of cm was missing in the student's answer. Her response is presented in Figure 4.46. Figure 4.46 shows that she noticed that the student did not use the length measures in the result. In addition, she explained the student's solution strategy by giving all the details and using mathematically correct language, such as adding the lengths of the sides around the shape end to end through the ruler. Thus, *her attending skill* was coded at a robust level of evidence. Considering her pre- and post-test responses, it could be said that her *attending skills*

progressed from lacking to robust levels throughout the intervention. In the following section, the evidence for improvement in *attending skills* during the intervention is presented.

The student's strategy is correct, but her expression of the result is wrong. She found the perimeter of the Figure by calculating the total length formed by adding the lengths of the sides around the shape end to end through the ruler. She may have forgotten to mention units of measure out of excitement. This solution method can be used in lessons.

Figure 4.46. Evidence of *attending skills* from Maya's response in the pretest.

4.2.1.1 Evidence for Improvement of *Attending Skill* during the Model

As stated above, Maya's *attending skill* regarding the perimeter concept was enhanced as she was involved in activities in the nodes of the model. In this section, the evidence that she developed this skill during her participation in the intervention of the current study is shared, paying attention to the sequence of activities in each node of the model, as well as the sequence of the nodes of the model, which are called *Pre-Preparation, Preparation, Implementation,* and *Post-preparation,* respectively. The excerpts from the current intervention's video transcriptions and individual responses to classroom activities were shared to describe Maya's progress in *attending skills* during the intervention.

As displayed in Table 4.21, evidence for the development of Maya's *attending skills* started to be observed when engaging in the activities in the first node of the model, called *Pre-preparation*. In this node, the preservice teachers were asked to examine the activities called Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes*, and Activity 2: *Let Us Examine the Dialogue*, first individually and then collectively. Specifically, the preservice teachers were asked to respond to all questions individually in the first lesson hour and then explain

their reasoning in the second lesson hour of implementing the first node. In the first activity, the preservice teachers were asked to reason with fifth-grade students' difficulties with the perimeter concept, considering the given problem as a part of the fourth question, as shown in Table 4.3. Such question aimed to enable preservice teachers to identify students' possible difficulties regarding the perimeter concept. Maya responded to such a question: "*Students count the dots twice and reach the incorrect result."* As understood from her response, she mentioned a few details of the student's strategy, such as counting dots, and evaluated the correctness of the student's strategy. Then, in the second hour of the implementation of the first node, the preservice teachers discussed their predictions on students' misconceptions of the perimeter concept. Meanwhile, they shared some details of potential students' strategies involving misconceptions as follows:

> **Preservice Teacher 2:** Students could count the dots of the shape by closing the shape.
>
> **Preservice Teacher 4:** Students can be confused about where they started. Thus, I would tell them to mark the starting point…
>
> **Maya:** …They can count the squares of the units in this question (Evidence of *attending skills*).

As shown in the dialogue above, Preservice 2 mentioned student strategy details such as counting the dots. Moreover, Preservice Teacher 4 emphasized that students confused the starting point in the perimeter calculation. Then, Maya expressed the details of the student's strategy, such as counting unit squares. Since they mentioned misconceptions, they did not need to evaluate their accuracy. Reasoning the students' possible misconceptions and discussing the details of students' strategies contributed to the development of Maya's *attending skills*. Then, within the scope of Activity 2 (see Figure 4.46) in the first node of the model, the following dialogue between the

student and the teacher regarding the perimeter concept was shared with the preservice teachers as part of the representation of the practice.

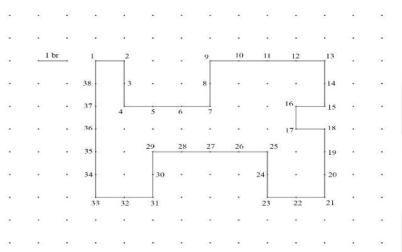
Student: Teacher, the answer is 38.

Teacher*:* How did you calculate Aleyna? Can you explain?

Student: Teacher, I chose a reference point for myself, so I started from one point and reached that point while passing 38 points.

Teacher*:* Can you show your solution on the board?

Student*:* Of course, my teacher. I counted the points to calculate the shape's perimeter and got 38.



Teacher: How do we calculate the perimeter of the shape, Aleyna?

Student*:* We count the dots around the shape. Specifically, we select a starting point around the shape and count the points until we reach the same point, teacher.

Teacher*:* So, do we count the points when calculating the perimeter, Aleyna?

Student: Yes, teacher.

Teacher: How would you calculate if I did not place the dotted paper under the shape and asked you this question?



Student: (16 + 4). x 2 = 40 cm teacher.

Teacher: How did you calculate?

Student*:* This shape is rectangular. To calculate the perimeter of the rectangle, I use the formula (a + b) x 2.

Figure 4.47. The dialogue used in Activity 2.

In this activity, the student returned to a point from which she had previously started, and in this way, she obtained the correct results even if her solution was incorrect. The pre-service teachers were expected to identify the mathematically significant details in the given student's thinking, such as counting dots, knowing the abbreviations for length units, calculating the shape's perimeter, and evaluating the correctness of the student's solution. The analysis of Maya's responses to such a question was, *"The student counted the dots*." Her response showed that she only mentioned counting points as the student's strategy. However, she could have pointed out other details, such as the fact that the student did not close the shape when counting the dots. After the discussion on Activity 1, the preservice teachers were asked to identify the student's strategies in Activity 2 collectively. At this point, Maya started the discussion by evaluating the procedural correctness of the student's strategy. The preservice teachers then considered whether the logic behind the student's strategy was mathematically correct. In other words, they discussed the mathematics in the student's strategy. For example,

> **Researcher:** What do you think about the student's strategy? Do you think it's true?
>
> **Maya:** Wrong, my teacher.
>
> **Researcher:** Why?
>
> **Maya**: She did not count the length between two points but counted the points.
>
> **Preservice Teacher 3**: I disagree with you. The solution was right because when we count the points of a closed shape like this, we calculate its perimeter since the number of points equals the number of line segments.
>
> **Researcher***:* Do you agree with Preservice Teacher 3's opinion?

Preservice Teacher 4*:* I was very conflicted, but I thought the student's solution was wrong while answering. However, I tried what PST1 said in other shapes, which could be true.

Maya: I do not know, but for example, when calculating the perimeter, we say the sum of the sides. So, we need to know the distance. So, length. So, we do not need to know the sum of the lengths of the sides for the perimeter. We need to know the distance between two points, that is, how much so we can find the perimeter. We need the distance between the points, not the points. So, we need to have the length.

Preservice Teacher 2: Since dots are not units.

Maya: I do not think counting the point is used in perimeter calculation…

The extract above shows that Maya started the discussion by saying that the student's strategy was wrong. After the facilitator asked her to explain her reasoning, she mentioned that counting dots does not mean counting the distance. However, Preservice Teacher 3 disagreed with Maya's idea and claimed that the student's answer was acceptable if the student followed such a way within a certain logic that the number of dots gives the shape's perimeter. Meanwhile, Preservice Teacher 4 claimed that Preservice Teacher 3's idea could be correct. However, Preservice Teacher 2 and Maya concluded that the strategy was mathematically incorrect as counting lengths or distances to calculate the perimeter was necessary. Through this fruitful discussion of the student's strategy, Maya reviewed her mathematical knowledge of the concept of perimeter. In addition to examining the student's approach from a mathematical perspective, Maya continued the discussion by focusing on the pedagogy in the student's strategy as follows:

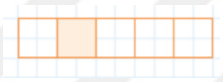
Maya: Also, if we accept the student's solution method is correct, like Preservice Teacher 3 says, the students and we can generalize it to other situations. For example, let us image a rectangle. They could say that the

rectangle's perimeter is four because it has four corners. So, I think this solution method was wrong, too.

Preservice Teacher 5: You are right.

Maya's explanation showed that she questioned the correctness of the student's strategy from a pedagogical point of view. She claimed that counting dots to find the shape's perimeter might pave the way for other students to make such inferences that the perimeter of the square and rectangle must also be four because they have four corners. In this way, the pre-service teachers focused on the pedagogy of the given strategy about the concept of perimeter. At the end of the discussion, they all agreed that the student's solution was incorrect by discussing the details of the student's strategy from a mathematical and pedagogical point of view as Maya examined the details of the student's strategy from a mathematical and pedagogical point of view and learned to identify the details of the student's strategy correctly. Thus, such a collective analysis of the student's strategy contributed to the development of Maya's *attending skills* on the perimeter concept. Considering all the evidence, it could be said that she had a chance to develop her *attending skills* while engaging in the activities in the *Pre-Preparation* node of the model organized to improve preservice teachers' mathematical knowledge and professional noticing skills related to the perimeter concept. The following table summarizes Maya's developmental process regarding her *attending skills* during this node. In addition to the first node of the model, called *Pre-Preparation*, some activities in the second node of the model, called *Preparation*, which were mainly designed to prepare the preservice teachers to make instructional decisions, were provided as evidence of the development of her *attending skills*. Specifically, at the end of the first node, Maya volunteered to rehearse perimeter teaching in the simulated classroom environment as part of the approximation of the practice. She was then asked to prepare a lesson plan to teach the concept of perimeter for the next lesson by setting up dialogues between the teacher and the fifth-grade students. Thus, it could be said that she decomposed and approximated the teaching practices while preparing the lesson plan. In this context,

in the first lesson of the implementation of the second node, the preservice teachers collectively reviewed Maya's lesson plan to improve the quality of her teaching as part of Activity 1: *Preparing for the Rehearsal.* While creating the dialogues, Maya was expected to produce possible fifth-grade solution strategies for the problems in her lesson plan. Although the strategies in the dialogues generally involved common misconceptions about the concept of perimeter or expected correct solutions rather than high-level thinking, talking about the details of the student's strategies while creating such a dialogue contributed to developing her *attending skills.* The following dialogue shown in Figure 4.48 was taken from Activity 1, called *Prepare for the Rehearsal.*



Maya (teacher): The rectangle in the Figure on the board comprises congruent squares. Calculate the perimeter of the painted square since the longest side of the rectangle is 15 cm. Who wants to solve the question?

Student (fictitious student): My teacher. May I solve it?

Maya (teacher): Sure.

Student (fictitious student): Divide 15 by five. We get three cm. If the side of the square is three cm, I find the perimeter of 12 cm.

Maya (teacher): Well done.

Figure 4.48. Evidence of *attending skills* from Maya's initial lesson plan.

Table 4.23 Sample evidence of the development of Maya's *attending skills* during the Pre-Preparation node of the model.

Activity Name	Excerpts from Maya's responses or explanations	Inferences about Maya's *attending skills*
Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes (Her response).*	Students count the dots twice and reach the incorrect result.	She identified students' possible difficulties regarding the perimeter concept.
Activity 2: *Let Us Examine the Dialogue (Her response).*	The student counted the points.	She partially mentioned the details of the student's strategy in the given dialogue.
Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes* (*Her explanation during the discussio*n).	Preservice Teacher 2: Students could count the dots of the shape by closing the shape. Preservice Teacher 4: Students can be confused about where they started. Thus, I would tell them to mark the starting point… Maya: …They can count the squares of the units in this question. (Evidence of *attending skills*).	She reasoned students' possible difficulties regarding the perimeter concept.

Table 4.23 (Continued).

Activity Name	Excerpts from Maya's responses or explanations	Inferences about Maya's *attending skills*
Activity 2: *Let Us Examine the Dialogue* (*Her explanation during the discussion*).	Researcher: What do you think about the student's strategy? Do you think it's true? Maya: Wrong, my teacher (Evidence of *attending skills*). Researcher: Why? Maya: She did not count the length between two points but counted the points (Evidence of *attending skills*). … Maya: I do not know, but for example, when calculating the perimeter, we say the sum of the sides. So, we need to know the distance. So, length. So, we do not need to know the sum of the lengths of the sides for the perimeter. We need to know the distance (Evidence of *attending skills*). … Maya: Also, if we accept the student's solution method is correct, like Preservice Teacher 3 says, the students and we can generalize it to other situations. For example, let us image a rectangle. They could say that the rectangle's perimeter is four because it has four corners. So, I think this solution method was wrong, too (Evidence of *attending skills*).	She identified the details of the given student's strategy and evaluated the correctness of the student's result.

Figure 4.48 shows that Maya produced the student's solution strategy: "*Divide 15 by five, and we get three cm. If the side of the square is three cm, I find the perimeter of it as 12 cm*." While producing such a solution, Maya reasoned with the details of the students' strategies. Thus, producing such solution strategies helped Maya develop her *attending skills*.

Similarly, during the second and third lessons where the same node was applied, Maya conducted perimeter teaching in a simulated classroom environment by approximating the practice through Activity 2: *Rehearsing Perimeter Teaching*. During the perimeter lesson rehearsal, Maya, as the teacher, had the opportunity to encounter possible student strategies related to perimeter calculation that the other preservice teachers had produced as students.

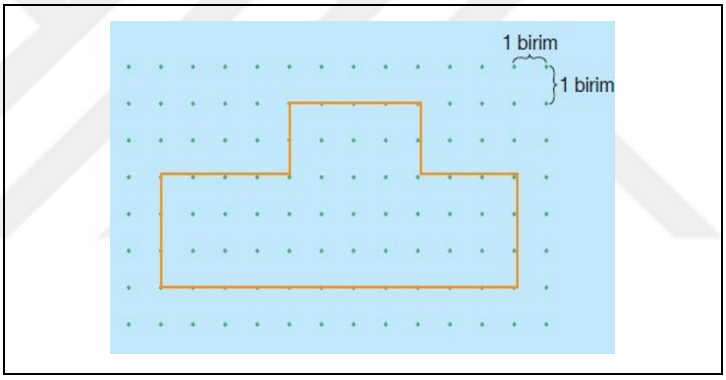


Figure 4.49. The figure used in the question.

Maya (in the teacher role): What is the shape's perimeter given on the dotted paper above? (The figure used in the question is presented in Figure 4. 38).

Preservice Teacher 3 (in the role of a student): I counted the units and found 32.

Maya (in the teacher role): Well done.

As understood from the dialogue above, Preservice Teacher 2 produced the solution way for the given problem, shown in Figure 4.49, as a student. Meanwhile, as the lesson's teacher, Maya attended to the student's solution strategy, which included counting the units around the shape, during the rehearsal. This experience provided a crucial opportunity for her to meet the potential students' thinking during the instruction. However, as seen in the dialogue, she did not correctly evaluate the student's strategy since the student did not express the length of units in his answer. If the preservice teachers meet such situations when collectively examining the video recordings of Maya's rehearsal teaching within part of *Activity 2: Let Us Examine the Rehearsal* in the fourth and fifth lesson hours of the implementation of the second node, they paid more attention to the details of students' strategy and or the procedural correctness of their strategy as part of the pedagogy of reflection. Concerning the above dialogue, they made the following comments:

> **Preservice Teacher 3:** You accepted the answer as correct when I told 32. Did you notice?
>
> **Maya:** Yes, it was.
>
> **Preservice Teacher 2:** Does using 32 in the perimeter calculation make sense?
>
> **Maya:** I should have emphasized units. You are right. I overlooked the units when Preservice Teacher 3 gave the correct answer.

The dialogue above shows that Preservice Teacher 3 reminded Maya to use units in her answers. As such, collective analysis enabled Maya to identify the details of students' strategies and to evaluate the correctness of these strategies in more detail. Consequently, it could be said that such analysis encouraged Maya to improve her *attending skills.* Then, considering the feedback she obtained so far, Maya was asked to revise her lesson plan as homework within the scope of Activity 4: *Prepare for Enactment,* located at the same node of the model. When examining her revised lesson plan as part of the decomposition and approximation of the practices, it was

seen that Maya detailed possible students' strategies. To illustrate, Maya prepared the dialogue shown in Figure 4.50 as part of Activity 1, *Prepare for the Rehearsal.* In this activity, she gave place to a potential student's solution strategy: "*Divide 15 by five. We get three cm. If the side of the square is three cm, I find the perimeter of 12 cm*." When examining her lesson plan as part of Activity 4: *Prepare for the Enactment*, she elaborated the solution strategy for the mentioned problem from the student's perspective, as shown in the figure. Table 4.24 summarizes Maya's developmental process regarding her *attending skills* during this node.



Maya (teacher): The rectangle in the Figure on the board comprises congruent squares. Calculate the perimeter of the painted square since the longest side of the rectangle is 15 cm. Who wants to solve the question? How do we calculate the shape's perimeter?

Student (fictitious student): *By adding all the side lengths of the shape.*

Maya (teacher): All sides of the shape? Can I add these sides exist inside the shape?

Student (fictitious student): *We can add these shapes existing around the shape.*

Maya (teacher): Well done. We should add the sides' lengths around the shape. So, how do we calculate the shaded shape's perimeter?

Student (fictitious student): *Divide 15 by five. We get three cm. If the side of the square is three cm, I find the perimeter of 12 cm.*

Maya (teacher): Well done.

Figure 4.50. Evidence of *attending skills* from Maya's final lesson plan.

Table 4.24 Sample evidence of the development of Maya's *attending skills* during the second node of the model.

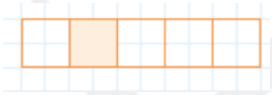
Activity Name	Excerpts from Maya's responses or explanations	Inferences about Maya's *attending skill*
Activity 1: *Prepare for the Rehearsal*	 Maya (teacher): The rectangle in the Figure on the board comprises congruent squares. Calculate the perimeter of the painted square since the longest side of the rectangle is 15 cm. Who wants to solve the question? Student (fictitious student): My teacher. May I solve it? Maya (teacher): Sure. Student (fictitious student): Divide 15 by five. We get three cm. If the side of the square is three cm, I find the perimeter of 12 cm. Maya (teacher): Well done.	She produced a solution strategy for the problem from the student's eyes.
Activity 2: *Rehearse the Perimeter Teaching!*	... Maya (in the teacher role): What is the shape's perimeter given on the dotted paper above? Preservice Teacher 3 (in the role of a student): I counted the units and found 32. Maya (in the teacher role): Well done.	Maya attended to the details of the student's strategy and evaluated it correctly in the rehearsal teaching.

Table 4.24 (Continued).

Activity Name	Excerpts from Maya's responses or explanations	Inferences about Maya's *attending skill*
Activity 3: *Let us Examine the Rehearsal*	Preservice Teacher 3: You accepted the answer as correct when I told 32. Did you notice? Maya: Yes, it was. Preservice Teacher 2: Does using 32 in the perimeter calculation make sense? Maya: I should have emphasized units. You are right. I overlooked the units when Preservice Teacher 3 gave the correct answer.	Maya was encouraged to pay attention to the details of the student's strategy, such as expressing units.
Activity 4: *Prepare for the Enactment*	 Maya (teacher): The rectangle in the Figure on the board comprises congruent squares. Calculate the perimeter of the painted square since the longest side of the rectangle is 15 cm. Who wants to solve the question? How do we calculate the shape's perimeter? Student (fictitious student): *By adding all the side lengths of the shape*. Maya (teacher): All sides of the shape? Can I add these sides exist inside the shape? Student (fictitious student): *We can add these shapes existing around the shape.* Maya (teacher): Well done. We should add the sides' lengths around the shape. So, how do we calculate the shaded shape's perimeter? Student (fictitious student): *Divide 15 by five. We get three cm. If the side of the square is three cm, I find the perimeter of 12 cm.* Maya (teacher): Well done.	Maya detailed the student's solution strategy, as mentioned in the previous line, as part of Activity 1: *Prepare for the Rehearsal.*

While conducting the perimeter teaching in the actual classroom environment within the context of Activity 1, *Enact the Perimeter Teaching* as part of the pedagogy of the enactment, under the third node of the model, called *Enactment*, Maya displayed exemplary behavior regarding *attending skills*. To display such a situation, the following excerpt is presented.

The question: The Figure (shown in Figure 4.51) below is made up of three equilateral triangles. If the perimeter of the painted equilateral triangle is 48 cm, what is the perimeter of the Figure?

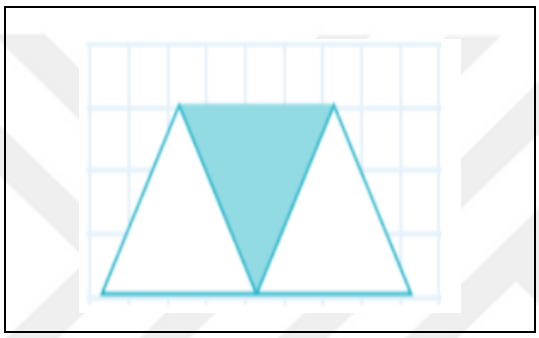


Figure 4.51. The question's figure is shown in the class.

Maya (as the teacher): Who wants to solve it?

Student 2 (in the actual classroom): May I solve it, teacher?

Maya (as the teacher): Sure. Okay. We are listening to Student 2.

Student 2 (in the actual classroom): The perimeter of the painted equilateral triangle is 48 cm. It is first divided 48 by three due to the three congruent lengths of the equilateral triangle. Then, we got 16 cm for each side length. The perimeter of the trapezoid was the sum of five sides. Thus, it was multiplied by 16 by five and found 80 cm of the perimeter of the trapezoid.

Maya (as the teacher): Well done. What did your friend do? Firstly, she divided 48 cm by 3 to obtain one of the side lengths of the equilateral triangle. Since the perimeter of the trapezoid consisted of five sides, one equal to 16

cm, she multiplied 16 by five and found 80 cm. Did you understand? (Evidence of *attending skills*).

Student 2 produced the correct solution strategy for the question in the dialogue above. Then, Maya evaluated the mathematical details of the student's strategy by saying "*well done*" and attended to the details of her strategy by paraphrasing what Student 2 said. Unfortunately, she sometimes missed or misjudged the details of students' strategies, especially when she encountered unexpected student thinking in her teaching. To illustrate, Student 3 produced an alternative solution to the same problem during the perimeter teaching after Student 2. The following dialogue took place:

Student 3 (in the actual classroom): Teacher, I found another method.

Maya (teacher): Yes, how did you solve it?

Student 3 (in the actual classroom): This 48, this 48, this 48 (she shows three congruent triangles inside the trapezoid). Then, we multiply 48 by three, so we get 144. We will remove the burden of the two sides. One side length is 16 cm. If we multiply 16 by two, we get 32. I found it wrong with this method.

Maya (teacher): What is it when you subtract?

Student 3 (in the actual classroom): When I subtract it, it becomes 144-32, and we get 112.

Maya (teacher): You subtracted 144 from 32 and got 112. So, we cannot find it that way, right?

Student 3 (in the actual classroom): Yes.

As the dialogue above shows, Student 3 produced an unexpected solution strategy for the problem. Since Maya, as the teacher of the lesson, was not expecting such a

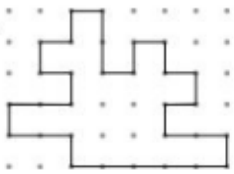
solution strategy, she missed some details of the student's strategy. She intervened without letting Student 3 complete it. Sometime later, Student 3 noticed the error in her thinking. She asked to speak again and came to the correct conclusion. This time, Maya was seen to be trying to skip over Student 3's answer as follows: " *Hmm. Yes, yes, you included them twice. You did it the correct way. Okay, let us turn to the other question*."

When examining the video recordings of her teaching as part of Activity 1, called *Let Us Examine the Enactment*, in the first and second lesson hours of implementing the fourth node, called *Post-preparation*, Maya and others obtained chances to investigate some details of students' strategies collectively through the pedagogies of reflection. For example, they made the following comments regarding the above dialogue:

> **Maya:** I was shocked. The child actualized her mistake when she thought out loud. It is good that she noticed; otherwise, I would have said "wrong" to her answer.
>
> **Preservice Teacher 4:** We must give children more chances to express themselves.
>
> **Maya:** And we should not just say wrong. We should give the students a way to think of other solutions.
>
> **…**

Her responses in the dialogue implied that she could not have judged the student's strategy as correct or incorrect without allowing her to explain herself. Such analysis enabled her to identify the details of students' strategies to prepare her for further teaching, which contributed to improving her *attending skills*. Meanwhile, she reviewed or created the dialogues in her lesson plan, considering her experiences as part of Activity 2, *Prepare for Further Teaching*, located at the same node. This activity was given as a homework assignment to enable her to detail the students'

solution strategies and the teacher's actions by decomposing and approximating the teaching practice. For example, she added the following question to her lesson plan by identifying the details of the student's strategy, as shown in Figure 4.52.



Question: The distance between both dots is 3 cm on the above-dotted paper. What is the perimeter of this shape, then?

Maya (teacher): What is the perimeter of the given shape?

Student 1 (fictitious student): *93*

Maya (teacher): How did you calculate it?

Student 1 (fictitious student): *I multiply three by 31*.

Maya (teacher): Why?

Student 1 (fictitious student): *I counted the points and obtained 31. Then, I multiplied 31 by three and obtained 93.*

Maya (teacher): Are there any solutions?

Student 2 (fictitious student*): Teacher, I counted the distance between two points and found 30. I know that the distance between two points is 3 cm, and the perimeter of the given shape is 90 cm.*

Maya (teacher): *Yes, but it is not 90, right? It is 90 cm. We need to emphasize the units in expressing the result.*

Student 2 (fictitious student): You are right, teacher.

Maya (teacher): What did your friend do? He added the lengths of the sides around the shape to find its perimeter.

Figure 4.52. Evidence of *attending skills* from Maya's final lesson plan.

As shown in Figure 4.52, Maya detailed the student's strategy in her lesson plan, which did not include length measures, by encouraging the student to explain herself.

Creating such dialogues helped develop her *attending skills* on the perimeter concept. The following table summarizes Maya's developmental process and *attending skills* during the third and fourth nodes of the model, *Implementation,* and *Post-Preparation.*

Overall, thinking about the potential difficulties of the fifth graders with the perimeter concept and examining the fifth graders' strategies in the given dialogue helped her to identify the details of the students' strategies regarding the perimeter concept. In addition, producing students' strategies regarding the perimeter concept when preparing lesson plans and encountering possible students' strategies in her rehearsal lesson allowed Maya to pay attention to students' strategies before conducting the perimeter lesson in the classroom. In other words, these activities prepared Maya for the student thinking she might encounter and identify in the classroom. However, Maya sometimes encountered unexpected student strategies in her teaching and missed some details, including rehearsal and actual teaching. When Maya missed such details, the preservice teachers analyzed the video recordings of her teaching and addressed the students' strategies together. Considering these, it could be said that she improved her *attending skills* as she had the opportunity to familiarize herself with the students' strategies concerning the perimeter concept during the intervention process. Below, the information on the development of the *interpreting skill* regarding the perimeter concept while participating in the current study's model is presented in detail.

Table 4.25 Sample evidence of the development of Maya's *attending skills* during the Implementation and Post-Preparation nodes of the model.

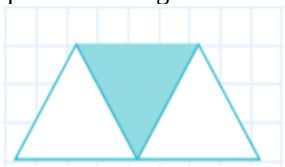
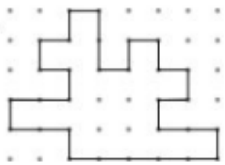
Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *attending skills*
Activity 1: *Enact the Perimeter Teaching!* (in the third node called *Implementation*).	The question: The Figure below is made up of three equilateral triangles. If the perimeter of the painted equilateral triangle is 48 cm, what is the perimeter of the Figure?  Maya (as the teacher): Who wants to solve it? Student 2 (in the actual classroom): May I solve it, teacher? Maya (as the teacher): Sure. Okay. We are listening to Student 2. Student 2 (in the actual classroom): The perimeter of the painted equilateral triangle is 48 cm. It is first divided 48 by three due to the three congruent lengths of the equilateral triangle. Then, we got 16 cm for each side length. The perimeter of the trapezoid was the sum of five sides. Thus, it was multiplied by 16 by five and found 80 cm of the perimeter of the trapezoid. Maya (as the teacher): *Well done. What did your friend do? Firstly, she divided 48 cm by 3 to obtain one of the side lengths of the equilateral triangle. Since the perimeter of the trapezoid consisted of five sides, one equal to 16 cm, she multiplied 16 by five and found 80 cm. Did you understand?* (Evidence of *attending skills*).	She recalled what the student said by identifying the details of the solution strategy.
Activity 1: *Let us Examine the Enactment* (In the fourth node called *post-Preparation).*	Maya: I was shocked. The child actualized her mistake when she thought out loud. It's good that she noticed; otherwise, I would have said "wrong" to her answer. Preservice Teacher 4: We must give children more chances to express themselves. Maya: And we should not just say wrong. We should give the students a way to think of other solutions. …	She was encouraged to pay attention to understanding the details of the student's strategy.

Table 4.25 (Continued)

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *attending skills*
Activity 2: *Prepare for Further Teaching* (in the fourth node called post-preparation).	 Question: The distance between both dots is 3 cm on the above-dotted paper. What is the perimeter of this shape, then? Maya (teacher): What is the perimeter of the given shape? Student 1 (fictitious student): *93* Maya (teacher): How did you calculate it? Student 1 (fictitious student): *I multiply three by 31*. Maya (teacher): Why? Student 1 (fictitious student): *I counted the points and obtained 31. Then, I multiplied 31 by three and obtained 93.* Maya (teacher): Are there any solutions? Student 2 (fictitious student*): Teacher, I counted the distance between two points and found 30. I know that the distance between two points is 3 cm, and the perimeter of the given shape is 90 cm.* Maya (teacher): *Yes, but it is not 90, right? It is 90 cm. We need to emphasize the units in expressing the result.* Student 2 (fictitious student): You are right, teacher. Maya (teacher): What did your friend do? He added the lengths of the sides around the shape to find its perimeter.	She detailed the students' strategies by mentioning the essential points.

4.2.2 Development of *Interpreting Skill*

In this section, how Maya made sense of the students' understandings as reflected in their strategies was examined within the context of *interpreting skills*. As Table 4.21 indicates, Maya's interpreting skill was enhanced from limited to robust evidence from the pre-assessment to the post-assessment. To examine her interpreting skills, she was asked to interpret the student's mathematical understanding of the perimeter concept, considering her solution as in Table 4.22.

The question "*Based on the student's approach to the question, what can you say about her mathematical understanding of the perimeter concept? Explain in detail* "was asked on the pre/posttests. According to her response in the pretest, her interpreting skill was coded at a limited level because she commented on the student's understanding of the perimeter concept by considering the student's strategy. Specifically, she responds to one such question: "*The student understood that calculating the perimeter of a shape means adding the side lengths*." As can be seen from her explanation, she based the student's understanding of the concept on addition and made a superficial explanation, considering the student's strategy. Still, she focused on the student's understanding of the perimeter concept rather than making general inferences without considering the student's strategy. However, she could have made other inferences since the student did not specify the length of the measures in the solution and could not internalize the measurement process. Because Maya made one correct inference considering the student's strategy, although she could have made it in other comments, her *interpreting skill* was coded at a limited level before the model.

After being involved in the model, it was observed that she began to make more interpretive inferences about the student's understanding of the perimeter concept. Her response on the post-test for the same question mentioned above is presented in Figure 4.53.

The student knows that the line segment formed by combining the side lengths around a closed shape gives the shape's perimeter. The perimeter can be calculated with a different and more straightforward method without adding up the sides or following the formula. She notices the perimeter is the particular form of the length measurement since she uses the ruler to calculate the perimeter. I think she understands the perimeter and length measurement concept. She also knows cm because she used a ruler but forgot it out of excitement.

Figure 4.53. Evidence of *interpreting skills* from Maya's responses in the posttest.

Figure 4.53 shows that she used a more interpretive stance to describe the student's understanding of the perimeter concept. For example, she stated that the student was conceptually aware of the meaning of the perimeter concept and calculated the perimeter of the shapes without the following formula. She claimed the student knew that the perimeter is a length because the student used the ruler. Thus, she expressed that the student understood the connections between measuring length and the concept of perimeter. In this way, considering the student's strategy, she made inferences about how the student could internalize the definition of the concept. In addition, she made inferences about why the student could not use the units in her outcome. In addition, she used mathematically correct language in explaining her reasoning. Thus, because she made inferences considering the student's strategy about the perimeter concept and calculated the perimeter using mathematically correct language, her *interpreting skills* about the perimeter concept after her intervention were coded with robust evidence. Considering her responses in the pre-and-post tests, it could be said that her *interpreting skills* progressed from limited to robust during the intervention. The section below provides evidence for improving *interpreting skills* during the intervention.

4.2.2.1 Evidence for Improvement of *Interpreting Skills* during the Model.

As stated above, Maya improved her *interpreting skills* regarding the perimeter concept during the intervention. In this section, the evidence that she developed this skill during her participation in the model of the current study is shared, paying attention to the sequence of activities in each node of the model, as well as the sequence of the nodes of the model, which are called *Pre-Preparation, Preparation*, and *Post-Preparation,* respectively. The excerpts from the current intervention's video transcriptions and individual responses to classroom activities were shared to describe Maya's progress in *interpreting skills* during the intervention.

As displayed in Table 4.21, evidence for developing her *interpreting skills* was observed in the first node of the *pre-preparation*. In this node, the preservice teachers were asked to examine the activities called Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes*, and Activity 2: *Let Us Examine the Dialogue*, first individually and then collectively. Indeed, the preservice teachers were asked to respond to all questions individually in the first lesson hour and then explain their reasoning in the second lesson hour of implementing the first node. In the first activity, the preservice teachers were asked to reason about the possible sources of students' misconceptions, considering the problem in the fourth question, as shown in Table 4.3. Such a question aimed to enable preservice teachers to make inferences about students' possible understanding of the perimeter concept. When Maya's response was analyzed, it was observed that she did not emphasize possible sources of students' difficulties in her answer. During the collective analysis of possible sources in the second lesson hour of the implementation of the first node, it was observed that she started to examine such sources as follows:

> **Preservice Teacher 2:** Students could count the dots of shape by closing the shape. In such a situation, they do not understand the perimeter…
>
> **Preservice Teacher 4:** Students can be confused about where they started. Thus, I would tell them to mark the starting point…

Maya: Students could confuse the area and the perimeter if they do not internalize the formulas... *(*Evidence of *interpreting skills).*

As you can see from the excerpt above, Maya began to think about why students were confusing the perimeter and area formulas. She realized students might have this misconception because they did not fully understand the concepts. This situation could be considered exemplary behavior regarding *interpreting students' understanding* of the perimeter concept. Meanwhile, in the first hour of the node application, the preservice teachers were asked to interpret the student's understanding of the perimeter concept by decomposing practice in the dialogue given within the scope of Activity 2. In this dialogue, the student did not understand the concept of length, the meaning of the perimeter, and the importance of units in expressing the results. Maya's implications for this student's mathematical understanding are as follows: "*I think the student may have thought she could count the points for perimeter calculation*." As understood from Maya's response, she tended to describe the student's strategy rather than making inferences about how the student might have thought. During the collective analysis of this activity in the second lesson hour of the implementation of the first node, it was seen that she took more interpretive stances in interpreting the student's mathematical understanding of the perimeter concept. To illustrate:

Preservice Teacher 4: She did not count the lengths. She counts dots. I do not think she understood the length measurement, too.

Preservice Teacher 2: The student could not develop a mathematical understanding of the perimeter. She probably misunderstood her teacher's definition of the perimeter, such as the perimeter length from a point until it returned to that point. She thought that the points would count to calculate the perimeter.

Maya: She was aware of the perimeter. However, I do not think she had adequate knowledge of it. Counting the distance between two points would

be more accurate in calculating the perimeter. *(*Evidence of *interpreting skills).*

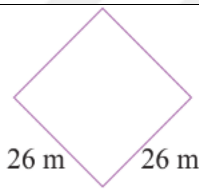
As seen in the dialogue above, Preservice Teacher 4 first mentioned that the student did not understand the concept of perimeter. Preservice Teacher 2 added that the student did not understand the length measurement. Preservice Teacher 2 asserted that the student was counting points rather than distances, most likely because of the emphasis on defining the perimeter by tracing the point back to where it started. Preservice Teacher 2 made inferences about the situations that might cause the student's lack of understanding. These inferences showed that Maya was interpreting the student's understanding of the perimeter concept in light of the student's strategies. She also emphasized that although the student knew something about the perimeter concept, the student's knowledge was limited. In addition, Maya used mathematically correct language to express herself. This type of analysis enabled Maya to make more detailed inferences using mathematically correct language considering the details of the student's strategy, which supported the development of her *interpreting skills*. As understood from her explanation, when she put the concept on conceptual foundations, she produced ways to cope with difficulties faced by students. The following table summarizes Maya's developmental process of *interpreting skills* during the first node of the model, called *Pre-Preparation*.

Table 4.26 Sample evidence of the development of Maya's *interpreting skills* during the Pre-Preparation node of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *interpreting skills*
Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes* (Her explanation of 4th question during the discussion).	… Maya: Students could confuse the area and the perimeter if they do not internalize the formulas *(*Evidence of *interpreting skills).*	She made inferences about the students' possible difficulties with the perimeter.
Activity 2: *Let Us Examine the Dialogue* (Her response to the fourth prompt).	The student may have thought she could count the points for perimeter calculation.	She made inferences about the student's understanding of the perimeter in the given dialogue.
Activity 2: *Let Us Examine the Dialogue* (Her explanation during the discussion *of the fourth prompt*).	Preservice Teacher 4: She did not count the lengths. She counts dots. I do not think she understood the length measurement, either. Preservice Teacher 2: The student could not develop a mathematical understanding of the perimeter. She probably misunderstood her teacher's definition of the perimeter, such as the perimeter length from a point until it returned to that point. She thought that the points would count to calculate the perimeter. Maya: She was aware of the perimeter. However, I do not think she had adequate knowledge of it. Counting the distance between two points would be more accurate in calculating the perimeter. *(*Evidence of *interpreting skills).*	She was encouraged to make accurate inferences about the student's understanding of the perimeter concept.

In addition to activities in the first node of the model, the sample behaviors regarding the development of her *interpreting skills* were observed while engaging in Activity 1: *Prepare for the Rehearsal*, in the second node called *Preparation*, and Activity 2: *Prepare for the Future Teaching* in the fourth node of the model called *post-Preparation.* Standard features of these activities included allowing Maya to make inferences about the student's understanding of the perimeter concept by decomposing the practice. While preparing the lesson plan, she was asked to consider possible sources of the students' strategies she produced in the dialogues. In this way, it was aimed to learn how to make inferences about the student's understanding of the perimeter concept. For example, Table 4.27 shows Maya's implications for students' understanding in light of the strategies produced as part of Activity 1, called *Prepare for the Rehearsal,* in the same node of the model.

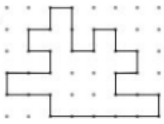
Table 4.27 Evidence of *interpreting skills* from Maya's initial lesson plan.

The Dialogue	The Possible Reasons for The Student (s).'s Strategy
 Maya (teacher): Let us calculate the perimeter of the rhombus-shaped garden above. Student (fictitious): My teacher, I multiply 26 m by four. I obtained 104 m. Maya (teacher): You *added the lengths of the sides around the shape* to find the perimeter.	The student understood the perimeter.

As understood from Table 4.27, Maya made inferences about the student's mathematical understanding of the perimeter concept, considering her solution strategy. Still, it can be seen in the dialogue she made superficial implications about the student's understanding. While preparing Activity 2: *Prepare for the Future Teaching* in the fourth node of the model, called *post-preparation*, it was seen that

she detailed her inferences about the student's understanding of the concept. Table 4.28 presents a sample excerpt from Maya's lesson plan called Activity 2.

Table 4.28 Evidence of interpreting skills from Maya's revised lesson plan.

The Dialogue	The Possible Reasons for The Student (s).'s Strategy
 Question: The distance between both dots is 3 cm on the above-dotted paper. What is the perimeter of this shape, then? Maya (teacher): What is the perimeter of the given shape? Student 1 (fictitious student): 93 Maya (teacher): How did you calculate it? Student 1 (fictitious student): I multiply three by 31. Maya (teacher): Why? Student 1 (fictitious student): I counted the points and obtained 31. Then, I multiplied 31 by three and obtained 93. Maya (teacher): Are there any solutions? Student 2 (fictitious student): Teacher, I counted the distance between two points and found 30. I know that the distance between two points is 3 cm, and the perimeter of the given shape is 90 cm. Maya (teacher): Yes, but it is not 90, right? It is 90 cm. We need to emphasize the units in expressing the result. Student 2 (fictitious student): You are right, teacher. Maya (teacher): What did your friend do? He added the lengths of the sides around the shape to find its perimeter.	*Student 1 did not understand the perimeter concept. She did not know the concept's meaning since she counted the dots to calculate the shape's perimeter.* *Student 2 used units in her results, which implied he had an idea about the length of the measures. He also knew how to calculate the shapes' perimeter.*

As shown in Table 4.28, Maya explained the students' understanding of the perimeter concept, considering their strategies. This situation allowed her to make inferences by putting the students' strategies at the center of her inferences, which helped improve her *interpreting skills*. Maya expressed her opinions on the role of lesson planning in her *interpreting skills* during the post-interview as follows:

> I noticed that students could take different approaches when they do not understand a subject or understand it well. I learned to make sense of their understanding by looking at their solution.

Her explanations implied that the planning process helped her to infer students' understanding of the concept by considering their strategies. The following table summarizes Maya's developmental process of *interpreting skills* during the *Preparation* and *Post-Preparation* nodes of the model.

Table 4.29 Sample evidence of the development of Maya's *interpreting skills* during the Preparation and Post-Preparation nodes of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *interpreting skills*
Activity 1: *Prepare for the Rehearsal* (*In the Preparation node of the model*).	The student understood the perimeter.	She made inferences about the student's possible understanding of the concept.
Activity 2: *Prepare for Further Teaching* (I*n the Post-Preparation*).	Student 1 did not understand the perimeter concept. She did not know the concept's meaning since she counted the dots to calculate the shape's perimeter. Student 2 *used units in her results, which implied he had an idea about the length of the measures.* He also knew how to calculate the shapes' perimeter.	She made inferences about the students' possible understanding of the concept.

In summary, it can be said that reasoning on the possible sources of students' mistakes and discussing them by considering the student's understanding strategy nurtures Maya's *interpreting skills*. In addition, she developed the ability to interpret reasoning about why students use such solution strategies while preparing the lesson plan. In these ways, she developed her *interpreting skills* while engaging in activities at the intervention nodes. Below, detailed information on developing *deciding skills* regarding the concept of perimeter during the intervention is presented.

4.2.3 Development of *Deciding Skills*

In this section, Maya's intended responses, considering the first two components of the framework, to extend and support students' thinking on the perimeter concept were examined within the context of *deciding skills*. Table 4.21 shows that Maya progressed her *deciding skill* from lack to robust levels from the pre-assessment to the post-assessment. To examine her *deciding skills*, she was asked how to continue the lesson, considering the student's mathematical understanding of the perimeter concept, as mentioned in Table 4.22.

For this purpose, the question "What kind of path would you follow after the student's solution? What questions or questions would you ask her? Why? Figure out how you would continue the lesson in detail" was directed to her in the pre-post-tests. Her response to the pre-test is presented in Figure 4.54.

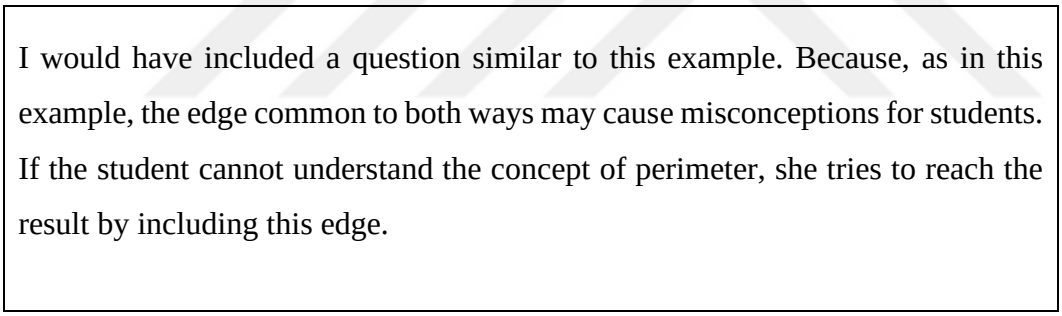

I would have included a question similar to this example. Because, as in this example, the edge common to both ways may cause misconceptions for students. If the student cannot understand the concept of perimeter, she tries to reach the result by including this edge.

Figure 4.54. Evidence of *deciding skills* from Maya's response in the pretest.

As understood in Figure 4.54, she intended to display some unresponsive actions, such as asking similar questions, without aiming to improve the student's understanding of the perimeter concept. Meanwhile, she made general expressions without considering the student's performance and based her explanations on her previous experiences. Consequently, she tended to display actions by ignoring the student's thinking. For these reasons, her *deciding skill* level was lack of evidence before attending the study.

After she participated in the intervention, it was seen that she tended to show actions to advance the student's understanding of the perimeter concept. For instance, her response to the post-test is presented in Figure 4.55.

> Then, I want her to calculate the circumference of the shape. In the meantime, I keep the rope in the classroom. From here, I move on to the circumference calculation of circle shapes. I will calculate the circle's circumference with a rope and a ruler and give the circle's radius. My goal is to enable the student not to limit the perimeter with the quadrilaterals. Besides, I allow her to notice units. Then, I asked if there was a relationship among the circles whose circumferences she calculated (to discover pi and radius).

Figure 4.55. Evidence of *deciding skills* from Maya's response in the posttest.

As can be seen in Figure 4.55, she was trying to extend the student's understanding by connecting the perimeter of the quadrilaterals to the perimeter of the circle. In other words, since she noticed that the student understood the meaning of perimeter conceptually, she tried to find ways to extend the student's understanding of the concept of perimeter by asking about the circumference of the circle. She also offered to use a ruler and some calculations about the circle's circumference to make the student aware of her error in expressing units. Consequently, she tended to show responsive actions to promote the student's understanding of the perimeter concept. Therefore, her *deciding skills* were coded at a robust level after her involvement in the intervention. Considering her responses in the pre-and-post tests, it could be said that Maya's *deciding skills* progressed from lacking to robust when involved in the intervention. The following section provides evidence for improving *deciding skills* during the intervention.

4.2.3.1 Evidence for Improvement of *Deciding Skills* during the Model

As stated above, Maya improved her *deciding skills* regarding the perimeter concept during the intervention. In this section, the evidence that she developed this skill during her participation in the intervention of the current study is shared, paying attention to the sequence of activities in each node of the model, as well as the sequence of the nodes of the model, which are called *Pre-Preparation, Preparation, Implementation,* and *Post-Preparation,* respectively. The excerpts from the current intervention's video transcriptions and individual responses to classroom activities were shared to describe Maya's progress in *deciding skills* during the intervention.

As displayed in Table 4.21, evidence for the development of Maya's *deciding skills* started to be observed when engaging in the activities in the first node of the *Pre-Preparation*. In this node, the preservice teachers were asked to examine the activities called Activity 1: *What is the perimeter? Let us Look at the Concept through your and the Student's Eyes,* and Activity 2: *Let Us Examine the Dialogue,* first individually and then collectively. Indeed, the preservice teachers were asked to respond to all questions individually in the first lesson hour and then explain their reasoning in the second lesson hour of implementing the first node. The first activity was designed to sensitize the preservice teachers to consider students' mathematical understanding of the concept of the perimeter while responding to their thoughts. In this regard, they were asked to explain how to overcome students' difficulties with the perimeter concept in the fifth question in Activity 1. Although Maya did not mention how to deal with them in her response, it was seen that during the collective analysis of the responses in the second lesson on implementing the same node, she began to produce some actions, even if they were mainly unresponsive. For example,

> **Preservice Teacher 2:** Students could count the dots of the shape … I would solve more examples.
>
> **…**

> **Maya**… They can count the squares of the units in this question. If I meet such situations, I could remind students about defining the perimeter and the area (Evidence of *deciding skills*).

From her explanation, Maya tended to remember the definition of the concept in response to students who were confused about the perimeter and area formula. Although she did not show responsive action at this stage, she at least began to focus on the student's thinking in the middle through discussion, which helped develop her *deciding skills*. Meanwhile, as part of Activity 2: *Let Us Examine the Dialogue*, the preservice teachers were expected to approximate their responses about how they would continue the lesson if they were the student's teachers. The student mentioned in Activity 2 had difficulty calculating the shape's perimeter, which was drawn on the dot paper, in which the distance between two points is one centimeter, and counted the dots to calculate its perimeter. Meanwhile, the student forgot to express units in her response. However, when the teacher directed her to another question that could be solved using the formula, the teacher observed that the student quickly solved this question with the formula. When asked how to continue the lesson considering the student's strategy, Maya responded, "*I could ask similar questions as a drill.*" Her response indicated she ignored the student's thinking while adopting the lesson. In other words, Maya focused on drills or similar questions without giving reasons to reinforce the student's understanding of the perimeter.

During the collective analysis of the responses in the second lesson on implementing the first node, it was seen that Maya started to produce responsive actions to extend the student's understanding of the perimeter concept. For example,

> …
>
> **Preservice Teacher 3**: The teacher could use irregular shapes and say the distance between two points is 2 cm. Then, the student was asked to calculate the perimeter of the given shape. …Then, the teacher should observe whether

she counts the dots again or focuses on length. Such examples aimed to make the student realize her mistake and notice that the distance between two points does not always represent 1 cm or 1 unit.

Maya: I would like the student to think that between two points, each point is a city, and there are two cities. When calculating the distance between these two cities, we do not take the distance as two by counting the cities. We calculate the distance between two cities, for example, how many kilometers the distance between two cities is. It also has value. In other words, one started to travel to Istanbul and returned to Istanbul by making a circular route to other cities by moving from Istanbul, Ankara, and Bursa. We calculate that distance in this pathway by looking at the total distance among cities. If we look at Ayşe's method, we might say three as an answer (Evidence of *deciding skills: advance).*

As can be seen from the quotations above, Preservice Teacher 3 tended to exhibit actions aimed at furthering the student's mathematical understanding of the concept of perimeter. Preservice Teacher 3 observed that the student had difficulty calculating the perimeter of the figure, which consists of points, and the distance between two points is one cm. With this in mind, he proposed to ask the student a question with a distance of two cm between two points and consisting of dots, aiming to enable the student to realize her mistake and understand the concept of perimeter. Upon Preservice Teacher 3's comment, Maya proposed a comprehensive question to advance the student's understanding of the perimeter concept. Mainly, Maya emphasized the role of distance in length measurement by modeling the perimeter concept with daily life examples. In this way, she aimed to connect length measurement and perimeter concept. Considering the excerpt above, it could be said that the collective analysis of how to continue the lesson, considering the student's strategy, promoted improving Maya's *deciding skills*. Besides, her explanation during the post-interview confirmed this inference. She stated:

> For example, I may not have been able to answer this student before. I might not have been able to give a definite answer if the student had counted the dots. However, doing such an activity and talking about it both helped me to make sense of it and also helped me to reason how to respond to the student's thinking correctly…It was very effective for me to see this dialogue, understand the concept and the student's thinking, and find productive ways to respond to the student's understanding.

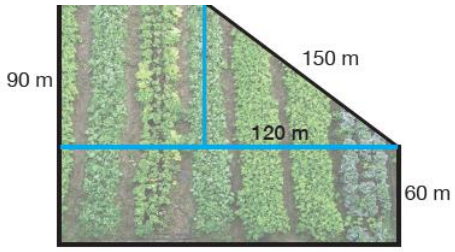
Her explanations implied that after she developed her understanding of the concept during the collective analysis as part of Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes;* in the first node of the model, she started to produce ways to respond to the student's thinking during the collective analysis of Activity 2: *Let Us Examine the Dialogue.* Considering the evidence, it could be said that Maya developed her *deciding skills* while engaging in the activities in the first node of the model, called *Pre-preparatio*n. The following table summarizes Maya's developmental process regarding *deciding skills* during this node.

At the end of this node, Maya was asked to prepare a lesson plan as homework to teach the perimeter concept to fifth-grade students by creating dialogues between the teacher (herself) and fictitious students as part of Activity 1: *Prepare for the Rehearsal*. This activity under the second node of the model, *Preparation*, was mainly designed to prepare preservice teachers for perimeter teaching and making instructional decisions by decomposing and approximating the teaching practices. Then, during the first lesson hour of the implementation of the second node, the preservice teachers examined Maya's lesson plan collectively, aiming to improve the quality of her decisions. In doing so, they focused on what actions Maya displayed to respond to the student's thinking and gave her feedback if necessary. When examining her lesson plan collectively, it was seen that she did not always

successfully respond to the student's thinking. She commonly said "*well done*" to confirm students' understanding, as stated in Figure 4.56.

Table 4.30 Sample evidence of the development of Maya's *deciding skill* during the Pre-Preparation node of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *deciding skills*
Activity 1: *What is the perimeter? Let us look at the concept through your and the student's eyes (Her explanation of the fifth question during the discussion).*	… Maya… They can count the squares of the units in this question. If I meet such situations, I could remind students about defining the perimeter and the area (Evidence of *deciding skills*).	She displayed actions to remind the student of the formula of the perimeter calculation.
Activity 2: *Let Us Examine the Dialogue (Her response to the fifth prompt).*	I could ask similar questions as a drill.	She displayed actions to improve the student's procedural fluency.
Activity 2: *Let Us Examine the Dialogue* (Her explanation during the discussion *of the fifth prompt*).	Maya: I want the student to think that between two points, each point is a city, and there are two cities. When calculating the distance between these two cities, we do not take the distance as two by counting the cities. We calculate the distance between two cities, for example, how many kilometers the distance between two cities is. It also has value. In other words, we can think that one started to travel to Istanbul and returned to Istanbul by making a circular route to other cities by moving from Istanbul, Ankara, and Bursa. We calculate that distance in this pathway by looking at the total distance among cities. If we look at the student's method, we might say three as an answer (Evidence of *deciding skills).*	She took action to realize her mistake to the student and understand her mathematical thinking of the perimeter concept.



Maya (teacher): On the right, some side lengths of a garden consisting of square, triangular, and rectangular sections are given. How many meters of wire would it take to run two rows around this garden?

Student (fictitious student): Teacher, I found it. There is a square there. It goes down to 120. The bottom is 210. The opposite is 60. Now 60+90+90+150+60+210=660 m.

Maya (teacher): *Yes, well done.*

Figure 4.56. Evidence of *deciding skills* from Maya's initial lesson plan.

During the collective analysis, the preservice teachers encouraged Maya to demonstrate actions to engage other students in the dialogue. About the dialogue above, they suggested:

> **Preservice Teacher 2:** I think you should ask whether or not there were other solution methods.
>
> **Preservice Teacher 4:** We may ask another student to summarize what this student has said.

As understood in the dialogue above, Preservice Teachers 2 and 4 suggested involving other students in the dialogue. These suggestions might contribute to the development of Maya's *deciding skills.* In the second and third lessons of the second node implementation, Activity 2: *Rehearse the Perimeter Teaching* was completed with some of the preservice teachers in the role of students and Maya in the role of the teacher in the simulated classroom environment by approximating the teaching

practice. During the rehearsal, the preservice teachers were in the role of students and asked, thought, and questioned like 5th-grade students. Meanwhile, they devised solution strategies for the given problem, including operational errors or common misconceptions about the concept of environment. In this way, Maya responded to students' mathematical thinking during rehearsal teaching and widely exhibited actions to explore the details of students' strategies. In other words, it was seen that he evaluated them instead of advancing them. The following excerpt is taken from the rehearsal to reveal such a situation. Italicized words in parentheses describe Maya's actions.

Maya (in the teacher role): Can you calculate the perimeter of this shape?

Preservice Teacher 2 (in the role of the student): My teacher, the answer is 33.

Student-Teacher: How did you find it? *(Explore).*

Preservice Teacher 2 (in the role of the student): Teacher, I counted the dots.

Maya (in the teacher role): Were we counting the points when calculating the perimeter? *(Evaluative revoice).*

Preservice Teacher 2 (in the role of the student): Yes, teacher.

Student-Teacher: Were we counting the dots, guys? (*Evaluative revoice*).

Preservice Teacher 3 (in the role of the student): Teacher, can I answer? We were calculating the length between two points. Thus, the answer would be 32 units.

Maya (in the teacher role): Well done. Did you understand? *(Evaluate).*

Preservice Teacher 2 (in the role of the student): Yes, teacher.

As understood in the excerpt above, Maya tried to understand Preservice Teacher 2's strategy by asking her to explain the way of the solution. This action implied that Maya was trying to understand Preservice Teacher 2's way of finding a solution. Later, Maya tended to evaluate Preservice Teacher 2's strategy by asking her a question with an evaluative voice. However, because Maya could not get the correct answer from Preservice Teacher 2, she invited other preservice teachers to participate in solving the problem. When Preservice Teacher 3 reached the correct answer, Maya evaluated her response by saying, "Well done." Then Maya asked if Preservice Teacher 2 understood how to solve the problem by waiting for her confirmation. Maya guided Preservice Teacher 2 to the correct result and made the students aware of their mistakes. While examining Activity 3, called *Let Us Examine the Rehearsal*, which was applied in the fourth and fifth lesson hours of the same node and involved the analysis of video recordings of Maya's rehearsal teaching through the pedagogies of reflection, the preservice teachers collectively questioned Maya's actions to promote students' thinking about the concept of perimeter. By way of illustration, they made the following suggestions concerning the above dialog in the rehearsal.

> **Preservice Teacher 5:** You could have asked why the student counts the dots to calculate the perimeter (*Invite to Explore).*
>
> **Preservice Teacher 3:** I expected you to ask the student to define the perimeter to understand her concept image (*Invite to Explore).*
>
> **Preservice Teacher 2:** Besides, you could have wanted the student to draw shapes representing the perimeter (*Invite to Explore).*
>
> **Preservice Teacher 3***:* You could have measured the circumferences of the shapes by wrapping the string around the shape and have had students discover with this length on the ruler that the circumference is a particular case of measuring length (*Invite to Connect).*
>
> **Maya***:* You are right. I could have asked.

As seen in the dialogue above, Preservice Teachers 2, 3, and 5 suggested asking research questions to explore the student's understanding of the concept of perimeter. For example, Preservice Teacher 2 suggested that the teacher ask the student to draw the concept to gain information about the student's understanding of the concept of perimeter. Next, Preservice Teacher 3 claimed that Maya should create a learning environment in which students connect length measurement and perimeter to promote students' understanding of the concept of perimeter. Thus, it can be said that such a fruitful discussion helped Maya think about how to respond to student's ideas about the concept of perimeter. As a result, it can be said that the opportunity to respond to students' thoughts and to examine his teaching in a simulated classroom environment allows Maya to develop her *deciding skills.* A further example was shared below from the rehearsal.

> **Maya (in the teacher role):** The rectangle in the Figure below comprises congruent squares. Since the long side of the rectangle is 15 cm, let us find the perimeter of the painted square (The figure shown in Figure 4.57).

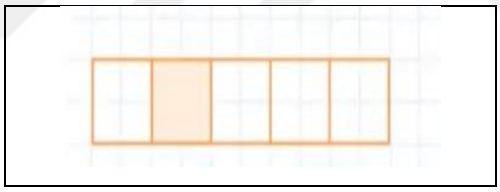


Figure 4.57. Used question’s figure.

> **Preservice Teacher 2 (in the role of a student):** Teacher, can I answer, is it 8 cm in perimeter?
>
> **Maya (in the teacher role):** Can you tell me how you did it? *(Explore).*
>
> **Preservice Teacher 2 (in the role of a student):** My teacher asked about the perimeter of the painted square since the sides are short, two cm, and the long side is 10 cm. Then I add all the sides and get two, two, two, two, to eight since one side is two cm.

Maya (in the teacher role): Could you reread the question? The question indicated 15 cm of the long side of the rectangle. So, it did not give information that each little square's side is one cm. Let us do it together. Since the shape consisted of five congruent squares, we should divide 15 by 5. Then, we get 3 cm, the small square's side length. Then, to find the perimeter of the painted shape, we add three, three, three, three, and get 1two cm. Do you understand now? *(Explain).*

Preservice Teacher 2 (in the role of a student): Yes, teacher, I did not see 15 cm.

The dialogue above between Maya acting as the teacher of the lesson and Preservice Teacher 2 acting as the lesson's student occurred in the rehearsal stage teaching. As understood from the dialogue, Preservice Teacher 2 followed a solution strategy for the given problem by focusing on the visual properties of the figure. Thus, she reached the incorrect conclusion. At that point, it was seen that Maya tried to figure out why Preservice Teacher 2 solved the question by asking her to explain her solution strategy. Unfortunately, she continued to dialogue by explaining the solution to the problem directly. During the analysis of the rehearsal, it was observed that the preservice teachers collectively sought ways to support such students' thinking. To illustrate:

Preservice Teacher 4: You could have invited others to evaluate students' thinking *(Invite to Evaluate).*

Preservice Teacher 3: You could have provided a space for the student to think about the solution to the problem after you said 15 cm. In the meantime, you can help the little one *(Invite to Scaffold).*

Maya: Yes, I could have asked.

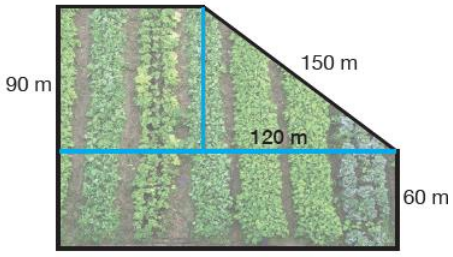
As the excerpt above shows, Maya was encouraged to provide a space for the student to think about a solution to the problem. Mainly, Preservice Teacher 4 offered to invite other students to comment on the mentioned student's strategy. In addition, Preservice Teacher 3 suggested scaffolding the student's thinking by providing time to think about the problem. Since both actions regarding understanding the student's strategy and making the student notice their mistake, it could be said that such suggestions contributed to her knowledge of Maya's *deciding skill*. Besides, Maya shared her thoughts during the post-interview as follows:

> I responded to students' thoughts in the rehearsal as yes or no. Then, the situation changed a bit more in the classroom enactment. Analysis of video recordings of the rehearsal and enactment teachings helped me to learn ways to support the students' understanding.

As understood from her explanations, such activities helped her learn how to respond to students' thinking, which contributed to developing her *deciding skills.* The following table summarizes Maya's developmental process *deciding skills* during the second node of the model, which was called *Preparation.*

Afterward, Maya conducted perimeter teaching in one of the fifth-grade classrooms as part of the pedagogy of enactment. This teaching was conducted in the first and second lesson hours of implementation of the third node of the model, called *Implementation,* within the context of practical hours of the model. During her lesson, she was able to respond to the fifth-grade students' thinking about the perimeter. During this time, it was observed that she showed actions to understand students' thinking, to make students notice their mistakes, or to ignore their thinking because of technical problems. For example,

Table 4.31 Sample evidence of the development of Maya's *deciding skills* during the Preparation node of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *deciding skills*
Activity 1: *Prepare for the Rehearsal*	 Maya (teacher): On the right, some side lengths of a garden consisting of square, triangular, and rectangular sections are given. How many meters of wire would it take to run two rows around this garden? Student (fictitious student): Teacher, I found it. There is a square there. It goes down to 120. The bottom is 210. The opposite is 60. Now 60+90+90+150+60+210=660 m. Maya (teacher): *Yes, well done*	She confirmed the student's saying.
Activity 2: *Rehearse the Perimeter Teaching!*	Preservice Teacher 2 (in the role of a student): Teacher, can I answer, is it 8 cm in perimeter? Maya (in the teacher role): Can you tell me how you did it? *(Explore).* …	She explored the details of the student's strategy.
Activity 3: *Let us Examine the Rehearsal*	Preservice Teacher 4: You could students' thinking *(Invite to Evaluate).* Preservice Teacher 3: You could have provided a space for the student to think about the solution to the problem after you said 15 cm. In the meantime, you can help the little one *(Invite to Scaffold).* Maya: Yes, I could have asked.	She was encouraged to be involved in the student process.

Student 5 (in the actual classroom): I found 16 cm by multiplying four by four.

Maya (teacher): How did we find the perimeter of the square? Student 5 says we get 16 cm by multiplying four by four. Besides, who wants to explain his solution to the problem once again? (*Revoice and invite to explain).*

Student 2 (in the actual classroom): Teacher, a square has all sides equal and has four sides, so we need to multiply four by four.

As seen in the dialogue above, Maya revoiced Student 5's solution and encouraged the others to explain the ways of their solution. While analyzing the video recording of Maya's perimeter teaching in Activity 1: *Let Us Examine the Enactment*, which took place during the first and second lesson hours of implementing the fourth node of the model called *Post-preparation*, the preservice teachers encouraged Maya to ask various productive questions to support the students' thinking. For example,

Preservice Teacher 2: You could have said, "Is there any other way to solve this problem?" rather than "Solve this question again." *(Invite to produce an alternative solution.).*

Preservice Teacher 3: You could have asked, "Could we apply this formula for calculating the perimeter of any shape?" *(Invite to Connect).*

Maya: Okay. I will pay attention to these issues for the next lesson.

As understood from the excerpt above, Preservice Teacher 2 suggested she revise her language by encouraging students to find alternative solutions to the problem. Besides, Preservice Teacher 3 recommended Maya display action to enable students to explore why they used such a formula to calculate the perimeter of all parallelograms. Other examples of dialogues between Maya and the students in the actual classroom and also between the other preservice teachers and Maya in the analysis of the perimeter teachings are presented below.

Maya (teacher): What is the perimeter of the painted triangle?

(The question: Figure 4.58 comprises three equilateral triangles. If the perimeter of the painted equilateral triangle is 48 cm, what is the perimeter of the Figure?

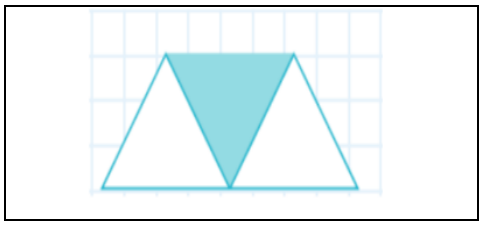


Figure 4.58. The figure used in the question.

Student 2 (in the actual classroom): The perimeter of the painted equilateral triangle is 48 cm. It is first divided 48 by three due to the three congruent lengths of the equilateral triangle. Then, we got 16 cm for each side length. The perimeter of the trapezoid was the sum of five sides. Thus, it was multiplied by 16 by five and found 80 cm of the perimeter of the trapezoid.

Maya (teacher): Well done *(Evaluate).*

Student 3 (in the actual classroom): Teacher, I found another method.

Maya (teacher): Yes, how did you solve it? *(Explore).*

Student 3 (in the actual classroom): This 48, this 48, this 48 (she shows three congruent triangles inside the trapezoid). Then, we multiply 48 by three, so we get 144. We will remove the burden of the two sides. One side's length is 16 cm. If we multiply 16 by two, we get 32. I guess I found it wrong with this method.

Maya (teacher): What is it when you subtract? *(Explore).*

Student 3 (in the actual classroom): When I subtract it, it becomes 144-32, and we get 112.

Maya (teacher): You subtracted 144 from 32 and got 112 (*Revoice).* So, we cannot find it that way, right? *(Ignore).*

Student 3 (in the actual classroom): Yes.

Maya (teacher): You included some of the edges here (shows inside of the shape)., but we included the edges around the shape, right? *(Insist).*

(A few seconds later).

Student 3 (in the actual classroom): I understand now, my teacher. We count twice these two lengths. We need to multiply these two by these two. The length of one side is 16. We have a fur side on the inside. So, we need to multiply 16 by four and get 64. Then, we needed to subtract 64 from 144 and get 80 cm (Figure 4.59).

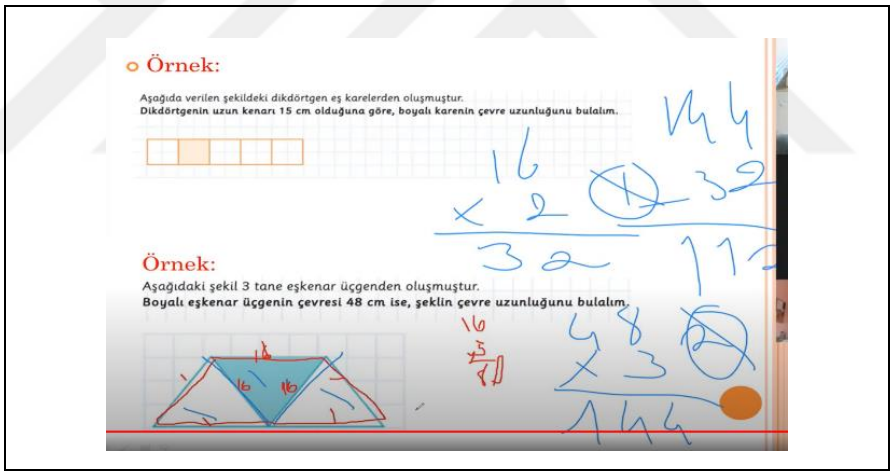


Figure 4.59. Student 3's solution to the question.

Maya (teacher): Hmm. Yes, yes, you included them twice. You did it the correct way. Okay, let us turn to the other question (*Hesitate).*

As you can see from the dialog above, Maya approved Student 2's solution by saying "*Well done*" because it was a strategy that Maya expected students to use. Later, Student 3 proposed another unexpected solution to the given problem. Although

Student 3 had a good reason for solving the problem, she forgot to subtract the length of the sides from the total length, which led to the wrong result. Meanwhile, Student 3 did not seem to trust her solution method because she did not get the number Maya approved at the beginning of the conversation. At the same time, it could be seen that Maya tried to understand Student 3's solution method by giving her several hints. However, since this solution had not been brought up in previous nodes of the model, Maya did not make sense of it then. Also, since Student 3 could not reach the correct number, Maya thought this solution method was wrong. Maya was then inclined to ignore Student 3's way of thinking and insist on following the expected and known solution, as stated by Student 2. However, Student 3 realized her mistake a few seconds later. She then arrived at the correct result and explained her solution in detail. Then, it seemed that Maya had to end this conversation by hesitantly agreeing with the solution method. Consequently, it could be said that Maya did not display actions to advance Student 3's understanding of the perimeter concept at the moment of the instruction.

During the analysis of the recordings of Maya's perimeter teaching through the pedagogies of reflection, it was seen that the preservice teachers offered to display some actions to Maya to support Student 3's understanding of the perimeter concept. To illustrate:

> **Maya:** I was shocked. The child actualized her mistake when she thought out loud. It's good that she noticed; otherwise, I would have said "wrong" to her answer (*Ignore and evaluate).*
>
> **Preservice Teacher 4:** We need to give children more chances to express themselves (*Invite to explain).*
>
> **Maya:** And we should not just say wrong. We should give the students a way to think of other solutions (*Invite to find alternative solutions).*
>
> ...

Preservice Teacher 2*:* We should ask to calculate the shape's perimeter formed by overlapping the side midpoints of a rectangle with the other equal rectangle *(Invite to advance).*

Maya: I will pay attention to these points in my other lessons.

As the dialogue above can be understood, Maya stated that if Student 3 did not continue to solve the question, Maya would ignore the student's way of thinking and evaluate it as wrong, even though it was correct. While examining the recordings of teaching, Maya noticed the details of the student strategy that she missed in her teaching. Meanwhile, Preservice Teacher 4 suggested that Maya explore the student's strategy and encourage students to generate alternative solutions. Preservice Teacher 2 suggested asking questions to encourage the student's thinking about the perimeter concept. Afterward, Maya ended the discussion by stating that she would use them in later lessons. Such fruitful discussions might contribute to the development of Maya's *deciding skills*. Furthermore, Maya expressed her thoughts on this situation during the post-interview as follows:

> …It was an answer (she implied the Student'3 strategy mentioned above) that I did not expect, and at that moment, I could not display actions to the student's response. I did not know what to do to continue the lesson. I could not respond to the student correctly. Nevertheless, then I thought differently. I saw the situation in the reflection (during the analysis of the enactment). … When we encountered the student's thinking that we expected, it made us happy to think that "aa, the student thought like this, there was indeed, and we could have responded in this way." I have an answer to this thinking (in the moment of the teaching). I have an answer that will enable the student to understand it. Observing this contributed a lot and made me happy and motivated…Analysis of enactment allowed me to see such points that I could not answer during the lesson or that my friend could not answer. For example, while watching, I questioned, "Do I have an answer to this question?" or "How can I advance the student's thinking? "All helped me to be prepared for

the next lesson…At the same time, we answered how I could teach the concept.

As her explanations explain, it was difficult for Maya to manage unexpected student strategies at the time of teaching. However, while examining the teaching records, he had the chance to notice the details of the students' strategy, which gave Maya an idea of how to respond to students' mathematical ideas. Additionally, Maya stated that she was happy and prepared when encountering possible student thoughts discussed in other nodes of the model. Considering all of these, it could be said that the intervention helped her to reason how to advance the student's understanding of the perimeter concept. These situations contributed to developing her *deciding skills.*

In addition, unfortunately, during the intervention nodes, it was assumed that some situations negatively influenced the development of Maya's *deciding skills*. Indeed, since the lesson was held on a Zoom program due to the COVID-19 pandemic, Maya and her students had difficulty managing the whiteboard on the Zoom program, as stated before. It was seen that such kind of situations shaped Maya's actions in her teaching. For example, Student 1 did not use the pencil on the whiteboard, which is the property of the Zoom program. Then, Maya opened the pencil and intervened with Student 1's solution as follows:

Student 1 (in the actual classroom): Teacher, my answer is 32.

Maya (teacher): How did you find it? *(Explore).*

Student 1 (in the actual classroom): I counted.

Maya (teacher): How did you count? Could you show it on the board? *(Explore).*

Student 1 (in the actual classroom): (He tried to use a pencil on the Zoom Program but failed.).

Maya (teacher): Let us count together. We select a starting point, right? and we need to close the shape by going around the shape. Let us count. 1,2 (She drew the following drawings shown in Figure 4.60).

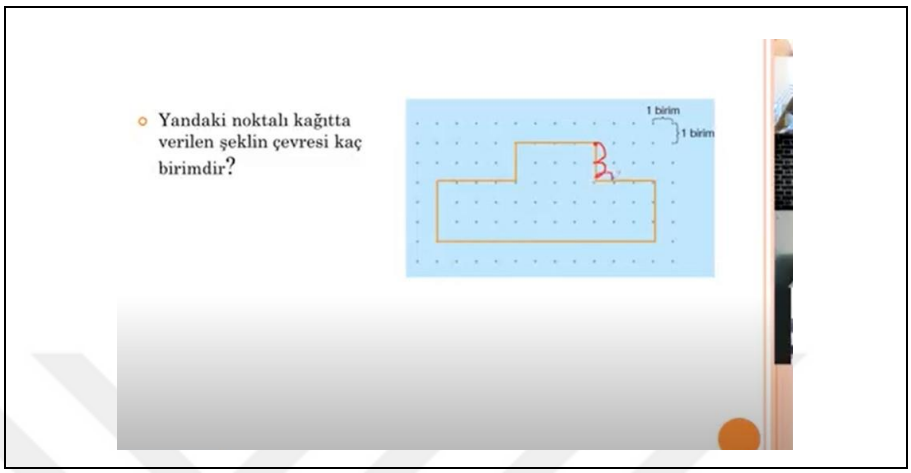


Figure 4.60. Student 1's first drawing to the question.

Student 1 (in the actual classroom): 2,3.....30

Maya (teacher): 30,31

Student 1 (in the actual classroom):32.

Maya (teacher): 32, yes. We found 32 units and counted all units. Is it clear?

Student 1 (in the actual classroom): If my teacher asked about the perimeter, it would be 32 cm, right? (She drew Figure 4.61).

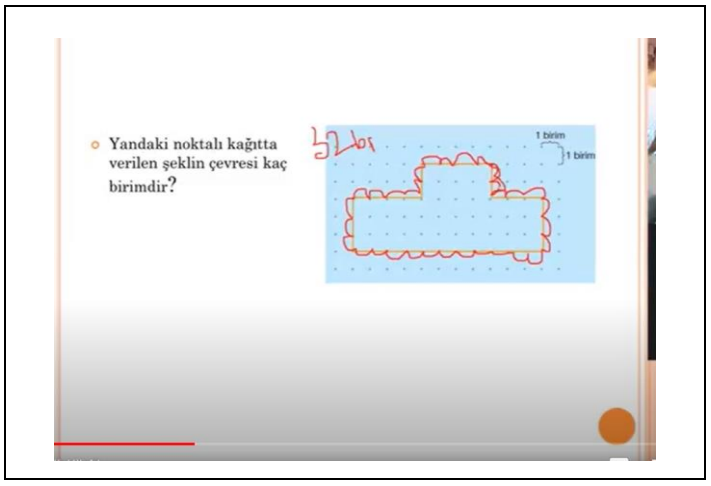


Figure 4.61. Student 1's second drawing to the question.

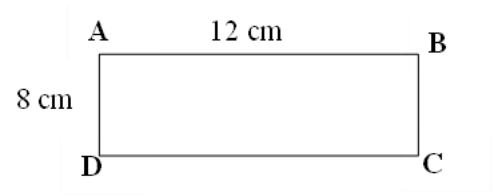
> **Maya (teacher):** Here, the distance is not given as cm. It is provided as a unit (*Explain).*

As shown in the dialogue above, although Maya tended to display moves to understand the student's strategy at the beginning of the conversation, she tended to explain the solution to the problem directly when Student 1 faced a technical problem. As understood, such a problem affected Maya's instructional decisions; otherwise, she could have displayed more responsive actions in her teaching. She expressed what she felt at that moment during the post-interview as follows:

> In my teaching, I wanted to understand the students' solutions as much as possible. However, technical problems on the internet made me panic about the instructions because I was afraid of being unable to set the time.

As understood from her explanations, she panicked due to managing time when she faced technical problems, which could affect her decisions. In addition to the activity mentioned, at the same node of the model, when Maya was asked to review the lesson plan last time by decomposing and approximating it Activity 2: *Prepare for Further Teaching* as a homework assignment, Maya appeared to display more exemplary behavior regarding *deciding skills*. The sample excerpt from her lesson plan is presented in Figure 4.62.

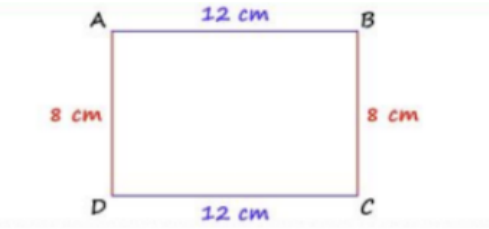
Teacher (Maya): What is the perimeter of the given shape?



Student 2 (fictitious student): Teacher, the answer is 40 cm.

Teacher (Maya): How did you find it?

Student 2 (fictitious student): I added 12 and 8 and found 20 cm. Then, I multiply it by two.

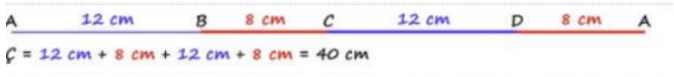


Teacher (Maya): Yes, well done. Are there any solution ways?

Student 1 (fictitious student): I added 12, 8, 12, and 8 and found a 40 cm teacher.

Teacher (Maya): You could also solve the question in these ways (I will open GeoGebra software).

First way:



The first way included adding the sides lengths of shape by adding end to end.

Second way:



The second way included adding together equal lengths.

Third way:



The third way included the following formula.

Figure 4.62. Evidence of *deciding skills* from Maya's revised lesson plan.

As shown in Figure 4.49, even if she continued to explain the solution to the problem, she preferred to use GeoGebra to connect the length measurement and the perimeter, aiming to advance the student's mathematical thinking on the perimeter concept. Similarly, she stated in the interview that she would give students time to explore such connections independently. She said: "*I have restricted communication with fictitious students in preparing lesson plans. When I conduct the perimeter teaching, I will provide more chances for students to focus on the process."* Table 4.32 summarizes Maya's developmental process regarding her *deciding skills* during the third and fourth nodes of the model, called *Implementation* and *Post-Preparation*, respectively.

To sum up, it could be said Maya had a chance to develop her *deciding skills* during the activities in the nodes of the current study's intervention, which consisted of four nodes: *Pre-Preparation*, *Preparation*, *Implementation*, and *Post-Preparation*. Specifically, it could be said she had a chance to decide how to continue the lesson considering the student's strategies mentioned in the dialogue.

Table 4.32 Sample evidence of the development of Maya's *deciding skills* during the Implementation and Post-Preparation nodes of the model.

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *deciding skills*
Activity 1: *Enact the Perimeter Teaching!* (in the third node called *Implementation).*	… Student 3 (in the actual classroom): Teacher, I found another method. Maya (teacher): Yes, how did you solve it? *(Explore).* Student 3 (in the actual classroom): This 48, this 48, this 48 (she shows three congruent triangles inside the trapezoid). Then, we multiply 48 by three, so we get 144. We will remove the burden of the two sides. One side's length is 16 cm. If we multiply 16 by two, we get 32. I guess I found it wrong with this method. Maya (teacher): What is it when you subtract? *(Explore).* …	She explored the details of the students' strategy.
Activity 1: *Let us Examine the Enactment* (in the fourth node called *Post-Preparation*).	**Maya:** And we should not just say wrong. We should give the students a way to think of other solutions (*Invite to find alternative solutions).*	Maya encouraged herself to support the students in finding alternative solutions.

Table 4.32 (Continued).

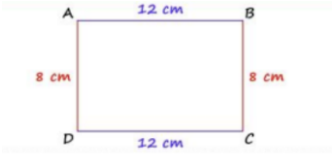
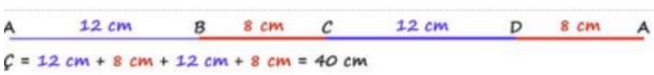
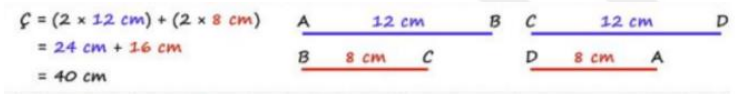
Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *deciding skills*
vActivity 2: *Prepare for Further Teaching* (in the fourth node called *Post-Preparation*).	… Student 2 (fictitious student): Teacher, the answer is 40 cm. Teacher (Maya): How did you find it? Student 2 (fictitious student): I added 12 and 8 and found 20 cm. Then, I multiply it by two.  Teacher (Maya): Yes, well done. Are there any solution ways? Student 1 (fictitious student): I added 12, 8, 12, and 8 and found a 40 cm teacher. Teacher (Maya): You could also solve the question in these ways (I will open GeoGebra software).	She tried to extend or support the students' mathematical thinking on the perimeter concept through virtual manipulatives.

Table 4.32 (Continued).

Activity Name	Excerpts from Maya's explanations or responses	Inferences about Maya's *deciding skills*
	First way:  The first way included adding the sides lengths of shape by adding end to end. Second way:  The second way included adding together equal lengths. Third way:  The third way included the following formula.	

Maya then created dialogues between herself and fictional students as she prepared lesson plans for perimeter concepts, which helped her approximate her decisions before the lesson. She also responded to students' thoughts in her lessons in simulated and real classrooms. During her teachings, she mainly displayed actions to confirm the students' correctness or explore the details of the students' strategies. Then, the preservice teachers collectively discussed Maya's actions to improve the quality of her actions. In this regard, they provided some suggestions for how Maya could support or advance students' thinking about the concept of perimeter. All contributed to the development of Maya's *deciding skills* during the intervention.

4.3 Summary of Findings

The previous sections of this chapter present the findings regarding the improvement of preservice teachers' mathematical knowledge and professional noticing skills on the perimeter concept through the development of the model grounded on the cycle of enactment and investigation within the framework of design-based research. The preservice teacher's mathematical knowledge of the perimeter concept was presented under *the overt subject knowledge, using terminology, identifying errors,* and the *theoretical underpinning of the pedagogy* knowledge types under the Foundation dimension of the KQ Model by Rowland et al. (2005). The preservice teacher's professional noticing skills on the perimeter concept were examined using the same data under attending, interpreting, and deciding components of the professional noticing framework put forward by Jacobs et al. (2010).

The model was designed to promote the development of preservice teachers' mathematical knowledge and professional noticing skills to improve their responsiveness with four nodes: *Pre-Preparation, Preparation, Implementation,* and *Post-Preparation*. Meanwhile, to explore their progress, the pre-and post-test and the post-interviews were applied before and after participating in the intervention. In each node, different pedagogies were utilized throughout the activities. Based on the

data obtained from the pre-posttest, the interview, classroom activities, and the video recordings of the models, it could be claimed that the preservice teacher developed her mathematical knowledge and professional noticing skills on the perimeter concept.

In the first node of the model, *Pre-Preparation*, during the shared exploration of the concept's meaning as part of Activity 1: *What is the Perimeter? Let Us Examine the Concept from Your and the Student's Eyes*; it was observed that the preservice teacher developed her understanding of the concept and her mathematical language related to it. These were accepted as evidence that she was developing *overt subject knowledge* and *using terminology* knowledge regarding the perimeter concept. Then, the preservice teachers collectively examined the dialogue in Activity 2: Let *Us Examine the Dialogue* as part of the practice's representation, decomposition, and approximation. During this analysis, it was observed that Maya explored the concepts from a mathematical and pedagogical perspective. Maya and the others made inferences about the student's conceptual understanding and approximated their instructional decisions in light of the student's strategy. Such inquiry contributed to developing her *overt subject knowledge*, *using terminology*, *identifying errors*, the *theoretical underpinning of the pedagogy* knowledge, and *attending*, *interpreting*, and *deciding skills.*

In the second node of the model, *Preparation*, Maya prepared the lesson plan by creating dialogues between herself and fictional students by decomposing and approximating the practices. After preparation, the preservice teachers examined Maya's discourse, concept definition, instructional strategies, potential student difficulties, and Maya's responses to students in the dialogues. This examination and creation of dialogues contributed to developing her *overt subject knowledge*, *using terminology*, *identifying errors*, the *theoretical underpinning of the pedagogy* knowledge, and *attending*, *interpreting*, and *deciding skills.* Next, Maya approximated teaching practice through rehearsal as part of Activity 2: *Let Us Rehearse the Perimeter Teaching.* During this activity, she had the opportunity to

attend to the details of the students' strategies, make instructional decisions, identify students' errors, and apply instructional strategies for perimeter instruction, which provided evidence of the development of her *attending skills*, *deciding skills*, the *overt subject knowledge*, *using terminology*, *identifying errors*, the *theoretical underpinning of the pedagogy* knowledge. Later, the pre-service teachers examined the video recordings of the rehearsal in the light of the pedagogy of reflection as part of Activity 3: *Let Us Examine the Rehearsal*. Thanks to this activity, the preservice teachers and Maya reviewed their instructional strategies, mathematical language, students' mathematical strategies, and Maya's instructional decisions regarding these strategies. In this way, Maya had the opportunity to develop her *overt subject knowledge*, *using terminology*, *identifying errors*, the *theoretical underpinning of the pedagogy* knowledge, and *attending* and *deciding skills*. At the same time, Maya reviewed her lesson plan, considering her experience so far in Activity 4: *Let Us Prepare for Enactment*. During this activity, it was observed that Maya improved her concept definition and mathematical language, detailed students' possible strategies, her inferences about students' mathematical understanding, and her instructional choices to promote students' mathematical understanding. This revision contributed to developing her *overt subject knowledge*, *using terminology*, *identifying errors*, the *theoretical underpinning of the pedagogy* knowledge, and *attending, interpreting,* and *deciding skills.*

In the third node of the model, *Implementation*, Maya enacted the perimeter teaching with the actual students as part of the pedagogy of enactment. This activity was called "Let Us Enact the Perimeter Teaching! Through this activity, Maya had the opportunity to adjust her teaching strategies and instructional choices in the moment of teaching. Such an examination allowed her to demonstrate exemplary behaviors regarding *overt subject knowledge*, *using terminology*, *identifying errors*, the *theoretical underpinning of the pedagogy* knowledge, *attending,* and *deciding skills.*

In the fourth node of the model, *Post-Preparation*, Maya and the preservice teachers examined the video recordings of the enactment in light of the pedagogy of reflection

as part of Activity 3: *Let Us Examine the Rehearsal*. Thanks to this activity, the preservice teachers and Maya reviewed their instructional strategies, mathematical language, students' mathematical strategies, and Maya's instructional decisions regarding these strategies. In this way, Maya had the opportunity to develop her *overt subject knowledge*, *using terminology*, *identifying errors*, the *theoretical underpinning of the pedagogy* knowledge, *attending*, and *deciding skills*. At the same time, Maya reviewed her lesson plan, considering her experience so far in Activity 4: *Let Us Prepare for Further Teaching*. During this activity, it was observed that Maya improved her concept definition of mathematical language, detailed students' possible strategies, her inferences about students' mathematical understanding, and her instructional choices to promote students' mathematical understanding. This revision contributed to developing her *overt subject knowledge*, *using terminology*, *identifying errors*, the *theoretical underpinning of the pedagogy* knowledge, and *attending*, *interpreting*, and *deciding skills*. In the following section, the critical findings of this study will be discussed through a critique of the existing literature.

CHAPTER 5

CONCLUSION, DISCUSSION, AND IMPLICATION

This study aims to support the improvement of prospective middle school mathematics teachers' mathematical knowledge and professional noticing skills on the perimeter concept through the development of a model based on the cycles of enactment and investigation within the framework of design-based research.

Following this purpose, the main points mentioned in the findings section are reviewed and discussed in terms of references to previous research in the literature. Along with the research questions of this study, the conclusions of the research findings were discussed in two main sections. In the first section, the development of preservice teachers' mathematical knowledge of the perimeter concept through the participation of the current study's model is discussed in relation to the previous studies. The second section discusses the development of preservice teachers' professional noticing skills regarding the perimeter concept through the participation of the current study's model. Finally, recommendations and implications for further studies and educational practices are identified, along with the limitations of the study.

5.1 Development of Mathematical Knowledge Regarding the Perimeter Concept

This section discusses the findings of the current study with respect to the development of prospective teachers' mathematical knowledge of the perimeter through the development of a model based on the cycles of enactment and investigation within the framework of design-based research. In this context, the following sections first summarize the preservice teacher's (Maya) existing

mathematical knowledge of the concept in terms of *overt subject knowledge, use of terminology, identifying errors,* and *theoretical underpinnings of pedagogy* based on her responses on the pre-posttest and post-interview and discuss them in the light of previous studies. Then, it is discussed how this development in the mathematical knowledge about the concept of the perimeter in the current study was supported through the model grounded on the cycle of enactment and investigation.

In accordance with the findings related to *overt subject knowledge*, it could be inferred that the preservice teacher lacked mathematical knowledge in explaining the meaning of the perimeter concept, and she based her explanation of the perimeter calculation on the addition operation before participating in the current study. Furthermore, she misused the mathematical terms related to the perimeter concept, such as unit squares. At the same time, the results of this study showed that although the preservice teacher had difficulty in defining the concept, she drew the examples correctly during the practice in a way that showed the perimeter, not the perimeter. When asked how she drew them, she said that she saw such drawings in the textbooks. These findings are consistent with the results of previous studies, such that preservice teachers memorize perimeter definitions and rules without questioning their meaning (e.g., Baturo & Nason, 1996; Fuller, 1997; Kellogg, 2010; Livy et al., 2012; Menon, 1998; Reinke, 1997; Yenilmez & Koza Çiftçi, 2014; Yew, 2010). These results were reasonably expected since procedural knowledge is generally emphasized in the teaching of measurement-related concepts (Barrett et al., 2006; Stoddart et al., 1993). This means that preservice teachers may focus on memorizing definitions, procedures, operations, and rules rather than questioning their underlying meanings. In other words, focusing on the product by following the rules instead of focusing on the process by questioning the meanings of mathematical concepts, rules, or procedures at the same time may be considered sufficient for preservice teachers during undergraduate education (Huang & Witz, 2011). Therefore, such difficulties may be due to the fact that the concept of perimeter is not taught conceptually during undergraduate education as stated by Kellogg (2010).

In contrast to the pretest results, Maya was found to have well-developed mathematical knowledge in terms of *overt subject knowledge* in the post-test and post-interview results. For example, she updated her explanation of the definition of perimeter by placing the concept on the conceptual foundation. Indeed, Maya defined it by emphasizing the concept's meaning and using mathematically correct language. At the same time, she grounded the perimeter calculation on a conceptual base without depending on the formula. For example, in response to the post-test question about the definition of the perimeter calculation, her answer was as follows: *"We can find the perimeter of a shape by adding the outside lengths or measuring the total length of the shape's boundary*." As understood from her response, she emphasized some critical terms such as "the boundary," "total length around the shape," and "adding outside of the shape" in her explanations. These terms are also used in explaining the concept of the perimeter and the perimeter calculation in the literature (e.g., Cathcart et al., 2006; Haylock, 2001; Long & DeTemple, 2003). In conclusion, she had a conceptual understanding of the perimeter concept and noticed that it is a kind of measurement after participating in the current study. Therefore, critical developments were observed in her *overt subject knowledge* after participating in the current study, and the possible reasons for such developments are discussed in the following paragraphs.

When the findings related to the *using terminology* knowledge were analyzed, the pretest results showed that Maya misused the mathematical language related to the concept of perimeter. To illustrate, she explained the perimeter calculation: "*We can calculate the perimeter by adding up the lengths that cover a shape or object."* As can be seen from her explanation, she misused the term "cover" for the perimeter concept, even though "cover" is used for the area concept. In addition, she ignored the use of length units and expressions in her responses before participating in the current study.

After participating in the current study, it was inferred that Maya's knowledge of *using terminology* was improved. In particular, as she developed her knowledge of the unit, she paid attention to units of measurement as "a central, unifying idea underlying all measurement" (Hiebert, 1981, p. 38). Moreover, she correctly used the terms regarding the perimeter concept. To illustrate, when asked to identify the details of the student's strategy and evaluate its correctness, although the student did not specify the measure of length in her response, Maya accepted the solution as mathematically correct before participating in the study. The results obtained from the post-test indicated that she noticed that the student did not express the measure of length in her answer. Meanwhile, she used mathematical terms to define the concept correctly, such as "*total length...around the shape*." In light of previous research studies (e.g., Purpura et al., 2017), it could be inferred that she was developing her mathematical knowledge regarding the concept of perimeter and noticed that the perimeter is a kind of linear measurement; consequently, she developed her *using terminology* knowledge and vice versa after participating in the current study. Put another way, such a conclusion was to be expected since her mathematical language reflected her *overt subject knowledge* (Lansdell, 1999; Purpura et al., 2017). As her *overt subject knowledge* was low at the beginning of the study, it was expected that her *use of terminology* knowledge would also be low, as Lansdell (1999) emphasized the role of mathematical knowledge in the mathematical language. Thus, it could be said that the improvement in Maya's *overt subject knowledge* regarding the perimeter concept was reflected positively in her mathematical language and *using terminology* knowledge. Moreover, she also explained her point of view on this issue during the post-interview as follows:

> We examined the meaning of the perimeter concept first. Indeed, we discussed what the perimeter means, which property it has, and so on. This way, I noticed that the perimeter is a length and belongs to a closed shape. Later, when I noticed it was a length, I started to use or pay attention to using

units in my response. Besides, as I grasped the meaning of the perimeter concept, I used the correct terms to define the concept.

As can be seen from her explanation, studying the concept's meaning allowed her to develop her mathematical language and terminology related to the concept of perimeter since it was seen that she noticed some properties of measurement, such as conservation when placing the length of the unfolded version of the shape on the plane. Moreover, it was observed that she noticed the importance of using linear units in expressing results during the collective analysis of the activities. Consequently, critical developments were observed in her *using terminology* after participating in the current study, and the possible reasons for such developments are discussed in the following paragraphs.

Regarding the findings related to knowledge of *identifying errors*, the analysis of the pretest scores showed that Maya had difficulty identifying errors in students' responses and difficulties that students might have with the perimeter concept. As mentioned in the previous paragraphs, although the student did not specify the measure of length in her response, Maya accepted it as mathematically correct: "*I think the student's solution is correct because she made the side lengths of the given shape a straight line*." As her response implied, she had difficulty detecting the student's error, namely, not indicating the length measure, before attending the present study. This result coincides with the findings of previous studies such that the preservice teachers struggle to detect students' difficulties regarding the length measurement and perimeter concepts (e.g., Caylan-Ergene & Işıksal-Bostan, 2021; Girit-Yıldız et al., 2022; Güner, 2017). In addition, the current study's findings indicated that Maya lacked knowledge about the possible causes of students' difficulties with the perimeter concept. Indeed, before participating in the study, she associated students' potential difficulties with perimeter with their lack of knowledge about quadrilaterals. Although the literature supports the idea that it is essential to have an idea about students' difficulties and their sources to support students'

mathematical understanding (e.g., Carpenter et al., 1996; Fennema et al., 1996; Rowland et al., 2005; Simon & Blume, 1994; Tirosh, 2000), similar to the finding of the current study, research studies have shown that preservice teachers and even teachers have difficulties with respect to them (e.g., Baş, 2013; Goldsmith & Seago, 2011; Kazemi & Franke, 2004; Wallach & Even, 2005). One of the possible explanations for these situations may be related to the lack of understanding of length measurement. For example, it could be inferred that Maya did not understand that calculating the perimeter is related to giving a numerical value regarding the attribute of the object. She did not connect measuring the circumference with measuring the length. Consequently, she did not pay attention to the use of units of length in the results of the problems and did not care that her students did not use units either. Additionally, research studies have shown that there are differences between preservice teachers' and teachers' predictions about students' mathematical thinking and students' actual mathematical thinking (e.g., Baş et al., 2011; Bergqvist, 2005; Guner & Akyuz, 2017) due to preservice teachers' and teachers' lack of knowledge (e.g., Didiş et al., 2015; Zeytun et al., 2010). In the concept of the perimeter, Dogan-Coskun (2017) stated that if teachers do not rely on their mathematical knowledge of the concept of the perimeter and quadrilaterals or lack knowledge about the concepts, they have difficulty overcoming them even if they can correctly identify students' misconceptions and difficulties about the perimeter.

The analysis of the post-test and post-interview results indicated that Maya correctly identified the mistake in the student's response after attending the present study, as mentioned in the previous paragraphs. Moreover, she expressed that the difficulties stemmed from a lack of understanding of the concept, such as the perceived relationship between the perimeter and the area. She stated it differently; she associated them with students' inability to understand the meaning of length, length measurement, perimeter, and area. Considering these, it could be said that Maya developed knowledge of *identifying errors* after attending to the current study. One of the possible explanations for this situation could be related to the development of

her *overt subject knowledge*. In light of these studies, it can be said that as Maya's mathematical knowledge of the concept of perimeter developed in connection with her *overt subject knowledge*, she was able to identify students' difficulties with the perimeter and the possible sources of these difficulties more easily. In other words, the development of her *overt subject knowledge* may have nurtured the development of her *identifying error knowledge.* It was also observed that as Maya had an idea about the possible difficulties that students might have with perimeter, such as confusing perimeter and area formulas, during the study, she was able to express students' mistakes more easily. Possible reasons for such developments are discussed in the following paragraphs.

Regarding the findings related to knowledge of the *theoretical underpinnings of pedagogy*, the analysis of the pretest revealed that Maya had a lack of knowledge about how she planned to use representations and materials to teach the perimeter concept during the lesson. For example, she only mentioned that she would use manipulatives such as GeoGebra in her lesson without justifying how this would support her instruction. In addition, Maya did not tend to provide space for students to construct knowledge about the concept. Instead, she focused on gaining students' procedural fluency through direct instruction before participating in the study. She tended to use the perimeter formula, such as 2 (a x b), where a and b are the side lengths of the shapes. However, this formula is only used to calculate the perimeter of regular quadrilaterals such as squares and rectangles. It cannot be used to calculate the perimeter of triangles, trapezoids, irregular shapes of a circle, etc. Thus, it was assumed that Maya would not include these shapes in her teaching and restrict the perimeter with regular quadrilaterals. This finding is consistent with previous research findings (e.g., Kellogg, 2010), which show that preservice teachers ignore the perimeter calculations of irregular shapes (e.g., Kellogg, 2010). Furthermore, her discourse was often I-oriented. She planned to use these materials from her explanations to demonstrate how she would teach the concept through direct instruction by means of the above formula. This situation could be due to a lack of

knowledge about the concept of perimeter since they did not associate the perimeter as a kind of length measurement before participating in the intervention. Given these, it could be argued that she had lack of knowledge of the *theoretical underpinnings of perimeter pedagogy* before participating in the current study. This finding is consistent with the findings of previous studies (e.g., Friedrichsen & Dana, 2005; Mellado, 1998; Tekin-Sitrava, 2014), which indicate that preservice teachers' teaching strategies are deficient, and they tend to conduct their teaching in a teacher-centered manner. In particular, this finding was reasonably expected since previous studies indicated that teachers tended to design teacher-centered instruction and present procedural information when they lacked mathematical knowledge (e.g., Bragg & Outhred, 2000; Stoddart et al., 1993). Put another way, preservice teachers need to have well-developed mathematical knowledge (Stoddart et al., 1993) to be able to teach the mathematical concepts conceptually and support their teaching with various strategies. Since Maya's *overt subject knowledge* of the perimeter concept was low, her knowledge of the *theoretical underpinning of the pedagogy* was also expected to be low or underdeveloped at the beginning of the study.

The analysis of her responses and explanations in the post-test and post-interview showed that she tended to provide more comprehensive explanations of how she planned to use representations and materials in teaching the perimeter concept. Meanwhile, she planned to provide space for students to construct knowledge about the perimeter in a particular form of length measurement. For example, her response in the post-test about how she would use representations in her classroom to teach the perimeter was as follows:

> …Besides, I use GeoGebra to allow them to explore the perimeter with various shapes. In this way, they explore that the perimeter is not limited to just a square and rectangle but that the perimeter can be calculated when the shape is closed.

Her explanation indicated that she would use manipulatives such as GeoGebra in her lesson by allowing students to construct their knowledge of the concept. In this way, she tended to use conceptual teaching strategies in her lesson by demonstrating a student-centered approach and helping students notice that irregular shapes and circles also have perimeters. Hence, it could be said that she developed her knowledge of *the theoretical underpinnings of pedagogy* after participating in the study. One of the possible explanations for such development could relate to the development of her *overt subject knowledge*. As previous studies have indicated, having a conceptual understanding of the concepts plays a crucial role in their instructional decisions (e.g., Doğan-Çoşkun, 2017; Karagöz-Akar, 2016; Liston, 2012; Maher et al., 2022) and instructional strategies (e.g., Ball et al., 2008; Grossman, 1990; Rowland et al., 2003). At the same time, previous studies showed that when teachers have ideas about students' understanding, they can enhance their instructional strategies (e.g., Jacobs & Empson, 2016). Similarly, the findings of the current study showed that as Maya's *identifying error* knowledge was developed about the perimeter concept during the study, she started to get an idea about students' difficulties regarding the perimeter concept. Consequently, she could enrich her perimeter teaching with various instructional strategies to support students' mathematical understanding of the perimeter concept. In conclusion, it could be claimed that improvements in her *overt subject knowledge* and *identifying errors* could have nurtured the development of her *theoretical underpinning of pedagogy* knowledge.

In conclusion, it could be claimed that the model grounded on the cycle of enactment and investigation supported the development of preservice teachers' mathematical knowledge about the concept of perimeter, considering the development of *overt subject knowledge, using terminology, identifying errors, and theoretical underpinning of the pedagogy* under Foundation Dimension of the KQ model. This conclusion coincides with the findings of the previous studies such that appropriate interventions supported the development of teachers and preservice teachers'

knowledge of such as proportional reasoning (e.g., Özgün-Koca et al., 2020), triangles (e.g., Uygun, 2016), and multiplicative reasoning (e.g., Fauskanger & Bjuland, 2019). However, there is a limited study in the literature to promote such development in the knowledge of the perimeter through the model grounded on a cycle of enactment and investigation. Therefore, this study differs from existing research in that it defined the process by which prospective teachers' mathematics knowledge of the perimeter concept was developed.

In the present study, one of the underlying reasons for this development can be attributed to the nature of the model of the current study. Firstly, the prospective teachers co-constructed their knowledge, practices, and skills and shared information by using several pedagogies such as approximation of the practice (Grossman et al., 2009a), pedagogies of investigation and reflection (Grossman et al., 2009b), and the enactment (Kazemi et al., 2009; Lampert et al., 2013; McDonald et al., 2013). In other words, the collective and repeated engagement in various pedagogies in teacher education programs enabled them to develop their knowledge and skills similar to the findings of the previous studies (e.g., Arbaugh et al., 2019; Lampert et al., 2013; McDonald et al., 2013). The difference between other studies grounded on the cycle of the investigation and enactment and the present study, which will be discussed in the following, can be related to the nature of activities in the current study's model.

In the current study, different than other studies, the cycle of enactment and investigation was integrated by multiple pedagogies by using similar activities rather than using the same instructional activity throughout the model. Thus, the nature of the activities could be another explanation for such development. For example, different from the studies of Lampert et al. (2013) and McDonald et al. (2013), the model started with working on Activity 1: *What is the Perimeter? Let Us Look at the Concept from your and the Students' Eyes* in the first node of the intervention, called *pre-Preparation*. In this activity, several questions enabled prospective teachers to co-construct their knowledge of the perimeter concept by explaining their reasoning

about their drawings regarding the perimeter. Indeed, they questioned the perimeter is a particular form of length measurement, learned to use appropriate units for length measurement (linear units rather than square units), and some concepts regarding the measurement such that when moving or opening a closed shape, its perimeter does not change, related to the conversation property of the measurement. Afterward, it was observed that Maya paid attention to some terms related to the concept of perimeter and not perimeter in the others' explanations of the preservice teachers' arguments about their drawings regarding the perimeter and not perimeter, such as "being closed," "being linear and nonlinear," "the boundary of the shapes," and so on. In this way, she realized that calculating the circumference of a circle is also related to calculating the circumference of closed shapes. Thus, instead of memorizing the definition of the concept of perimeter, basing the perimeter calculation on the addition operation, or memorizing concept images of the perimeter, she had the opportunity to co-construct their meaning in depth while being a part of the learning community. In other words, Maya gained learning opportunities in the group discussion to share, review, compare, and debate her mathematical ideas with her peers, which helped her to re-and co-construct her mathematical knowledge of the concept and her concept images related to the concept (Hiebert et al., 2003). In this way, she was able to conjecture alternative mathematical ideas in light of her peers' arguments about the perimeter. In this regard, Maya had the opportunity to share ideas with her peers and to critique her ideas and those of her peers (Wenger, 1998; 2000), which could be influential in improving mathematical knowledge. In other words, it could be inferred that such collective analysis of the activities created an imbalance between Maya's existing knowledge of the concept and the new knowledge she gained from the study's model, which could contribute to the development of the mathematical knowledge of the perimeter concept.

Furthermore, unlike other studies (Lampert et al., 2013; McDonald et al., 2013), the model of the current study allowed the preservice teachers' space to individually respond to the questions before the collective analysis took place, similar to Arbaugh

et al.'s (2019) study. For example, as part of Activity 2: *Let Us Examine the Dialogue* in the *pre-Preparation* node of the model, the prospective teachers deconstructed a given dialogue between a teacher and a student about the perimeter computation in light of the components of the professional noticing framework (Jacobs et al., 2010), first individually and then collectively. Specifically, the representation and decomposition of the practices in the given dialogue (Grossman et al., 2009a) in light of the components of the professional noticing framework (Jacobs et al., 2010) allowed Maya to speculate about the details of the student's mathematical strategies and alternative instructional strategies to facilitate student thinking, taking into account the ideas of her peers and the student's way of thinking. In addition, the prospective teachers collectively approximated their decisions about continuing the lesson in light of the given student's strategies. When Maya's responses to the questions in the activity were compared with her ideas in the conversations with the preservice teachers, it was observed that Maya correctly participated in the details of the student's strategy from mathematical aspects such as counting points rather than lengths or units and tended to take the necessary precautions to prevent misconceptions from occurring in the classroom by considering the student's way of thinking during the collective analysis such as not accepting the result even if it was correct because the student's solution was wrong, which could contribute to the development of her mathematical knowledge of the perimeter concept. One of the possible explanations for this situation might be related to discussing the conceptual meaning of the perimeter based on the preservice teachers' drawings on the perimeter and not the perimeter as part of Activity 1, as mentioned above. During such a discussion, Maya put the perimeter on a conceptual ground and grasped that the perimeter is a kind of measurement. Afterward, during the discussion on Activity 2, the preservice teachers collectively questioned the details of the student's strategy from a conceptual point of view within the framework of their existing current knowledge. State it differently; this may have contributed to the preservice teachers' discussion of the perimeter concept through mathematical details in the student's solution. Meanwhile, it was observed that Maya tended to produce conceptual

strategies to teach the concept as a part of intended responses. In fact, she tried to help students realize that when the perimeter of the shape is opened and placed on the plane, the perimeter length does not change, which is related to the conservation property of measurement, and that the perimeter is a one-dimensional measure. Moreover, she was also trying to make students realize that the calculation of the circumference of a circle is also a kind of perimeter calculation. In this way, she tended to provide a space for the students to construct the knowledge of the perimeter concept as a particular form of length measurement. As stated before, her conceptual understanding developed on the perimeter and associated the perimeter as a kind of measurement. Thus, she tended to display instructional decisions to support students' mathematical thinking of the concept, which is consistent with the findings of the previous research (e.g., Stoddart et al., 1993).

In addition to these activities, Maya was asked to prepare a lesson plan for perimeter teaching by creating dialogues between herself and fictitious students for each problem or activity as part of Activity 1: *Let Us Prepare for the Rehearsal* in the *preparation* node of the model. Unlike other studies (e.g., Lampert et al., 2010), this step was added to this study to prepare the preservice teacher for individual teaching. State it differently; during the preparation of this activity, she decomposed the teaching practices considering the possible solution strategies of the students and approximated her decisions considering the students' actions before the teaching occurred. For example, she assumed that students might be confused about the formulas for perimeter and area. With this in mind, she tended to emphasize appropriate units, such as linear units rather than square units, to help students realize that the perimeter is actually a length and to notice the difference between the perimeter and the area. Consequently, as Arbaugh et al. (2019) pointed out, decomposing and approximating practices contributed to the development of a repertoire of instructional strategies by allowing for a close look at possible students thinking and possible teacher actions (including themselves) to such thinking before the actual teaching took place. Additionally, in the context of the same activity, the

preservice teachers collectively examined the appropriateness of the questions and teaching strategies, the mathematical language and expressions, and the appropriateness of the teacher's actions. For example, the prospective teachers examined Maya's mathematical language in defining perimeter in her dialogues and sometimes suggested that she revise her language in light of the concept's meaning. At the same time, they sometimes suggested adding some activities, such as GeoGebra, to allow students to discover the meaning of the concept rather than memorize the rules and associate it with the length measurement. In this way, Maya reviewed her mathematical knowledge of the perimeter concept and redesigned her lesson plan based on such comments. In this way, she was given another opportunity to deconstruct the teaching practices and review her definitions, mathematical language, and the details of the students' strategies, which could contribute to the development of the mathematical knowledge of the perimeter concept.

Additionally, Maya was asked to review her lesson plan and redesign the dialogues in light of the experience as part of Activity 2: *Prepare for Further Teaching* in the final node of the model called *post-Preparation*. In this way, she had one last chance to deconstruct the teaching practices and review her definitions, the mathematical language, the details of the students' strategies she produced, and her responses to those strategies to prepare for further perimeter teaching. For example, it was observed that she diversified her perimeter teaching strategies in her lesson plan, considering actual and potential students' way of thinking, using GeoGebra by emphasizing the main concepts of measurement such as partitioning, iteration, conservation, and accumulation of distance. At that point, similar to the findings of the current study, Güner (2017) emphasized the role of planning the lessons, considering students' potential solution strategies, on teachers' teaching strategies, aiming to respond to students' needs.

Furthermore, having different roles in a learning community also contributed to this development and enhanced mutual engagement (Lave & Wagner, 1991; Wegner,

1998; McDonald et al., 2013). In the present study, the participants in the current study took on different roles during the study. For example, Maya was in the role of the teacher, while the others were in the role of evaluating the lesson plans during the lesson plan preparation. In addition, Maya had the role of teacher, while others had the role of students during the rehearsal. Besides, Maya had the role of teacher, while the others had the role of observer during the classroom enactment. During the analysis of the video recordings, all of them were in the role of observers and made comments to improve the quality of Maya's perimeter teaching. Such a change of roles in the community might enable the preservice teachers to look at the perimeter concept and the perimeter teaching from different aspects, such as from learners' and teachers' perspectives, which might contribute to the development of mathematical knowledge of the perimeter.

Another reason for the development of Maya's mathematical knowledge could be attributed to authentic experiences through the current study's model. For example, Maya rehearsed the perimeter lesson in a simulated classroom environment as part of Activity 2: *Rehearsing the Perimeter Teaching* in the second node of the model, *Preparation*. The rehearsal activity situated in the approximation of the practice allowed her to teach the concept and demonstrate her instructional choices and strategies in a simulated classroom environment. Indeed, approximating the teaching practices provided "opportunities (for preservice teachers) to rehearse and enact discrete components of complex practice in settings of reduced complexity" (Grossman et al., 2009a, p.283). While Maya was in the role of teacher, the others questioned and queried the perimeter concept and the problems like five-grade students. As a result, Maya had to benefit from their understanding of the perimeter concept, point out errors, and present her ideas to others during rehearsal. Thus, this result was expected because the interactions between Maya and others in the simulated environment contributed to the development of Maya's mathematical knowledge, similar to the findings of previous research (e.g., Ghousseini, 2017).

Besides, Maya implemented the perimeter lesson with actual students as part of Activity 1: *Enact the Perimeter Teaching* under the third node of the model called *Implementation*. Implementing the perimeter lesson in a classroom allowed her to demonstrate her instructional choices with actual students. In other words, Maya had the experience of identifying students' difficulties and solutions, using materials, defining the concept, and using mathematical language while teaching the concept of perimeter to actual students. In this way, she also had a chance to reflect on her learning during the model of the study in her teaching. For example, Maya paid attention to expressing linear units in her teachings and emphasized the meaning of the perimeter from conceptual aspects. Thus, she demonstrated exemplary behaviors regarding mathematical knowledge in terms of *overt subject knowledge*, *use of terminology*, *identifying errors*, and the *theoretical underpinnings of pedagogy* knowledge during her teaching. Such finding confirms Pişkin-Tunç & Çakıroğlu's (2020) conclusion that providing authentic experiences has contributed to knowledge development.

Furthermore, when Maya failed to demonstrate exemplary behaviors related to mathematical knowledge, the prospective teachers, including Maya, collectively reviewed her teaching and reflected on their feedback to improve the quality of teaching while reviewing recordings of her rehearsal lessons and actual lessons as a part of the pedagogy of investigation and reflection. More specifically, Activity 2: *Let Us Examine the Rehearsal* under the preparation node of the model and Activity 1: Let Us Examine the Enactment under the post-preparation of the model focused on the collective analysis of Maya's teaching practices. For example, it was seen that the preservice teachers encouraged Maya to pay attention to using the units in her explanations if she did not mention units of length in her teaching. In addition, they suggested that Maya should not limit the perimeter to quadrilaterals and should also mention circles. In the literature, researchers have generally focused on investigating the video recordings of other teachers' classroom implementation to review their teaching practices (van Es & Sherin, 2008). Unlike such studies, Maya had the

opportunity to collectively review her definitions, language, units of measurement, teaching strategies, details of students' strategies, and instructional decisions in her teaching experiences (including in the simulated and actual classroom settings). Put another way, collectively analyzing her teaching practice by receiving feedback from peers helped her learn from her practice (Johnson & Cotterman, 2015) because, as McDonald et al. (2013) stated, "the reflective and analytical work that novices do together is an important aspect of giving meaning to the practices they are working on" (p. 383).

In addition, the development of preservice teachers' mathematical knowledge of the concept of perimeter may be due to the content and nature of courses in current teacher education programs. Specifically, as mentioned in the Methodology section, the preservice teachers do not have the opportunity to take courses as a bridge between theory and practice. Instead, the preservice teachers have mainly focused on how the perimeter is treated in the middle school curriculum and how to teach the concept without having a chance to teach the concept. For example, the course called "Geometry and Measurement Teaching" in the current teacher education programs includes the discussion of length by organizing the course content with the use of appropriate teaching materials and strategies, and so on. In addition, another course related to geometry and geometry teaching called “Foundation of Mathematics II” includes a discussion of the basic concepts and properties of length measurement and its relationship to the goals, principles, and program of middle school mathematics education. A course that allowed the co-construction of the meaning of the concept of perimeter through productive discussions using the model of the study could support the development of the mathematical knowledge of preservice teachers, which is consistent with the findings of previous research, such that the collective analysis of practices contributed to the development of knowledge (e.g., Aguirre-Garzón, 2018; Mitchell & Appleget, 2021). At the same time, the results of the current study showed that providing more practice-based experiences would be an effective way to improve preservice teachers' mathematical knowledge of the

perimeter concept. This is consistent with the findings of previous research, such that practice-based experiences contribute to knowledge development (e.g., Pişkin-Tunç & Çakıroğlu, 2022).

To sum up, all could contribute to the development of Maya's mathematical knowledge of the perimeter concept concerning *overt subject knowledge, using terminology, identifying errors*, and *the theoretical underpinnings of pedagogy* knowledge regarding the perimeter concept by using several activities grounded on various pedagogies. In other words, the prospective teachers' mathematical knowledge of the perimeter developed through the model grounded in the cycle of enactment and investigation. In a broader context, the findings support the idea that professional development programs that include participation in authentic experiences supported by examination and reflection on the work of teaching can positively impact participants' mathematical knowledge development (e.g., Guskey, 2003; Hawley & Valli, 1999; Wilson & Berne, 1999; Pişkin-Tunç & Çakıroğlu, 2022; van Es, 2012).

5.2 Development of Professional Noticing Skills on the Perimeter Concept

This section discusses the findings about improving the preservice teacher's professional noticing skills (attending, interpreting, and deciding) on the perimeter through the development of the model grounded on the cycle of enactment and investigation. In the following paragraphs, the preservice teacher's (Maya) existing professional noticing of the perimeter concept in the current study in terms of *attending, interpreting,* and *deciding* skills based on her responses on the pre-posttest and post-interview are summarized and discussed, respectively, in light of previous studies. Then, it is discussed how this development of the professional noticing skills on the perimeter in the current study was supported through the model grounded on the cycle of enactment and investigation.

Attending skill is related to identifying "what is mathematically significant and skill in finding those mathematically significant indicators in children's messy, and often incomplete, strategy explanations" (Jacobs et al., 2010, p.194). The current study's model was designed to allow the preservice teachers to identify the essence of students' mathematical strategies regarding the perimeter concept and produce possible students' strategies regarding the concept through collective analysis, lesson planning, rehearsal and classroom implementations, and reflection of the teaching practices through several pedagogies. Following the findings of this study, it was observed that Maya's *attending skill* was promoted from lacking to robust levels after participating in the current study. Indeed, before the study, while Maya did not correctly evaluate the student's outcomes and did not identify the details of the student's strategy, she was observed to attend and evaluate the outcomes correctly after participating in the intervention. Therefore, it could be said that the results of the study showed that the development of preservice teachers' *attending skills* about the perimeter should be supported with appropriate interventions, which was consistent with the findings of previous research (e.g., Jacobs et al., 2010, Sherin & Van Es, 2009, Star & Strickland, 2008).

Interpreting skills are related to preservice teachers' or teachers' "sufficient understanding of the mathematical landscape to connect how those strategies (students' strategies) reflect an understanding of mathematical concepts" (Jacobs et al., 2010, p. 195). The current study's model was designed to allow the preservice teachers to make sense of the students' mathematical strategies regarding the perimeter concept and predict possible students' mathematical understanding of the concept through collective analysis and lesson planning processes. Before participating in the model, Maya's *interpreting skills* were coded as low because she described the student's mathematical understanding of perimeter superficially. After participating in the current study, it was observed that she detailed the student's mathematical understanding of the perimeter concept; consequently, her interpreting skill was coded at a robust level. This finding confirms Jacobs et al.'s (2010)

conclusion that interpreting students' mathematical understanding is challenging for preservice teachers, and appropriate support is needed to improve preservice teachers' interpreting skills. This is consistent with the findings of the previous empirical research studies (e.g., Sherin & Van Es, 2006; Vondrova & Zalaska, 2013).

Deciding skill includes "knowledge about children's mathematical development to identify a reasonable next step and the ability to facilitate that next step by selecting a problem that will be accessible yet also challenge children's thinking" (Jacobs et al., 2010, p. 195). The current study's model was designed to allow the preservice teachers to display how to respond to students' mathematical thinking regarding the perimeter concept through various pedagogies, from collective analysis to implementing lessons. The current study indicated that Maya's *deciding skills* were promoted from lack to robust levels. More specifically, before participating in the current study's model, her *deciding skill* was coded at a lack of evidence level because she tended to ignore the student's mathematical understanding of the perimeter concept while responding to the student. After participating in the model, it was found that she tended to promote the student's mathematical understanding of the concept by modeling the concept. Therefore, her *deciding skills* were coded at a robust level. State it differently; she progressed her *deciding skills* from lack to robust level as she participated in the current study. This result of the current study is essential since researchers often observed no significant improvement in *deciding skills* was observed (e.g., Krupa et al., 2017; Morris, 2006; Sánchez-Matamoros et al., 2019; Schack et al., 2013).

In conclusion, considering such instances, it could be inferred that preservice teacher's professional noticing skills should be promoted with appropriate interventions, which is consistent with the findings of previous studies (e.g., Girit-Yildiz et al., 2022; Güner & Akyüz, 2020; Roth McDuffie et al., 2014; Ulusoy & Çakıroğlu, 2021; Wieman & Webel, 2019). While these studies contribute to the literature on improving noticing through appropriate interventions over one or more

semesters, this study presents a difference with a short-term professional development program that lasted less than one semester and focused on the development of preservice teachers' professional noticing skills through collectively examination and reflection on the work of teaching and having authentic experiences. In this regard, the current study makes a valuable contribution to the literature from the point of view of how preservice teachers' professional noticing skills with respect to *attending, interpreting*, and *deciding skills* could be supported through the model situated in the cycle of enactment and investigation.

Such improvements may stem from the nature of activities in the current study's model, which allowed preservice teachers to learn in, from, and through teaching practice via authentic and collaborative experiences. To illustrate, Maya had the opportunity to encounter students' potential and actual strategies regarding the perimeter, such as counting dots rather than units, and to demonstrate proximal and actual instructional choices, such as the emphasis on the perimeter is the type of length, in response to students' ideas in simulated and actual classroom settings through Activity 2: *Rehearse the Perimeter Teaching* in the *Preparation* node and Activity 1: *Enact the Perimeter Teaching* in the *Implementation* node of the model. During her teaching, it was observed that Maya tended to explore the details of students' thinking through multiple follow-up questions, even when she had difficulty in moving students' mathematical understanding forward. Even if she had difficulty in advancing students' ideas at the moment of teaching, it is observed that she learned to focus on students' mathematical understanding during the instruction. Thus, the current study's findings confirm McDonald et al. (2013) suggestion that having teaching experiences in different settings (controlled, designed, and authentic) has contributed to the development of teaching skills.

In addition, the current study's findings provided a unique lens through which to unpack how preservice teachers' instructional choices might differ from their written choices about promoting students' mathematical understanding. One such lens may

be recognizing the critical role of collective analysis in developing professional noticing skills (Barnhart & van Es, 2015; Johnson & Cotterman, 2015; Star & Strickland, 2008; van Es & Sherin, 2008). The collective analysis, based on a pedagogy of investigation and reflection, contributed to Maya's revision of her intended, approximate, and actual instructional decisions, paying more attention to the details of students' strategies. In other words, the prospective teachers, as a group, reframed the teacher's action by suggesting actions to clarify and support students' mathematical thinking in the given dialogues, lesson plans, rehearsals, and classroom practices. In this way, Maya could examine her own teaching experience as an outsider. She adopted different perspectives to support students' mathematical thinking while receiving feedback and discussing her ideas with her peers. For example, during Maya's perimeter teaching, it was observed that she hesitated about the correctness of the student's solution, and because of her hesitation, she skipped the student's solution. While examining Maya's teaching, it was observed that other prospective teachers encouraged Maya to ask some clarifying questions to the student to make sense of the student's solution instead of skipping the student's solution. The finding of the current study confirms van Es and Sherin's (2021) and Amador et al. (2023) conclusion that collectively examining the dialogues between students and teachers has contributed to noticing skills development.

Furthermore, the result was also consistent with the findings of previous studies in the literature, such that preservice teachers and teachers who actively participate in video clubs that include the analysis of teaching practices have a positive impact on the development of their noticing skills (e.g., Girit-Yildiz et al., 2022; Ivars et al., 2020; Ulusoy & Cakiroglu, 2021; van Es & Sherin, 2002; Warshauer et al., 2021). However, previous research has mainly provided preservice teachers with field experiences by analyzing the teaching experiences of other teachers (Ulusoy, 2020; Girit-Yıldız et al., 2022; Roth Mcduffie et al., 2014; Wager, 2014). This study takes existing research a step further by allowing Maya to make instructional decisions in various forms and then critiquing the videotapes of her instructional practice.

Therefore, it could be said that such experiences could contribute to the development of her professional noticing skills, especially the *deciding subskill*, which is the most challenging subskill among professional observation skills to develop (e.g., Jacobs et al., 2011; Krupa et al., 2017; Sánchez-Matamoros et al., 2019; Schack et al., 2013). Still, the current study's findings have highlighted that prospective teachers need more practical experience displaying responsive actions in the teaching perimeter to promote student understanding, even if they gave place responsive actions in their written responses.

Furthermore, as previous studies showed, planning is crucial in attending to students' mathematical thinking and responding to their mathematical understanding (Guner, 2017; Jakopovic, 2021; Lampert, 2001). The current study also allowed the prospective teacher to plan the instructions, estimating students' strategies and teachers' actions to such strategies. For example, Maya was allowed to generate possible student solutions to the problems or activities in the lesson plans by focusing on the students' underlying understandings and how she, as the teacher, intended to respond to those solutions by approximating teaching practices through the lesson planning activities such as Activity 1: *Prepare for Rehearsal* as part of *preparation* node of the model. In this way, Maya gained experience in observing, interpreting, and deciding how to proceed with the lesson through decomposition and approximation of the practice. That is to say, she reasoned students' potential conceptions and misconceptions about the perimeter, such as confusing units, while producing students' possible solutions ways regarding problems from students' eyes. At the same time, she questioned potential students' mathematical understanding of the perimeter concept, considering the ways of their possible solutions and producing actions to respond to their mathematical understanding of the perimeter concept from teachers' eyes, such as an emphasis on the perimeter is a kind of length measurement. This conclusion is consistent with the results of the studies of Amador et al. (2021) and Lee and Kim (2022), such that providing opportunities for preservice teachers

to analyze practices through several pedagogies has contributed to developing professional noticing skills.

In addition, the prospective teachers collectively questioned the meaning of the perimeter concept through learning from and with each other and explored pedagogical approaches to teaching the concept, taking into account the details of the students' strategies as part of Activity 2: *Let us Examine the Dialogue* in the *Pre-preparation* node of the model. As a result, they co-constructed or reviewed their knowledge of the concept of perimeter when they noticed that it is a particular form of length measurement and includes linear units, even though it is one of the most challenging concepts for preservice teachers to learn and teach. Although several studies in the literature had emphasized the role of collective analysis in developing "new knowledge that goes beyond any knowledge possessed by any one member in isolation" (Donato, 2004, p. 287), the results of the study showed that when Maya and others attended to the details of students' strategies for the perimeter concept, it made a difference in the development of mathematical knowledge as a whole while attending to students' mathematical strategies. From another perspective, studies have shown that teachers' knowledge has been accepted as a pre-requisite of teacher noticing skills (Copur-Gencturk & Tolar, 2022; Dick, 2017; Empson & Levi, 2011; Jacobs & Empson, 2016). To illustrate, in the study of Dick (2017), she emphasized the role of specialized content knowledge in developing professional noticing skills. In the present study, it was observed that Maya tended to attend to the mathematical details of students' strategies, interpret their mathematical understanding, and decide how to continue the lesson, considering the students' mathematical thinking on the perimeter concept as she developed her understanding of the concept. Therefore, it is argued that the development of Maya's mathematical knowledge might influence her professional noticing skills.

Besides, in the current studies, the activities to support the development of prospective teachers' *interpreting skills* were less than activities to support the

development of *attending* and *deciding skills*. But, there has been a significant improvement in Maya's interpreting skills. A possible explanation of this development could be related to the interrelated nature of professional noticing skills' components. Indeed, as Jacobs et al. (2010) expressed, developing one subskill of professional noticing could contribute to the development of the others. In the current study, Maya took more interpretive stances as she developed her *attending skills.* Hence, it could be inferred that the development of *attending skills* could nurture the development of *interpreting skills*, which is consistent with the findings of the previous research such that the role of the development of *attending skills* on the development of the *interpreting skills* (e.g., Magiera et al., 2021; Tekin-Sitrava et al., 2022; Weiland et al., 2014).

In addition, the development of preservice teachers' professional noticing skills related to the concept of perimeter may be due to the content and nature of courses in current teacher education programs. Specifically, as noted in the Methodology section, preservice teachers do not primarily have the opportunity to focus on students' mathematical thinking. Without the opportunity to teach the concept or provide hands-on experiences, preservice teachers have focused primarily on how perimeter is treated in the middle school curriculum and how to teach the concept. In the current study, the course supported by several practice-based pedagogies, such as rehearsal and enactment, may have contributed to the development of preservice teachers' professional noticing skills, similar to the findings of previous research (e.g., Amador et al., 2021; 2023; Estapa et al., 2018).

Finally, the findings of the present study showed that developments of the prospective middle school mathematics teacher's mathematical knowledge of the perimeter concept were observed in each component through participation in the model grounded on the cycle of enactment and investigation. Various studies in the literature examine the relationship between teachers' knowledge and professional noticing skills (e.g., Dick, 2017; Ersari, 2018; Jakobsen et al., 2014). More

specifically, as Jacobs et al. (2010) stated, "To interpret children's understandings, one must not only attend to children's strategies but also have sufficient understanding of the mathematical landscape to connect how those strategies reflect understanding of mathematical concepts" (p. 195). Similarly, Vondrova and Žalská (2013) stated that the "ability to notice was possibly a manifestation of mathematical knowledge for teaching" (p. 361). Thus, it could be concluded that teachers' knowledge and professional noticing are an integral part of each other and influence students' mathematical understanding. Although the attention of the current study was not to examine the relationship between knowledge and professional noticing skills, the results of the current study suggest that the improvement of Maya's mathematical knowledge of the perimeter concept and the improvement of her professional noticing skills on the perimeter concept could nurture each other. For example, it was observed that the development of Maya's *overt subject knowledge* and *use of terminology* related to the concept of perimeter could contribute to the development of her professional noticing skills. In fact, it was observed that as her mathematical understanding of the concept developed, she could more easily identify the mathematical details in the students' strategies regarding the perimeter concept by using mathematically correct language. As a result, it could be said that such development contributed her *attending skills*. Meanwhile, it was observed that as she paid attention to the essence of students' strategies regarding the perimeter, she could more easily identify students' errors during the instructions. That is to say, the development of her *attending skills* could contribute to the development of her *identifying errors* knowledge. These findings are consistent with the results of previous research, such that there is a positive relationship between teachers' knowledge and their professional noticing skills (e.g., Dick, 2017, Ersari, 2018; Thomas et al., 2017)

In summary, prospective teachers' professional noticing skills on the perimeter concept developed through the model grounded on the cycle of enactment and investigation. In a broader context, the findings support the idea that professional development programs that include participation in authentic and collective experiences supported by investigation and reflection on the work of teaching can positively impact the development of participants' professional noticing skills (e.g., Arbaugh et al., 2019; Amador et al., 2021; Girit et al., 2022).

5.3 Implications of the Study

In light of the current study's findings, some implications and suggestions are proposed for curriculum developers, researchers, and mathematics teacher educators interested in implementing professional development programs simulated in the cycle of enactment and investigation in both preservice and in-service teacher education.

First and foremost, the findings of this study revealed that integrating and incorporating among multiple pedagogies, such as the representation of practice, approximation of practice, and enactment, are an efficient way to promote mathematical knowledge and professional noticing skills on the perimeter concept which is the essential practical implication of the current study. Therefore, mathematics teacher educators and scholars in mathematics education could easily benefit from these pedagogies in their courses as an effective tool to promote preservice teachers' mathematical knowledge and professional notice skills on the perimeter concept. At the same time, teacher educators and researchers can also integrate the procedures of this study into their courses to promote preservice teachers' knowledge and professional noticing skills in various mathematical domains and branches of teacher education such as chemistry and so on. This important finding may open a new window for researchers to develop preservice teachers' knowledge and skills through a cycle of enactment and investigation.

Furthermore, another implication could relate to increasing the dissemination of various pedagogies through the cycle of enactment and investigation in prospective teacher education. In this regard, libraries of shared activities for different mathematical subjects could be formed that give mathematics educators access and integrate them into their courses whenever they want. In this way, the cycle of enactment and investigation approach could be used as a learning tool to develop the prospective teachers' professional visions in university methods courses.

Another implication concerns being a part of the community's crucial role in developing knowledge and skills. In the present study, the preservice teachers were provided opportunities to collectively analyze the meaning of the topics, alternative instructional strategies, and students' solution strategies and to reflect on how to promote students' mathematical understanding of the perimeter concept by reviewing instructional practices. Thus, the study findings showed that they co-constructed their mathematical knowledge and professional noticing skills on the perimeter concept. For example, in the current study, at the beginning of the first node, *Pre-Preparation*, the preservice teachers were asked to respond individually to the activities. They were then allowed to co-construct their mathematical knowledge and professional noticing skills by discussing their reflections on these activities as part of the pedagogy of investigation and reflection. Their improvement was striking when their collective analysis responses were compared with their individual analysis responses. They expanded their knowledge of how to teach, how to define the concept, how to use materials to meet students' needs, and their shared understanding as they were involved in the current study. Thus, scholars should use community learning settings in teacher education programs through the cycle of enactment and investigation to build shared understanding. As Frykholm and Glasson (2005) stated, "...the potential problems inherent in the student teachers' deficiencies in content knowledge dissipated as they collaborated, shared ideas and helped each other with fundamental concepts and procedures" (p. 138). Consequently, the current study's findings supported the idea that learning in a social

setting allows preservice teachers to discuss different perspectives and come together under a standard heading.

It is clear from the present study's findings that various activities in the cycle of enactment and investigation in preservice teacher education help support their mathematical knowledge and professional noticing skills both separately and simultaneously (Arbaugh et al., 2019). Therefore, instead of using the same instructional activities the whole learning cycle, like in the studies of Lampert et al. (2013) and McDonald et al. (2013), the types and number of activities could be differentiated throughout the model, taking into account the goal(s) of the study, which could be another implication of the current study. Put another way, the study's findings give new perspectives for researchers to develop mathematics, pedagogy, skills, and practices separately or simultaneously through multiple activities, considering the goal(s) of the research through the cycle of enactment and investigation. Moreover, considering the findings of the current study, it could be suggested that mathematics educators give more place to authentic and collective experiences in their methods courses with the aim of supporting the development of mathematical knowledge and professional noticing skills.

Building a stronger connection between teacher education and practices could be another implication of the current study. As previous studies have shown, having strong content and pedagogical content knowledge does not guarantee effective teaching (e.g., Dilberoğlu, 2016). Similarly, the results of this study showed that preservice teacher's ability to notice students' mathematical understanding and mathematical knowledge about the concept of perimeter were more developed in post-tests and post-interviews than in teaching practices. In other words, the preservice teacher sometimes had difficulties reflecting her existing knowledge and skills. Still, the results of this study give researchers hope that when preservice teachers become teachers, they will be able to incorporate the knowledge and skills into their teaching; therefore, the courses that include more authentic and practice-

based experiences should be added into undergraduate courses in teacher education programs.

As a final implication, researchers who wish to conduct studies to improve preservice or in-service teachers' responsiveness should simultaneously address the development of mathematical knowledge and professional noticing skills. Jacobs & Empson (2016) stated that knowledge and professional noticing skills are two critical determinants of teachers' responsiveness. As an essential point, the study's findings showed that improving mathematical knowledge about the concepts before preparing preservice teachers for teaching practices was beneficial and reasonable in noticing and responding to students' mathematical ideas. At the same time, integrating students' mathematical thinking into the content courses rather than providing knowledge-based experiences is essential to mathematical knowledge and professional noticing skills development (e.g., Baki, 2012).

5.4 Limitations and Suggestions for Future Studies

This study provided a picture of how Maya developed her mathematical knowledge and professional noticing skills on the perimeter concept through the model situated in the cycle of enactment and investigation. However, several limitations may influence the results of this study. These limitations and the various suggestions for further research based on the current study's findings and the relevant literature are detailed below.

One potential limitation may be that the ways of representing the practice were used in the study. Indeed, this study used written dialog and Maya's teaching experiences to represent practice. To overcome this limitation, it could be suggested that different representations, such as video clips of student teachers and other student teachers and video clips between teachers and students, could be used simultaneously to develop knowledge and professional observation skills. For example, the video recordings of the dialogues could be analyzed first, and then the student teachers

could be asked to note down the points that caught their attention in the videos. Then, the transcript of such a video could be distributed to the preservice teachers, and they could be asked to examine the dialogue and compare their notes again by analyzing the video recordings and written documents. In addition, they can also be presented with a video or written document in which the teacher gives incorrect information to the students or ignores their ideas and follows the above process. In this way, more opportunities can be created for preservice teachers to develop their mathematical knowledge and professional discernment skills. In addition, one-on-one interviews with teachers and students should be included in preservice teachers' professional development programs. In this way, preservice teachers can have the opportunity to examine the perspectives, knowledge, and pedagogical strategies of expert teachers. In addition, previous research has shown that designing professional development programs by integrating both preservice and in-service teachers also contributes to the professional development of preservice teachers (Hammerness et al., 2005). Thus, it is hoped that researchers, mathematics educators, and stakeholders can further use these findings to examine knowledge and professional noticing development via course design and redesign in teacher education and professional development programs.

In addition, there are several international and national studies related to the perimeter to determine preservice and in-service teachers' existing knowledge of the perimeter concept. This situation made it very helpful to identify learners' difficulties with the concept while designing the activities of the current study. Although the research findings unpacked the development of prospective teachers' mathematical knowledge of the perimeter, one of the most challenging subjects for learning and teaching, through the model grounded on the cycle of enactment and investigation, still little is known about how to support the development of mathematical knowledge of the perimeter concept through appropriate interventions. Therefore, further studies are needed to obtain in-depth information about how to support the prospective teachers' mathematical knowledge of the perimeter concept practically and effectively. Additionally, further studies could be conducted to obtain in-depth

information about the developmental process of mathematical knowledge and professional noticing skills on different mathematical concepts with the aim of evaluating the effectiveness of the model.

Furthermore, this study examined the mathematical knowledge of preservice middle school mathematics teachers about the perimeter concept using some components of the Foundation Dimension of Knowledge Quartet Model. Although existing research has focused more on the data obtained from preservice teachers' practices, in this study, the data were not only limited to practices but also confirmed by questionnaires and interviews. Thus, further studies can be suggested to examine all dimensions of the Knowledge Quartet Model regarding the perimeter concepts and different mathematical subjects with and without practices to understand the model's applicability in different settings.

Finally, the existing literature has shown that there is a relationship between teachers' knowledge and professional noticing skills (e.g., Dick, 2018; Ersari, 2018). However, little is known about the direction of this relationship. Although the purpose of the current study was to focus on the separate but simultaneous development of pre-service teachers' mathematical knowledge and professional noticing skills, the results of the current study showed that there was a bidirectional relationship between mathematical knowledge and professional noticing skills. Therefore, further research could be conducted to focus on the relationship between the development of components of the Foundation dimension, and also other dimensions of the Knowledge Quartet model, and components of professional noticing skills. Such a study would be a significant contribution to studies of teacher responsiveness, given the lack of studies examining the relationship between professional noticing skills and mathematical knowledge.

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APPENDICES

A. Post Interview Questions

Görüşme Soruları

1. Öğretmenlik mesleğini severek mi tercih ettiniz? (Şu anda seviyor musunuz?).
2. Lisans eğitiminiz boyunca matematik/ matematik eğitimine yönelik en çok sevdiğiniz dersiniz hangisi idi? Neden? En sevmediğiniz dersiniz hangisi idi? Neden?
3. Uyguladığımız modeli özetler misiniz? Aşamaları hatırlıyor musunuz?
4. Uyguladığımız modelin ilk aşamasında çevre kavramını üniversite ders saatinde tartıştık.
   4.1. Öncellikle genel olarak deneyimleriniz nelerdi?

   Sizce bu tartışmanın;

   4.2. Çevre kavramına yönelik tanımlarınız üzerinde etkisi oldu mu? Olduysa ne açıdan? (Çevre kavramına tanımlarken çizdiğinizin görseller üzerinde etkisi oldu mu? Olduysa ne açıdan?).
   4.3. Çevre kavramına diğer matematiksel kavramlarla/konularla ilişkilendirmeniz üzerinde etkisi oldu mu? Olduysa ne açıdan?
   4.4. Çevre kavramına yönelik kullandığınız matematiksel dilin (terminoloji kullanımı vs.) üzerinde etkisi oldu mu? Olduysa ne açıdan?
   4.5. Çevre kavramına ile ilgili öğrenci düşünüşlerini/zorluklarını tespit etme üzerinde etkisi oldu mu? Olduysa ne açıdan?
   4.6. Çevre kavramına ile ilgili öğrenci zorluklarının üstesinden gelmek için uyguladığınız/uygulamayı planladığınız stratejilerin üzerinde etkisi oldu mu? Olduysa ne açıdan?

5. Ardından bu basamakta sınıf içinde öğretmen ve öğrenci arasında gerçekleşen bir diyalog ve bu diyaloga yöneltilen soruları incelediniz ve size yöneltilen sorular üzerinden cevaplarınızı tartıştınız.

5.1. Öncellikle genel olarak deneyimleriniz nelerdi?

Sizce bu tartışmanın;

5.2. Çevre kavramına yönelik tanımlarınız üzerinde etkisi oldu mu? Olduysa ne açıdan? (Çevre kavramına tanımlarken çizdiğinizin görseller üzerinde etkisi oldu mu? Olduysa ne açıdan?).

5.3. Çevre kavramına yönelik kullandığınız matematiksel dilin (terminoloji kullanımı vs.) üzerinde etkisi oldu mu? Olduysa ne açıdan?

5.4. Çevre kavramına yönelik kullandığınız öğretim stratejileri üzerinde etkisi oldu mu? Olduysa ne açıdan?

5.5. Çevre kavramına ile ilgili öğrenci düşünüşlerini/zorluklarını tespit etme üzerinde etkisi oldu mu? Olduysa ne açıdan?

5.6. Çevre kavramına ile ilgili öğrenci zorluklarının üstesinden gelmek için uyguladığınız/uygulamayı planladığınız stratejilerin üzerinde etkisi oldu mu? Olduysa ne açıdan?

5.7. Çevre kavramına yönelik öğrencilerin stratejilerindeki matematiksel detayları tespit etmenizde etkisi oldu mu? Olduysa ne açıdan?

5.8. Çevre kavramına yönelik öğrencilerin matematiksel anlamalarını yorumlamanızda etkisi oldu mu? Olduysa ne açıdan?

5.9. Çevre kavramına yönelik öğrencilerin matematiksel düşüncelerine karşılık vermenizde etkisi oldu mu? Olduysa ne açıdan?

(Tümü için Not: İlgili öğretmen adayının ön ve son testindeki cevaplarından örnek verilebilir).

6. Modelin ikinci aşamasında belirtilen formatta ders planını hazırlayarak öğretim provasına hazırlandınız. Ardından öğretmen adaylarının bir kısmı 5. Sınıf öğrencisi, biri de öğretmen rolüne bürünerek öğretim provası gerçekleştirdiniz.

6.1. Genel olarak ders planı hazırlama deneyiminiz nasıldı?

- (Tümü için Not: İlgili öğretmen adayının ön ve son testindeki cevaplarından örnek verilebilir).

Yine bu basamakta öğretmen adaylarının bir kısmı 5. Sınıf öğrencisi, biri de öğretmen rolüne bürünerek öğretim provası gerçekleştirdiniz.

6.2. Genel olarak bu aşamada deneyiminiz nasıldı?

Bu öğretim provasının,

6.3. Çevre kavramını tanımlamanız üzerinde etkisi oldu mu? Olduysa ne açıdan?

6.4. Çevre kavramına yönelik kullandığınız matematiksel dilin (terminoloji kullanımı vs.) üzerinde etkisi oldu mu? Olduysa ne açıdan?

6.5. Çevre kavramına yönelik kullandığınız öğretim stratejileri üzerinde etkisi oldu mu? Olduysa ne açıdan?

6.6. Çevre kavramı ile ilgili öğrenci zorluklarını tespit etme üzerinde etkisi oldu mu? Olduysa ne açıdan?

6.7. Çevre kavramı ile ilgili öğrenci zorluklarının üstesinden gelmek için uyguladığınız/uygulamayı planladığınız stratejilerin üzerinde etkisi oldu mu? Olduysa ne açıdan?

6.8. Çevre kavramına yönelik öğrencilerin stratejilerindeki matematiksel detayları tespit etmenizde etkisi oldu mu? Olduysa ne açıdan?

6.9. Çevre kavramına yönelik öğrencilerin matematiksel anlamalarını yorumlamanızda etkisi oldu mu? Olduysa ne açıdan?

6.10. Çevre kavramına yönelik öğrencilerin matematiksel düşüncelerine karşılık vermenizde etkisi oldu mu? Olduysa ne açıdan?

6.11. Bu esnada beklediğiniz ve ders esnasında karşılık verdiğiniz/ verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

6.11.1. Bu durumda nasıl hissetiniz?

6.11.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

6.12. Bu esnada beklemediğiniz ve ders esnasında karşılık veremediğiniz/ verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

6.12.1. Bu durumda nasıl hissetiniz?

6.12.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

6.13. Bu esnada beklemediğiniz ve ders esnasında karşılık verdiğiniz/ verilen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

6.13.1. Bu durumda nasıl hissetiniz?

6.13.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

6.14. Bu esnada beklemediğiniz ve ders esnasında karşılık veremediğiniz/verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

6.14.1. Bu durumda nasıl hissetiniz?

6.14.2. Benzer bir öğrenci düşünüşü ile şimdi nasıl cevap verirdiniz?

- (Tümü için Not: İlgili öğretmen adayının cevaplarından örnek verilebilir.).

7. Bir sonraki basamakta bu öğretim provasını hep birlikte değerlendirdiniz.

7.1. Genel olarak bu aşamada deneyiminiz nasıldı?

Bu değerlendirmenin;

7.2. Çevre kavramını tanımlamanız üzerinde etkisi oldu mu? Olduysa ne açıdan?
7.3. Çevre kavramına yönelik kullandığınız matematiksel dilin (terminoloji kullanımı vs.) üzerinde etkisi oldu mu? Olduysa ne açıdan?
7.4. Çevre kavramına yönelik kullandığınız öğretim stratejileri üzerinde etkisi oldu mu? Olduysa ne açıdan?
7.5. Çevre kavramı ile ilgili öğrenci zorluklarını tespit etme üzerinde etkisi oldu mu? Olduysa ne açıdan?
7.6. Çevre kavramı ile ilgili öğrenci zorluklarının üstesinden gelmek için uyguladığınız/uygulamayı planladığınız stratejilerin üzerinde etkisi oldu mu? Olduysa ne açıdan?
7.7. Çevre kavramına yönelik öğrencilerin stratejilerindeki matematiksel detayları tespit etmenizde etkisi oldu mu? Olduysa ne açıdan?
7.8. Çevre kavramına yönelik öğrencilerin matematiksel anlamalarını yorumlamanızda etkisi oldu mu? Olduysa ne açıdan?
7.9. Çevre kavramına yönelik öğrencilerin matematiksel düşüncelerine karşılık vermenizde etkisi oldu mu? Olduysa ne açıdan?
7.10. Bu esnada beklediğiniz ve ders esnasında karşılık verdiğiniz/ verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?
    7.10.1. Bu durumda nasıl hissetiniz?
    7.10.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?
7.11. Bu esnada beklemediğiniz ve ders esnasında karşılık veremediğiniz/ verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?
    7.11.1. Bu durumda nasıl hissetiniz?
    7.11.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

7.12. Bu esnada beklemediğiniz ve ders esnasında karşılık verdiğiniz/ verilen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

7.12.1. Bu durumda nasıl hissetiniz?

7.12.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

7.13. Bu esnada beklemediğiniz ve ders esnasında karşılık veremediğiniz/verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

7.13.1. Bu durumda nasıl hissetiniz?

7.13.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

8. Modelin üçüncü basamağında içinizden biri, bir ortaokulda 5. Sınıf öğrencilerine çevre konusunun kazanımlarına yönelik bir öğretim gerçekleştirdi ve diğerleri de bu sırada dersi gözlemledi.

8.1. Genel olarak bu aşamada deneyiminiz nasıldı?

Bu öğretimin;

8.2. Çevre kavramını tanımlamanız üzerinde etkisi oldu mu? Olduysa ne açıdan?

8.3. Çevre kavramına yönelik kullandığınız matematiksel dilin (terminoloji kullanımı vs.) üzerinde etkisi oldu mu? Olduysa ne açıdan?

8.4. Çevre kavramına yönelik kullandığınız öğretim stratejileri üzerinde etkisi oldu mu? Olduysa ne açıdan?

8.5. Çevre kavramı ile ilgili öğrenci zorluklarını tespit etme üzerinde etkisi oldu mu? Olduysa ne açıdan?

8.6. Çevre kavramı ile ilgili öğrenci zorluklarının üstesinden gelmek için uyguladığınız/uygulamayı planladığınız stratejilerin üzerinde etkisi oldu mu? Olduysa ne açıdan?

9. Modelin dördüncü ve son aşamasında ise çevre kavramına yönelik kayıt altına aldığımız dersi izlediniz. Bu aşamada dersin daha etkili olması adına sınıf arkadaşınıza dönütler verdiniz veya aldınız.
   9.1. Genel olarak bu aşamada deneyiminiz nasıldı?

Bu değerlendirmenin;

   9.2. Çevre kavramını tanımlamanız üzerinde etkisi oldu mu? Olduysa ne açıdan?
   9.3. Çevre kavramına yönelik kullandığınız matematiksel dilin (terminoloji kullanımı vs.) üzerinde etkisi oldu mu? Olduysa ne açıdan?
   9.4. Çevre kavramına yönelik kullandığınız öğretim stratejileri üzerinde etkisi oldu mu? Olduysa ne açıdan?
   9.5. Çevre kavramı ile ilgili öğrenci zorluklarını tespit etme üzerinde etkisi oldu mu? Olduysa ne açıdan?
   9.6. Çevre kavramı ile ilgili öğrenci zorluklarının üstesinden gelmek için uyguladığınız/uygulamayı planladığınız stratejilerin üzerinde etkisi oldu mu? Olduysa ne açıdan?
   9.7. Çevre kavramına yönelik öğrencilerin stratejilerindeki matematiksel detayları tespit etmenizde etkisi oldu mu? Olduysa ne açıdan?
   9.8. Çevre kavramına yönelik öğrencilerin matematiksel anlamalarını yorumlamanızda etkisi oldu mu? Olduysa ne açıdan?
   9.9. Çevre kavramına yönelik öğrencilerin matematiksel düşüncelerine karşılık vermenizde etkisi oldu mu? Olduysa ne açıdan?
   9.10. Bu esnada beklediğiniz ve ders esnasında karşılık verdiğiniz/ verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?
      9.10.1. Bu durumda nasıl hissetiniz?
      9.10.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?
   9.11. Bu esnada beklemediğiniz ve ders esnasında karşılık veremediğiniz/ verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

9.11.1. Bu durumda nasıl hissetiniz?

9.11.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

9.12. Bu esnada beklemediğiniz ve ders esnasında karşılık verdiğiniz/ verilen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

9.12.1. Bu durumda nasıl hissetiniz?

9.12.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

9.13. Bu esnada beklemediğiniz ve ders esnasında karşılık veremediğiniz/verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

9.13.1. Bu durumda nasıl hissetiniz?

9.13.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

10. Ders planını bir sonraki ders için model boyunca düzenlediniz (öğretmen rolünde ise). Bu düzenlemenin,

10.1. Çevre kavramını tanımlamanız üzerinde etkisi oldu mu? Olduysa ne açıdan?

10.2. Çevre kavramına yönelik kullandığınız matematiksel dilin (terminoloji kullanımı vs.) üzerinde etkisi oldu mu? Olduysa ne açıdan?

10.3. Çevre kavramına yönelik kullandığınız öğretim stratejileri üzerinde etkisi oldu mu? Olduysa ne açıdan?

10.4. Çevre kavramı ile ilgili öğrenci zorluklarını tespit etme üzerinde etkisi oldu mu? Olduysa ne açıdan?

10.5. Çevre kavramı ile ilgili öğrenci zorluklarının üstesinden gelmek için uyguladığınız/uygulamayı planladığınız stratejilerin üzerinde etkisi oldu mu? Olduysa ne açıdan?

10.6. Çevre kavramına yönelik öğrencilerin stratejilerindeki matematiksel detayları tespit etmenizde etkisi oldu mu? Olduysa ne açıdan?

10.7. Çevre kavramına yönelik öğrencilerin matematiksel anlamalarını yorumlamanızda etkisi oldu mu? Olduysa ne açıdan?

10.8. Çevre kavramına yönelik öğrencilerin matematiksel düşüncelerine karşılık vermenizde etkisi oldu mu? Olduysa ne açıdan?

10.9. Bu esnada beklediğiniz ve ders esnasında karşılık verdiğiniz/ verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

10.9.1. Bu durumda nasıl hissetiniz?

10.9.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

10.10. Bu esnada beklemediğiniz ve ders esnasında karşılık veremediğiniz/ verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

10.10.1. Bu durumda nasıl hissetiniz?

10.10.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

10.11. Bu esnada beklemediğiniz ve ders esnasında karşılık verdiğiniz/ verilen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

10.11.1. Bu durumda nasıl hissetiniz?

10.11.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

10.12. Bu esnada beklemediğiniz ve ders esnasında karşılık veremediğiniz/verilmeyen bir öğrenci düşünüşü ile karşılaştınız mı? Örnek verebilir misiniz?

10.12.1. Bu durumda nasıl hissetiniz?

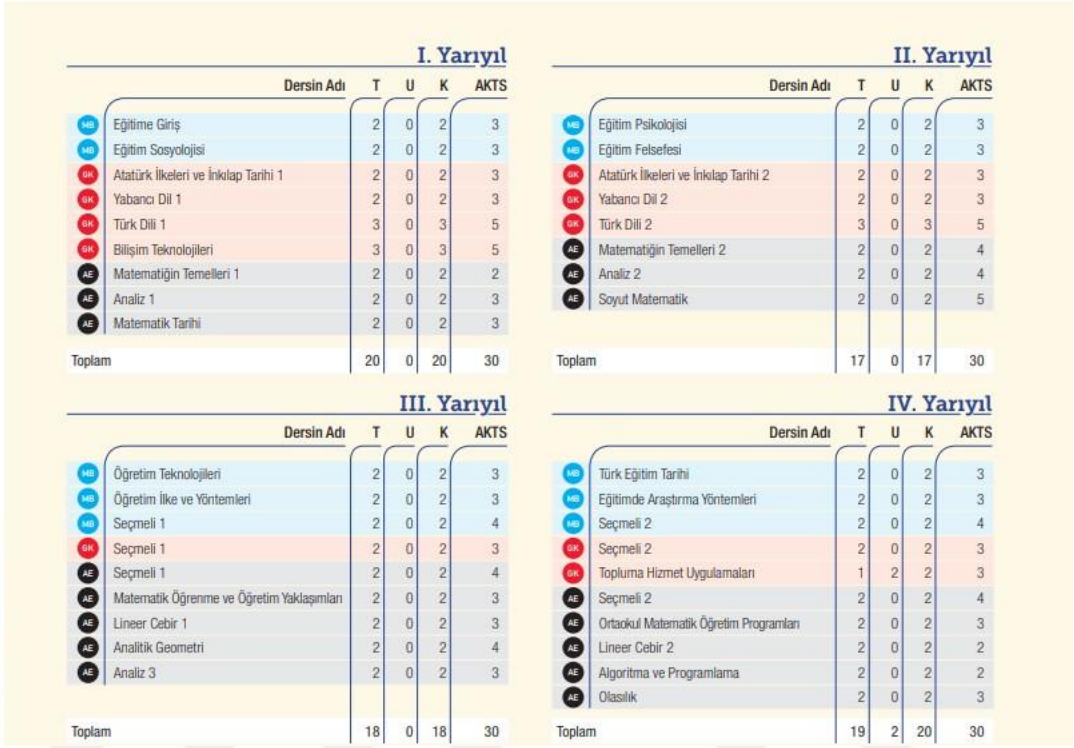
10.12.2. Benzer bir öğrenci düşünüşü ile şimdi karşılaşsanız nasıl cevap verirdiniz?

11. Tüm bu süreci göz önünde bulundurduğunuzda;

11.1. Üniversite hocanızın gelişiminize katkısı olduğunu düşünüyor musunuz? Olduysa hangi açılardan ve nasıl?

11.2. Sınıf arkadaşlarınızın gelişiminize katkısı olduğunu düşünüyor musunuz? Olduysa hangi açılardan ve nasıl?

B. The courses offered by the Higher Education Council in 2018.



I. Yarıyıl

	Dersin Adı	T	U	K	AKTS
MB	Eğitime Giriş	2	0	2	3
MB	Eğitim Sosyolojisi	2	0	2	3
GK	Atatürk İlkeleri ve İnkılap Tarihi 1	2	0	2	3
GK	Yabancı Dil 1	2	0	2	3
GK	Türk Dili 1	3	0	3	5
GK	Bilişim Teknolojileri	3	0	3	5
AE	Matematiğin Temelleri 1	2	0	2	2
AE	Analiz 1	2	0	2	3
AE	Matematik Tarihi	2	0	2	3
Toplam		20	0	20	30

II. Yarıyıl

	Dersin Adı	T	U	K	AKTS
MB	Eğitim Psikolojisi	2	0	2	3
MB	Eğitim Felsefesi	2	0	2	3
GK	Atatürk İlkeleri ve İnkılap Tarihi 2	2	0	2	3
GK	Yabancı Dil 2	2	0	2	3
GK	Türk Dili 2	3	0	3	5
AE	Matematiğin Temelleri 2	2	0	2	4
AE	Analiz 2	2	0	2	4
AE	Soyut Matematik	2	0	2	5
Toplam		17	0	17	30

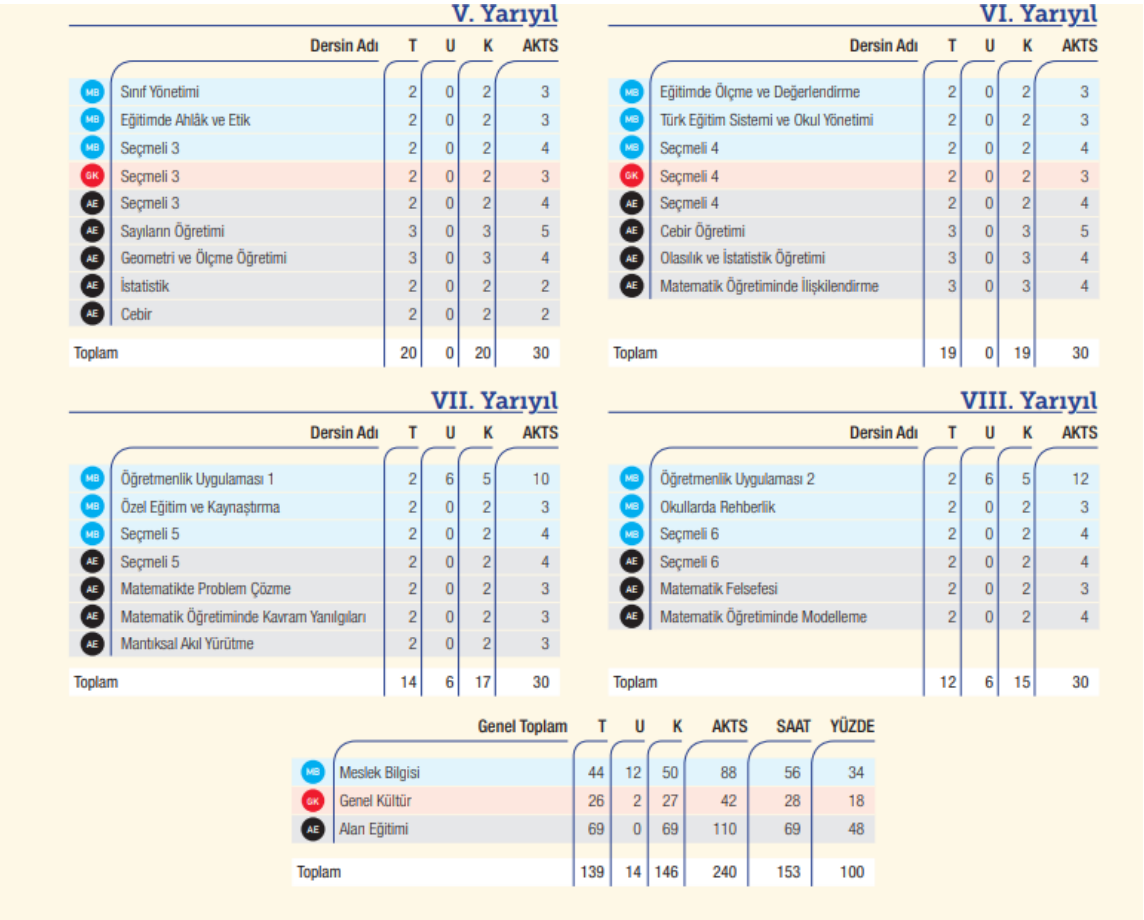
III. Yarıyıl

	Dersin Adı	T	U	K	AKTS
MB	Öğretim Teknolojileri	2	0	2	3
MB	Öğretim İlke ve Yöntemleri	2	0	2	3
MB	Seçmeli 1	2	0	2	4
GK	Seçmeli 1	2	0	2	3
AE	Seçmeli 1	2	0	2	4
AE	Matematik Öğrenme ve Öğretim Yaklaşımları	2	0	2	3
AE	Lineer Cebir 1	2	0	2	3
AE	Analitik Geometri	2	0	2	4
AE	Analiz 3	2	0	2	3
Toplam		18	0	18	30

IV. Yarıyıl

	Dersin Adı	T	U	K	AKTS
MB	Türk Eğitim Tarihi	2	0	2	3
MB	Eğitimde Araştırma Yöntemleri	2	0	2	3
MB	Seçmeli 2	2	0	2	4
GK	Seçmeli 2	2	0	2	3
GK	Toplumsal Hizmet Uygulamaları	1	2	2	3
AE	Seçmeli 2	2	0	2	4
AE	Ortaokul Matematik Öğretim Programları	2	0	2	3
AE	Lineer Cebir 2	2	0	2	2
AE	Algoritma ve Programlama	2	0	2	2
AE	Olasılık	2	0	2	3
Toplam		19	2	20	30

Figure A.1. The courses offered by the Higher Education Council in 2018.



V. Yarıyıl

	Dersin Adı	T	U	K	AKTS
MB	Sınıf Yönetimi	2	0	2	3
MB	Eğitimde Ahlâk ve Etik	2	0	2	3
MB	Seçmeli 3	2	0	2	4
GK	Seçmeli 3	2	0	2	3
AE	Seçmeli 3	2	0	2	4
AE	Sayıların Öğretimi	3	0	3	5
AE	Geometri ve Ölçme Öğretimi	3	0	3	4
AE	İstatistik	2	0	2	2
AE	Cebir	2	0	2	2
Toplam		20	0	20	30

VI. Yarıyıl

	Dersin Adı	T	U	K	AKTS
MB	Eğitimde Ölçme ve Değerlendirme	2	0	2	3
MB	Türk Eğitim Sistemi ve Okul Yönetimi	2	0	2	3
MB	Seçmeli 4	2	0	2	4
GK	Seçmeli 4	2	0	2	3
AE	Seçmeli 4	2	0	2	4
AE	Cebir Öğretimi	3	0	3	5
AE	Olasılık ve İstatistik Öğretimi	3	0	3	4
AE	Matematik Öğretiminde İlişkilendirme	3	0	3	4
Toplam		19	0	19	30

VII. Yarıyıl

	Dersin Adı	T	U	K	AKTS
MB	Öğretmenlik Uygulaması 1	2	6	5	10
MB	Özel Eğitim ve Kaynaştırma	2	0	2	3
MB	Seçmeli 5	2	0	2	4
AE	Seçmeli 5	2	0	2	4
AE	Matematikte Problem Çözme	2	0	2	3
AE	Matematik Öğretiminde Kavram Yanılgıları	2	0	2	3
AE	Mantıksal Akıl Yürütme	2	0	2	3
Toplam		14	6	17	30

VIII. Yarıyıl

	Dersin Adı	T	U	K	AKTS
MB	Öğretmenlik Uygulaması 2	2	6	5	12
MB	Okullarda Rehberlik	2	0	2	3
MB	Seçmeli 6	2	0	2	4
AE	Seçmeli 6	2	0	2	4
AE	Matematik Felsefesi	2	0	2	3
AE	Matematik Öğretiminde Modelleme	2	0	2	4
Toplam		12	6	15	30

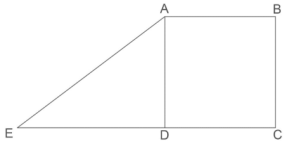
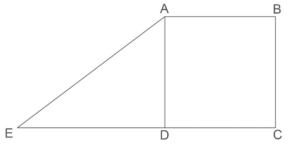
	Genel Toplam	T	U	K	AKTS	SAAT	YÜZDE
MB	Meslek Bilgisi	44	12	50	88	56	34
GK	Genel Kültür	26	2	27	42	28	18
AE	Alan Eğitimi	69	0	69	110	69	48
Toplam		139	14	146	240	153	100

Figure A.1. The courses offered by the Higher Education Council in 2018 (Continued).

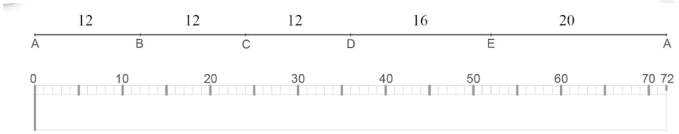
C. Table of specification of the items in the pre-and post-test based on the used frameworks.

Questions in Pre/Post Test	Evidence of the Foundation Dimension of the KQ Framework (Rowland et al., 2005; 2009).	Dimensions of Professional Noticing Framework (Jacobs et al., 2010).
1. Define the concept of the perimeter in detail.	*Evidence of overt subject knowledge* and *Evidence of using terminology*	
2. Describe the perimeter measurement in detail.	*Evidence of overt subject knowledge* and *Evidence of using terminology*	
3. What information do you need to calculate the perimeter of any shape? Please explain.	*Evidence of overt subject knowledge* and *Evidence of using terminology*	
4. How would you describe the perimeter to middle school students? Does your definition change according to the grade level? Please explain.	*Evidence of the theoretical underpinning of pedagogy* and *Evidence of using terminology*	
5. Which representation(s) (Table, model, diagram, picture, verbal problem, real-life situations, concrete materials, etc.) do you prefer to use in teaching the perimeter concept? How? Please explain.	*Evidence of the theoretical underpinning of pedagogy*	
c. Do you prefer to use technology use in teaching the perimeter concept? If your answer is yes, How? Please explain.	*Evidence of the theoretical underpinning of pedagogy*	
6. What could be middle school students' potential difficulties with the perimeter concept? Please explain in detail.	*Evidence of identifying errors*	
b. What could be the source of these difficulties? Please explain.	*Evidence of identifying errors*	

(Continued).

Questions in Pre/Post Test	Evidence of the Foundation Dimension of the KQ Framework (Rowland et al., 2005; 2009).	Dimensions of Professional Noticing Framework (Jacobs et al., 2010).
d.What could you do to overcome these difficulties? Please explain.	*Evidence of identifying errors* and *Evidence of the theoretical underpinning of pedagogy*	
7. ABCD is a square, and ADE is a right triangle. Since the perimeter of triangle ADE is 48 cm, IECI=28 cm, and IABI=12, calculate the perimeter of trapezoid ABCE as much as possible ways. 	*Evidence of overt subject knowledge* and *Evidence of using terminology*	
8. Let us imagine that you are teaching the concept of the perimeter to 5th grades, and you started the lesson by asking the following question: ABCD is a square, and ADE is a right triangle. Since the perimeter of triangle ADE is 48 cm, IECI = 28 cm, and IABI = 12, calculate the perimeter of trapezoid ABCE. 		

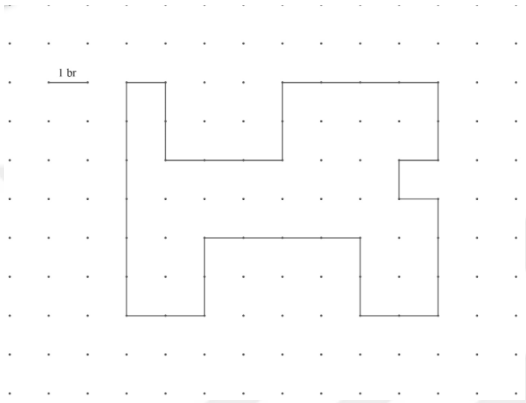
(Continued).

	Questions in Pre/Post Test	Evidence of the Foundation Dimension of the KQ Framework (Rowland et al., 2005; 2009).	Dimensions of Professional Noticing Framework (Jacobs et al., 2010).
	 While solving this question, your student took the ruler on the board and made some measurements.. Meanwhile, the following image appeared on the board. She said: "I found it in a very different way. My result is 72."		
d.	Explain how the student solved this problem (her strategy, etc.). Do you think her solution is correct? Why?	*Evidence of overt subject knowledge*, *Evidence of using terminology*, and *Evidence of identifying errors*	*Evidence of attending skill*
e.	Based on her approach to the question, what can you say about her mathematical understanding of the perimeter concept? Explain in detail.	*Evidence of overt subject knowledge*, *Evidence of use of terminology*, and *Evidence of identifying errors*	*Evidence of interpreting skill*
f.	What kind of path would you follow after the student's solution? What questions or questions would you ask her? Why? Figure out how you would continue the lesson in detail.	*Evidence of theoretical underpinnings of pedagogy*	*Evidence of deciding skill*

D. Activity 1: What is the Perimeter? Let Us Look at the Concept through Your and the Student's Eyes.

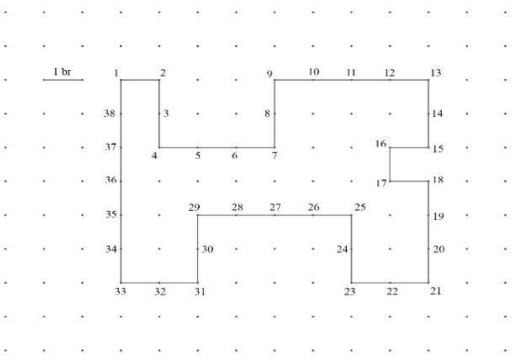
Activity 1: *What is the Perimeter? Let Us Look at the Concept Through Your and the Student's Eyes.*

1. Draw shapes that represent the perimeter concept.
2. Draw shapes that do not represent the perimeter concept.



3. Calculate the perimeter of the given shape as much as possible. Please indicate your solutions ways. Then, share your solution ways with your friends.
4. What mistakes do students make when solving the above question? What are the possible sources of them? Please think individually and then share your thoughts with your friends.
5. How do you handle such mistakes or prevent them from occurring such mistakes during the instruction? Please think individually and then share your thoughts with your friends.

E. Activity 2: Let us Examine the Dialogue

Activity 2: *Let Us Examine the Dialogue*	
Representation of Practice for the Perimeter. *Student:* Teacher, the answer is 38. *Teacher:* How did you calculate Aleyna? Can you explain? *Student:* Teacher, I chose a reference point for myself, so I started *from one point and reached that point while passing 38 points.* *Teacher:* Can you show your solution on the board? *Student:* Of course, my teacher. I counted the points to calculate the shape's perimeter and got 38.	Prompts regarding the Dialogues. a. What do you think about the student's first strategy? Do you think it is true?
 	b. What do you think about the student's second strategy? Do you think it is true?
Teacher: How do we calculate the perimeter of the shape, Aleyna? *Student:* We count the dots around the shape. Specifically, we select a starting point around the shape and count the points until we reach the same point, teacher. *Teacher:* So, do we count the points when calculating the perimeter, Aleyna? *Student*: Yes, teacher. *Teacher:* How would you calculate if I did not place the dotted paper under the shape and asked you this question?	c. What do you think about the students' mathematical understanding of the perimeter concepts? D. Why do you think the teacher asked the student the second question? e. If you were the student's teacher, how would you continue the lesson? Which questions could you ask the student to advance her learning of the concept?
16 cm  4 cm *Student:* (16 + 4) x 2 = 40 cm teacher. *Teacher:* How did you calculate? *Student:* This shape is rectangular. To calculate the perimeter of the rectangle, I use the formula (a + b) x 2.	

CURRICULUM VITAE

Surname, Name: Dışbudak Kuru, Özge

EDUCATION

Degree	Institution	Year of Graduation
MS	METU Elementary Mathematics Education	2017
BS	METU Elementary Mathematics Education	2014
High School	Savaştepe Anatolian Teacher High School	2008

WORKING EXPERIENCE

2015-2017 Mathematics Teacher-Beyciler Ortaokulu DÜZCE

2017-Present Research Assistant-Istanbul Medeniyet University/Faculty of Educational Sciences/ Mathematics and Science Department

FOREIGN LANGUAGES

Advanced English, Beginner Italian.

PUBLICATIONS

1. Dışbudak Ö., & Akyuz, D. (2018) Exploring Fifth Grade Students' Perceptions on Quadrilaterals. ICES-UEBK. Side-Türkiye.

2. Disbudak, O., & Akyuz, D. (2019). The Comparative Effects of Concrete Manipulates and Dynamic Software on the Geometry Achievement of Fifth-Grade Students. *International journal for technology in mathematics education*, *26*(1).

3. Dışbudak-Kuru, Ö., Toker, H. Ozdogan, Z. & Sevinc, S. (2020). Subjectivity in Participant-and Researcher-Generated Data in Qualitative Research. The Qualitative Report Conference. Fort Laudardale-USA

4. Dışbudak-Kuru, Ö., & Işıksal-Bostan, M. (2021). Öğretmen Adaylarının Matematiksel İlişkilendirme Bağlamında Kullandıkları Gösterim Şekilleri Seçimlerinin İncelenmesi. 14.Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi. Burdur-Türkiye.

5. Dışbudak-Kuru, Ö., Ucuzoğlu, A. N., Işıksal-Bostan, M., Yemen-Karpuzcu, S., & Tekin-Sitrava, R (2022). Ortaokul Matematik Öğretmenlerinin Mesleki Fark Etme Becerileri: Dikdörtgenler Prizmasının Hacmine İlişkin Problem Durumu. *Mersin Üniversitesi Eğitim Fakültesi Dergisi*, *18*(2), 154-174.

6. Dışbudak-Kuru, Ö., Ucuzoğlu A.N., Işıksal-Bostan, M., Yemen-Karpuzcu, S. & Tekin-Sitrava, R. (2021). Ortaokul Matematik Öğretmenlerinin Öğrencilerin Matematiksel Düşünüşüne Yönelik Mesleki Fark Etme Becerilerinin Belirlenmesi: Dikdörtgenler Prizması Örneği. 5. Uluslararası Türk Bilgisayar ve Matematik Eğitimi (TÜRKBİLMAT-5) Sempozyumu. Alanya-Türkiye.

7. Dışbudak-Kuru, Ö., & Işıksal-Bostan, M. (2023). Supporting the development of preservice teachers' mathematical connection skills in a teacher education programme. *International Journal of Mathematical Education in Science and Technology*, 1-27.

8. Yemen-Karpuzcu, S., Dışbudak-Kuru, Ö., Işıksal-Bostan, M., & Tekin-Sitrava, R. (2023). Design of an Online Professional Development Platform for Mathematics Teachers' Noticing of Students' Mathematical Thinking. the 13th Congress of the European Society for Research in Mathematics Education (CERME13). Budapest-Hungary.

SCIENTIFIC PROJECT

Project Title: Investigation of Middle School Mathematics Teachers' Professional Noticing Skills of Students' Mathematical Thinking and Transferring of these Experiences into Learning Environment

Project Length: 24 months (2020-2022)

Funding Institution: The Scientific and Technological Research Council of Türkiye

Project Leader: Prof. Dr. Mine Işıksal-Bostan

Role in the Project: Researcher

FELLOWSHIPS AND AWARDS

1. High Honor Roll (B.S.) – Middle East Technical University, Turkey, 2014.

2. Performance Award (The most successful student in the Ph.D. program of the Department of Elementary Education) Middle East Technical University, Türkiye, 2018-2019.

3. 2224-A Grant Program for Participation in Scientific Meetings Abroad Fellowship, The Scientific and Technological Research Council of Türkiye, 2019.

4. 1003 Primary Subjects R&D Funding Program Fellowship, The Scientific and Technological Research Council of Türkiye, 2020-2022.

5. 2250 - Performance-Based Scholarships Programme for Ph.D. and Post-Doc Scholars Fellowship, The Scientific and Technological Research Council of Türkiye, 2022.