JOINT DISTRIBUTIONS OF THE MAXIMUM DRAWDOWN AND MAXIMUM DRAWUP IN LÉVY PROCESSES

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ABSTRACT

JOINT DISTRIBUTIONS OF THE MAXIMUM DRAWDOWN AND MAXIMUM DRAWUP IN LÉVY PROCESSES

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The maximum drawdown and maximum drawup variables in a certain time period have attracted the attention of investors and have been used as dynamic risk measures. This thesis aims to derive the joint probability distributions of the maximum drawdown and maximum drawup amounts under a Lévy process framework. For the first time, the joint distributions are studied under the Lévy process model. Establishing the relationship between these variables through joint distributions enables the estimation of one variable given the other. The Doob-h transforms of the process from infimum to supremum and post-supremum process have been previously obtained in the literature via path decomposition conditioned on the extreme values of the Lévy process. However, characterizing the process up to the infimum and determining the distribution of the maximum drawdown within this region remains an open research question. This thesis addresses this gap by introducing novel methods, revealing how the maximum drawdown and maximum drawup vary with each other.

Keywords: Lévy Process, Maximum Drawdown, Maximum Drawup, Path Decomposition, Doob-h Transform, etc.

ÖZ

LÉVY SÜREÇLERİNDE EN BÜYÜK DÜŞÜŞ VE EN BÜYÜK ARTIŞ DEĞERLERİNİN BİRLİKTE DAĞILIMLARI

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Belirli bir zaman dilimindeki maksimum düşüş ve maksimum yükseliş değişkenleri, yatırımcıların ilgisini çekmiş ve dinamik risk ölçütleri olarak kullanılmıştır. Bu tez, Lévy süreçleri çerçevesinde maksimum düşüş ve maksimum yükseliş miktarlarının ortak olasılık dağılımlarını türetmeyi amaçlamaktadır. İlk kez, ortak dağılımlar Lévy süreç modeli altında incelenmiştir. Bu değişkenler arasındaki ilişkiyi ortak dağılımlar yoluyla belirlemek, bir değişkenin diğerine bağlı olarak tahmin edilmesini mümkün kılmaktadır. İnfimumdan supremuma ve supremum sonrasına kadar olan sürecin Doob-h dönüşümleri, literatürde Lévy sürecinin ekstrem değerlerine koşullu yol ayrıştırmalarıyla daha önce elde edilmiştir. Ancak, infimuma kadar olan sürecin karakterize edilmesi ve bu bölgede maksimum düşüşün dağılımının belirlenmesi açık bir araştırma sorusu olarak kalmıştır. Bu tez, yeni yöntemler geliştirerek bu boşluğu doldurmakta ve maksimum düşüş ile maksimum yükselişin birbiriyle nasıl etkileştiğini ortaya koymaktadır.

Anahtar Kelimeler: Lévy Süreçleri, Maksimum Düşüş, Maksimum Yükseliş, vd.

To my lovely family

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LIST OF ABBREVIATIONS

MDD	Maximum Drawdown
MDU	Maximum Drawup
RCLL	Right Continuous with Left Limit
SNLP	Spectrally Negative Lévy Processes
CED	Conditional Expected Drawdown

CHAPTER 1

INTRODUCTION

The decreases and increases in the financial markets are of great importance for investors. The difference between the historical maximum of a process and its current value is called drawdown. Similarly, the difference between the current value and the historical minimum value of a process is called the drawup. The maximum drawdown (MDD) and maximum drawup (MDU) variables in a certain time period have attracted the attention of investors and have been used as dynamic risk measures. They can be viewed as measures of relative regret and satisfaction, respectively. According to these measures, an investor may have an idea to change his position in his portfolio. Therefore, the maximum drawdown and maximum drawup variables of asset prices in finite or infinite time intervals have many application areas in finance. To illustrate, the investor may use maximum drawdown and drawup analysis to manage risk and return of his portfolio, decide which asset to invest and when, and determine the most appropriate dividend patterns. In addition, risk measures based on Maximum Drawdown and Maximum Drawup have been extensively used in recent studies and can serve as an alternative to the commonly used Value-at-Risk. In terms of a comprehensive risk assessment and pricing, not only the values of the biggest decrease and the largest increase, but also how long the decrease and increase will last are among research topics. Interest in Maximum Drawdown and Maximum Drawup has increased due to the recent financial crisis because large drawdowns and drawdowns can bring portfolio losses and liquidity imbalances. Companies make studies according to the maximum decrease and maximum increase values in the pricing of reserve insurances and in futures transactions aimed at reducing currency risk. They were willing to seek insurance against drawdowns and interest in hedging products

has increased. Therefore, another application area of Maximum Drawdown and Maximum Drawup is the fair pricing of derivative instruments, such as Russian Options, Crash Options, and Rally Options.

The fair price of a Russian Option, in which the option holder has the right to hold the option indefinitely, exercise the contract at any time, and, upon executing the contract, receive either the current price or the maximum price at which the option ever traded in the past, has been founded with the help of the maximum drawdown.

Moreover, the maximum drawdown itself can also be viewed as a contingent claim, and one can price and hedge it accordingly as a derivative contract. Trading drawdown contracts or replicating them by hedging would directly address the concerns of portfolio managers who would like to insure the market drops. Similar contracts can be written on the maximum drawup. More complex products, such as plain vanilla options on the maximum drawdown or drawup, or related barrier options on Maximum Drawdown and Maximum Drawup , which are called Crash and Rally Options, respectively, are also used for portfolio optimization. The reason why these options are needed is that currently traded contracts have only a limited ability to insure market drawdowns. For instance, deep out-of-the-money puts are optimal to buy when the market reaches its maximum, otherwise they lose their value. This happens when the market goes up before it drops down. Thus, timing is very important for put options. On the other hand, maximum drawdown is not sensitive to timing, the contract delivers the largest drop no matter when this drop happens. The maximum drawdown does not lose value when the underlying asset is resetting its new maximum as opposed to put options.

In finance, assets such as stocks in the market are generally modeled with a geometric Brownian motion (Black-Scholes model). But empirical study of financial data reveals the fact that in disaster situations like the financial crisis, stocks can suddenly fall or increase more than expected. The distribution of the log-return of the stock price exhibits features which cannot be captured by the normal distribution, such as heavy tails and asymmetry. For this reason, real data diverge from the Geometric Brownian Motion (Black-Scholes) model. Lévy processes are used to model such jump states. Lévy processes are processes whose increments in different intervals

have the property of stationarity and independence. If the price sequences show self-similarity, it can also represent this property by choosing its parameters. Although the Lévy model does not provide long-term dependence, it includes jumps that correspond to spikes and decreases in real-price data. Another feature is that it can capture the heavy-tailed right-sided distributions seen in the data, since they are not symmetric like Gaussian processes. In addition, since it is a Markov process, it provides convenience in theoretical calculations.

In this thesis, the aim is to find the joint probability distributions of the maximum drawdown/loss and the maximum drawup/gain under a Lévy process. For the first time, the joint distributions will be studied under the Lévy process model. Determining the relationship between the variables with joint distributions will enable the estimation of a variable given the other. Under the condition that infimum occurs before supremum, the Doob-h transforms of the process from infimum to supremum and post-supremum processes have been obtained in the literature by performing a path decomposition given the extreme values of the Lévy process. However, characterizing the process up to the infimum and finding the distribution of the maximum drawdown in this part is left as an open research question. The thesis will fill this gap by developing new methods, revealing how the maximum drawdown and drawup vary with each other. For the Brownian Motion, it is enough to characterize just the case infimum comes before supremum to find the joint distribution. However, for the Lévy processes, we need to characterize the case supremum comes before infimum to find the joint distribution. In this thesis, we will characterize the pre-supremum, intermediate, and post-infimum regions when supremum comes before infimum.

In the second chapter, the literature review is presented in three separate parts: existing results in the literature on the Lévy process drawdown, existing results on the time of drawdowns, and existing results on the variable Maximum Drawdown . In the third chapter, the mathematical foundations necessary to conduct these studies, such as the properties of Lévy processes, the Doob-h transform, and Scale functions, are provided. Chapter 4 develops the main methodology and outlines the mathematical tools employed in solving the stated research problems. In Chapter 5, the distributions of pre/post infimum and pre/post supremum parts, the distributions of the parts of the path will then be found under the condition that the supremum precedes the

infimum and the infimum precedes the supremum with existing and novel results. Chapter 6 presents the calculation of the maximum drawdowns in the case where the infimum precedes the supremum, and the maximum drawups in the case where the supremum precedes the infimum. In Chapter 7, the joint distribution will be derived by calculating and summing up the probabilities of disjoint events that determine the values that the maximum drawdown and drawup take together. Chapter 8 concludes the thesis by summarizing the main findings and discussing potential directions for future research.

CHAPTER 2

LITERATURE REVIEW

2.1 Existing Results in the Literature on the Drawdown Process of Levy Processes

The drawdown process is closely monitored over time by portfolio managers to determine and mitigate investment risk. [21] studied portfolio selection problems under certain constraints related to the drawdown process. [18] extended this work to multi-asset models. In their studies, [15] defined risk measures constructed using the drawdown variable for asset prices and applied these new risk measures in portfolio optimization. The drawdown process is often defined as a risk measure, and using the drawdown process, performance measures such as the Calmar ratio, Sterling ratio, and Burke ratio have been introduced to compare the performance of stocks [54].

[46] introduced a new Greek variable dependent on the drawdown variable, which was used to measure the sensitivity of an investment. [12] developed a European-style digital drawdown insurance contract and proposed its use in futures transactions to reduce the exchange rate risk of insurance claims. There are many more studies on drawdown insurance available in the literature such as ([44], [33], [34], [9], [12]). The Russian option, constructed based on the supremum values of its underlying asset, was defined in [55]. For this put option, the optimal stopping time T^* is found as the first time the drawdown exceeds a certain level k^*.

In their article, [3] determined the value function of the Russian option under specific Lévy process models by finding the distribution of the first passage time of the reflected process (the drawdown). [6] introduced a new scaling function Z for spec-

trally negative Lévy processes to determine the Laplace transform of the time to ruin. [11] generalized similar results to spectrally negative Markov additive processes.

By defining the W and Z scale functions and utilizing first passage times, problems related to drawdown process applications have been effectively solved in terms of these functions ([7], [25], [34], [2], [8]). The economic deficit caused by drawdowns has been analyzed in ([39], [34], [37]).

Another interesting application of the drawdown variable is the scenario in which the company's reserves fall below zero, requiring recovery from bankruptcy. Additionally, cases where taxes are levied, or dividends are distributed when the drawdown surpasses a certain threshold b, have been studied in ([32], [49]).

A recently introduced and intriguing risk model in the literature focuses on regime switching in investments whenever the drawdown exceeds a stress signal level d [34]. One of the key findings for this risk model relates to the severity of ruin for the process taken until generalized drawdown occurs ([37], [34]).

Furthermore, let τ be a random variable with an exponential distribution independent of the Lévy process. Define $G_\tau = \sup\{t < \tau : X_t = S_t\}$, which represents the last time the process X_t reaches its supremum before τ ([10], [30]).

The joint distributions of the variables $G_\tau, \tau - G_\tau, S_\tau$, and $S_\tau - X_\tau$ are determined as follows:

i) (G_τ, S_τ) and $(\tau - G_\tau, S_\tau - X_\tau)$ are independent.

ii) For all $\alpha, \beta > 0$ and $q > 0$,

$$\begin{aligned} &E\left(\exp\left\{-\alpha G_\tau - \beta S_\tau\right\}\right) \\ &= \exp\left(\int_0^\infty dt \int_{[0,\infty)} \left(e^{-\alpha t - \beta x} - 1\right) t^{-1} e^{-qt} P\left(X_t \in dx\right)\right), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &E\left(\exp\left\{-\alpha\left(\tau - G_\tau\right) - \beta\left(S_\tau - X_\tau\right)\right\}\right) \\ &= \exp\left(\int_0^\infty dt \int_{(-\infty,0)} \left(e^{-\alpha t + \beta x} - 1\right) t^{-1} e^{-qt} P\left(X_t \in dx\right)\right). \end{aligned} \tag{2.2}$$

For a spectrally negative Lévy process, [39] found the joint distribution of the following 6 variables :

1. The first passage time σ_a, at which the drawdown process first exceeds a level $a > 0$,
2. The last time before σ_a when the supremum was reached, $\bar{G}_{\sigma_a}$,
3. The infimum at σ_a,
4. The supremum at σ_a,
5. The undershoot, $a - X_{\sigma_{a-}}$,
6. The overshoot, $X_{\sigma_a} - a$.

2.2 Existing Results on the Duration of Drawdowns

From a risk management perspective, the value of the drawdown process alone is insufficient for comprehensive risk assessment. The frequency and duration of drawdowns are also critical factors in evaluating risk. Recent studies have started exploring the duration of drawdown trends, i.e., the speed at which drawdowns occur in stochastic processes.

The work of [48] investigates the distribution of a drawdown duration variable by defining it as the difference between the current time t and the last time the Lévy process achieved its global maximum up to t. Mathematically, for a Lévy process observed up to time $t > 0$, let

$$G_t := \sup\{0 \leq s \leq t : D_s = 0\} \tag{2.3}$$

denote the last time the drawdown process D reached the level 0. The study defines the drawdown duration as $t - G_t$, which measures the time difference between t and the last occurrence of the process's maximum.

In this study, the variable

$$\eta_b = \inf\{t \geq b : t - G_t \geq b\} \tag{2.4}$$

represents the first time the drawdown duration exceeds a threshold b. Its Laplace transform is derived as:

$$\mathrm{E}\left[e^{-q\eta_b^\epsilon}\right] = \frac{e^{-q\bar{b}} f_\epsilon^{(0)}(b)}{1 + E\left[e^{-q\tau_\varepsilon}\right] + f_\epsilon^{(0)}(b)}, \tag{2.5}$$

where η_b^ϵ is an approximation of η_b, constructed using only stopping times of excursions with heights exceeding ϵ.

Additionally, the auxiliary function $f_\epsilon^{(p)}(t)$ is defined for $p \geq 0$ as:

$$f_\epsilon^{(p)}(t) = \int_{[\varepsilon,\infty)} E\left[e^{-q\tau_\varepsilon} 1_{\{D_{\tau_\varepsilon} \in dy\}}\right] P\{S_{e_p \wedge t} \leq y\}. \tag{2.6}$$

For Brownian motion, the frequency of drawdowns has been determined by defining two types of drawdown time sequences, depending on whether the process revisits its previous maximum, as described in [35].

In [65], the Laplace transforms of the last time the maximum was reached before a drawdown, the duration of the drawdown, and the maximum value of the process are derived for general diffusion processes. The application of these results to pricing insurance claims based on drawdowns and drawdown durations is also discussed. The importance of drawdown durations extends to agreements designed to protect buyers or sellers from adverse conditions.

Mathematically, let $(\Omega, \mathcal{H}, \mathcal{P})$ represent the probability space and $\mathcal{F}$ the filtration under which X is a one-dimensional linear diffusion process. Assume W_t is a standard Brownian motion and $\mu(.)$ and $\sigma(.)$ are real-valued continuous functions. The model for asset prices is given as:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x. \tag{2.7}$$

Let D_t represent the drawdown process. Define the first time D_t reaches a certain level K as:

$$\tau_K^D = \inf\{t \geq 0 : D_t = K\}. \tag{2.8}$$

Here, τ_K^D is a stopping time. Before this stopping time, the last time the process achieves its maximum is denoted by:

$$\rho = \sup\{t \in [0, \tau_K^D] : S_t = X_t\}, \tag{2.9}$$

where $\mathcal{S} := \tau_K^D - \rho$ represents the duration of the drawdown. The variables $\rho, \mathcal{S}$, and X_ρ are analyzed and their Laplace transforms are derived as follows:

$$E_x \exp\left(-\alpha\rho - \beta\mathcal{S} - \gamma X_\rho\right) = \int_x^b \frac{e^{-x-s'(m)}}{g_\rho(m-K)h_\rho(m) - g_\rho(m)h_\rho(m-K)}$$
$$\cdot \exp\left(-\int_x^m \frac{g_\theta(u-K)h_\rho(u) - g_\theta(u)h_u(u-K)}{g_\theta(u-K)h_\theta(u) - g_a(u)h_Q(u-K)}du\right)dm, \quad \forall x \in (a,b),$$

where:

- $s(.)$ is the scale function of the diffusion process,
- $g_\lambda(.)$ and $h_\lambda(.)$ ($\lambda = \alpha, \beta$) are two independent solutions to the differential equation:
$$\frac{1}{2}\sigma^2(u)f'' + \mu(u)f' = \lambda f, \tag{2.10}$$
- The Wronskian of these solutions satisfies:
$$g_\lambda(u)h'_\lambda(u) - g'_\lambda(u)h_\lambda(u) = s'(u), \quad \forall u \in (a,b). \tag{2.11}$$

In the first study, the duration of drawdown is defined as the difference between time t and the last time the Lévy process reached its maximum up to t. The distribution of this duration was derived. The second study, however, focuses on diffusion processes and provides the joint distributions of the last time the process reached its maximum before the drawdown, the duration of the drawdown, and the maximum value of the process.

In [61], instead of considering an independent exponential time, the process up to α_d is analyzed, and the portion of the process from κ_{α_d} to α_d is examined. The Laplace transform of the drawdown duration is found as follows.

Proposition 2.1. *Given $\bar{X}_{\alpha_d} = m$, the process after κ_{α_d} until $\alpha_d < \tau_m^+$ is an h-transform of the spectrally negative Lévy process killed at time α_d, where $h(x)$ is given by:*

$$h(x) = 1 - e^{(m-x)\frac{W'(d)}{W(d)}} \quad \textit{for } x \leq m. \tag{2.12}$$

Furthermore, the Laplace transform of the drawdown duration $\mathcal{T}^d$ is:

$$\mathbb{E}\left[e^{-\gamma\mathcal{T}^d}\right] = \frac{W(d)}{W'(d)} \cdot \frac{Z^{(\gamma)}(d)W^{(\gamma)'}(d) - \gamma\left(W^{(\gamma)}(d)\right)^2}{W^{(\gamma)}(d)}, \tag{2.13}$$

where

- $W(d)$ *is the scale function of the Lévy process,*
- $Z^{(\gamma)}(d)$ *and* $W^{(\gamma)}(d)$ *are the scale functions associated with the spectrally negative Lévy process and the parameter* γ*,*
- $W^{(\gamma)'}(d)$ *denotes the derivative of* $W^{(\gamma)}(d)$ *with respect to* d*.*

The result obtained here aligns with ([35], Thm 3.1). Furthermore, when the process is fragmented and the post-supremum process remains below the level b, the drawdown duration $\mathcal{T}^d_{H_S,T}$ over the interval (H_S, T) is given as:

$$\mathcal{T}^d_{H_S,T} = \alpha_d(H_S, T). \tag{2.14}$$

Thus, the expectation of the Laplace transform of the drawdown duration is:

$$\mathbb{E}\left[e^{-\gamma\mathcal{T}^d_{H_S,T}} \mid S_T = b\right] = 1 - Z^{(\gamma)}(d)\frac{W^{(\gamma)'}(d)}{\phi(\gamma)W^{(\gamma)}(d)} - \frac{\gamma W^{(\gamma)}(d)}{\phi(\gamma)}. \tag{2.15}$$

Given the infimum and supremum values, the Laplace transform of the drawdown duration is:

$$\mathbb{E}\left[e^{-\gamma\mathcal{T}^d} \mid \bar{X}_{\alpha_d} = m\right] = \frac{W(d)}{W'(d)}\frac{Z^{(\gamma)}(d)W^{(\gamma)'}(d) - \gamma W^{(\gamma)}(d)^2}{W^{(\gamma)}(d)}. \tag{2.16}$$

Furthermore, when it is known that the infimum occurs before the supremum, the post-supremum drawdown duration $\mathcal{T}^d_{H_S,T}$ is derived from the distribution of the post-supremum process. It is also shown that:

$$\begin{aligned}
&\mathbb{E}\left[e^{-\gamma\mathcal{T}^d_{H_S,T}} \mid H_I < H_S, I_T = a, S_T = b\right] \\
&= 1 - \frac{\left(Z^{(\gamma)}(d) - 1\right)\frac{W^{(\gamma)'}(d)}{W^{(\gamma)}(d)} - \gamma W^{(\gamma)}(d)}{\left(Z^{(\gamma)}(b-a) - 1\right)\frac{W^{(\gamma)'}(b-a)}{W^{(\gamma)}(b-a)} - \gamma W^{(\gamma)}(b-a)},
\end{aligned}$$

where $0 < d < b - a$. The drawdown duration distribution provided in (8) is independent of the supremum value m. Similarly, when the process is considered up to α_d instead of T, the characterization of the process between κ_{α_d} and α_d provides the drawdown duration distribution, which is also independent of $\bar{X}_{\alpha_d}$.

2.3 Existing Results on the Maximum Drawdown Variable

For investors and companies, the distributions and applications of the maximum drawdown and range variables are as crucial as the drawdown variable itself. The maximum drawdown has been studied theoretically under various models and conditions. It has been considered a risk measure in finance ([57], [62], [63]).

In practical applications, the sensitivity of portfolios to maximum drawdowns was examined in [47]. The distributional properties of both drawdowns and maximum drawdowns were studied in ([19], [38], [52], [46]). Drawdown and drawup stopping times were analyzed from a probabilistic perspective in ([22], [46], [64]). Optimal investment strategies under drawdown constraints were investigated in ([15], [18], [21]).

The tail conditional expectation of the maximum drawdown random variable was proposed in the context of portfolio optimization by [15]. Beyond portfolio optimization, [38] explored the relationship between maximum drawdowns and average returns in the context of performance analysis.

In [36], models were estimated for three different stock indices (two from developed markets: the U.S. Nasdaq and U.K. FTSE, and one from a volatile market: Brazil's IBOVESPA). They investigated the relationship between maximum drawdown fluctuations and daily return GARCH volatility.

For directional trading, [62] analyzed the expected maximum drawdown of a market in contracts linked to maximum drawdowns. Maximum drawdown insurance research was conducted in ([12], [63], [24]). Risks related to maximum drawdowns have recently been formalized as the Conditional Expected Drawdown (CED), defined as the tail mean of the maximum drawdown distribution.

[50] refined the concept of CED, incorporating this information into drawdown risk measurement. A specific application investigated the problem of two paths sharing the same maximum drawdown, despite one path having an all-time minimum cumulative return significantly higher than the other, conditioned on the minimum cumulative returns of the paths.

In the Black-Scholes model, which is the special case of Lévy models, the logarithm of the asset price can be used, leading to a Brownian motion with a non-zero drift. The properties of Brownian motion with nonzero drift differ significantly from those of standard Brownian motion, where the drift is zero. In [60], the correlation between the maximum and minimum values of the standard Brownian motion and the Brownian motion with nonzero drift was calculated.

The joint distributions of the largest gain and the maximum loss, as well as their relationship for different drift values, were analyzed using simulation methods. The results suggest that the correlation between the maximum and minimum is positive, while the correlation between the largest gain and loss is negative. Furthermore, it was conjectured that the correlation reaches its minimum when $\mu = \pm 2$. Asymptotic results for the individual distributions of the largest gain and the maximum loss were also derived.

For the standard Brownian motion, [52] calculated the joint distribution of (M_T^-, M_T^+), where T is an independent exponential time. Furthermore, by stopping the process at the first passage time to a fixed level $b > 0$ and applying the excursion theory, they obtained the joint distributions of $M_{H_b}^-$ and $M_{H_b}^+$.

Building on this, [60] extended the analysis using the same techniques to spectrally negative Lévy processes, deriving the joint and marginal distributions of $M_{\tau_\beta}^-$ and $M_{\tau_\beta}^+$.

In [53], martingale theory was employed to obtain results for general diffusion models. Focusing on the first passage time to a level $b > 0$, denoted H_b, they derived the distribution of $M_{H_b}^-$ and showed that the joint distribution of $M_{H_b}^-$ and $M_{H_b}^+$ depends only on the scale function of the diffusion.

Furthermore, [53] showed that for $b > 0$, the maximum drawdown process $M_{H_b}^-$ is a pure jump Markov process. They also determined its generator and transition probabilities, providing a comprehensive characterization of its behavior.

Spectrally negative Lévy processes (without positive jumps) are popular due to their flexibility and tractability in numerous applications such as risk theory, mathematical finance, tail theory, and more ([6], [30], [5]).

In [60], under a spectrally negative Lévy process model, the joint distributions of the maximum increase $M^+_{\tau_\beta}$ and the maximum drawdown $M^-_{\tau_\beta}$ up to a fixed level β were obtained for $u \geq 0$ and $v \geq \beta$ as:

$$P\left\{M^-_{\tau_\beta} \leq u, M^+_{\tau_\beta} \leq v\right\} = \begin{cases} e^{-\beta W'_+(u)/W(u)} & u \leq v-\beta, \\ \frac{W(v-\beta)}{W(u)} e^{-(v-u)W'_+(u)/W(u)} & v-\beta \leq u \leq v, \\ \frac{W(v-\beta)}{W(v)} & v \leq u, \end{cases} \tag{9}$$

where W is the scale function, which is the inverse Laplace transform of $1/\psi(\lambda)$. This result coincides with that given in [52] for the special case of Brownian motion with $\mu = 0$ and $\sigma = 1$.

The marginal distributions of the maximum increase and maximum drawdown were also derived, and these results were shown to match those of the Brownian motion. For $\gamma > 0$, $a > 0$, and scale functions $W^{(\gamma)}$ and $Z^{(\gamma)}$, the results are:

$$\begin{aligned} P\{M^-_T > a\} &= Z^{(\gamma)}(a) - \gamma \left[W^{(\gamma)}(a)\right]^2 / W^{(\gamma)'}_+(a), \\ P\{M^+_T > a\} &= \frac{Z^{(\gamma)}(0)}{Z^{(\gamma)}(a)}. \end{aligned} \tag{2.17}$$

For general Markov processes, the independence of path segments decomposed at the lowest or highest level was demonstrated in [40]. Using the strong Markov property, this result was extended to random walk processes in ([10], Thm. 2.3). In continuous time, typical excursion theory is required depending on whether 0 is regular for the reflected process (see [10], Lem.VI.6.ii).

In ([13], [14]), the distributions of the processes before and after the infimum, conditioned to remain positive, were derived for spectrally negative Lévy processes starting at $x > 0$. These processes are described by the h-transform of the Lévy process based on its local time. The related work by [20] analyzed the drawdown process during an excursion of general Lévy processes, examining the supremum before and after the process.

Recently, [61] performed a path decomposition for spectrally negative Lévy processes using extreme values. The distributions of the processes before and after the supremum and after the infimum were expressed as Doob's h-transforms. Moreover, under the condition that the infimum occurs before the supremum, the distributions of the

process between the infimum and supremum, as well as after the supremum, were also derived.

Conditioned on $S_T = b$ for $b > 0$, the distribution of the maximum drawdown before the supremum M^-_{0,H_S} and after the supremum $M^-_{H_S,T}$ were given as:

$$P\left\{M^-_{0,H_S} < d \mid S_T = b\right\} = e^{-b\left(\frac{W^{(\gamma)\prime}(d)}{W^{(\gamma)}(d)} - \phi(\gamma)\right)}, \tag{2.18}$$

and

$$P\left\{M^-_{H_S,T} > d \mid S_T = b\right\} = 1 + \frac{\left(Z^{(\gamma)}(d) - 1\right)\left(W^{(\gamma)\prime}(d) - \left(W^{(\gamma)}(d)\right)^2\right)}{\phi(\gamma)W^{(\gamma)}(d)}. \tag{2.19}$$

The conditional distributions of $M^-_{H_I,H_S}$ and $M^-_{H_S,T}$ under the condition $H_I < H_S$, $S_T = b$ and $I_T = a$ are:

$$P\left\{M^-_{H_I,H_S} < d \mid H_I < H_S, I_T = a, S_T = b\right\} = \frac{W^{(\gamma)}(b-a)}{W^{(\gamma)}(d)} e^{-(b-a-d)\left(\frac{W^{(\gamma)\prime}(d)}{W^{(\gamma)}(d)} - \phi(\gamma)W^{(\gamma)}(d) + \phi(\gamma)\right)}, \tag{2.20}$$

and

$$P\left\{M^-_{H_S,T} < d \mid H_I < H_S, I_T = a, S_T = b\right\} = \tag{2.21}$$

$$\frac{\left(Z^{(\gamma)}(d) - 1\right)\frac{W^{(\gamma)\prime}(d)}{W^{(\gamma)}(d)} - \gamma W^{(\gamma)}(d)}{\left(Z^{(\gamma)}(b-a) - 1\right)\frac{W^{(\gamma)\prime}(b-a)}{W^{(\gamma)}(b-a)} - \gamma W^{(\gamma)}(b-a)}. \tag{2.22}$$

It has been observed that the supremum of the process following the infimum is also included in the distribution.

$$d \to \frac{\Phi(\gamma)\bar{W}^{(\gamma)}(d)}{W^{(\gamma)}(d)}, \tag{2.23}$$

where $d > 0$. The fact that such a function is a cumulative distribution function is a previously overlooked property. This result is significant in practice because it models a bull market (supremum) following a bear market. Another observation that emerged in this study is that

$$d \to \frac{\left(Z^{(\gamma)}(d) - 1\right)\left(W^{(\gamma)\prime}(d) - \left(W^{(\gamma)}(d)\right)^2\right)}{\Phi(\gamma)W^{(\gamma)}(d)}, \tag{2.24}$$

is also a cumulative distribution function. However, characterizing the process up to the infimum and finding the distribution of the maximum drawdown remained open

questions, as noted in [61]. This thesis aims to address this gap by developing new methods that reveal how maximum drawdown and maximum gain interact as part of a more comprehensive investigation. The correlation coefficient between them will be expressed or computed as a function of the parameters. As explained in the methodology section, characterizing the process before infimum serves as the novel result which leads to the solution.

CHAPTER 3

MATHEMATICAL FOUNDATIONS

In this chapter, we review the relevant material, some basic definitions, notation, and theorems of stochastic drawdown theory and stochastic calculus, which will be needed to solve the aimed problems. We mainly refer to Cont and Tankov [16], Kyprianou [29], Papapantoleon [45], Kuznetsov [28], Øksendal and Sulem [42]. Throughout this thesis, we work with the complete filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where Ω denotes the sample space, $\mathbb{F}$ is the sigma algebra, $(\mathcal{F}_t)_{t\geq 0}$ is filtration and $\mathbb{P}$ is a probability measure.

Brownian motion is a substantially important process that appears in most financial models. It is an example of diffusion process which is a solution to a stochastic differential equation. A diffusion process is a Markov process with continuous paths, which means that it does not exhibit jumps. Formally, a diffusion process X_t satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}X_t = \mu(X_t, t), \mathrm{d}t + \sigma(X_t, t), \mathrm{d}W_t, \tag{3.1}$$

where:

- $\mu(X_t, t)$ is the drift term, representing the deterministic trend of the process,
- $\sigma(X_t, t)$ is the diffusion term, controlling the random variability of the process,
- W_t is a standard Brownian motion, introducing randomness into the system.

Diffusion models are widely used in mathematical finance due to their tractable analytical properties. However, they cannot generate sudden discontinuities, which limits

their ability to fully capture real-world dynamics. Empirical observations in financial markets indicate that price movements often involve jumps, which makes pure diffusion models insufficient for describing such behavior. To address this limitation, models incorporating sudden discontinuities, such as jump-diffusion models or Lévy processes, are necessary to accurately represent the observed reality of financial markets.

In this thesis, financial models with negative jumps are studied. Since the jump-diffusion models contain the diffusion models, we proceed with the study of models with jumps. We can classify these models into two categories: jump-diffusion models and infinite activity models, which are discussed in Sections 2.1 and 2.2, respectively.

3.1 Jump-Diffusion Models

The first category consists of the *jump-diffusion* models which must contain a Brownian motion component and jumps at random times. That is, the process jumps at some times and has a continuous random path between jumps. Here, there are only finitely many jumps in every finite time interval, jumps are rarely appearing and represented by a compound Poisson process. In jump-diffusion models, since the distribution of jump sizes is known, they carry out quite well for a realistic description of price dynamics and market risks, moreover, jump-diffusion models are easy to simulate. In jump-diffusion models, the concept of characteristic function of a random variable is significant because while the densities are not known in closed form, the characteristic function is known explicitly. As an example of jump diffusion models, we can give the Merton model with the stock price $S_t = S_0 \exp\{X_t\}_{t\geq 0}$ and Gaussian distributed jumps.

A jump-diffusion process has the form of:

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s) + J(t), \quad t > 0 \tag{3.2}$$

where $J(t)$ is an adapted, right continuous pure jump process.

A *pure jump process* begins at zero, is constant between jump times, and has finitely many jumps in each finite time interval. The fundamental pure jump process is the

Poisson process.

Definition 3.1. (The Poisson Process)

Let $\{\tau_j\}_{j\in\mathbb{N}}$ be a sequence of independent exponentially distributed random variables with parameter λ, i.e., with cumulative distribution functions $P\{\tau_j \geq x\} = e^{-\lambda x}$ and $S(n) = \sum_{k=1}^{n} \tau_k$. Then, the process

$$N(t) = \sum_{n\geq 1} 1_{\{t\geq S_n\}} \tag{3.3}$$

is called the *Poisson process* with intensity λ.

The Poisson process $(N(t) : t \geq 0)$ counts the number of jumps that occur at or before time t because all jumps of a Poisson process are of size one. The τ_k random variables are called the interarrival times and they are exponentially distributed.

The arrival times are

$$S(n) = \sum_{k=1}^{n} \tau_k, \tag{3.4}$$

i.e., $S(n)$ is the time of the n^{th} jump.

Because the expected jump time between jumps is $\frac{1}{\lambda}$, the jumps are arriving at an average rate of λ per unit time.

Proposition 3.1. ***(Rama Cont, 2003, [16])*** *Let $\{N(t)\}_{t\geq 0}$ be a Poisson process.*

1. *For any ω, the sample path $t \mapsto N(t)$ is cádlág piecewise constant (Right continuous with left limit, RCLL for short).*
2. *The Poisson process $N(t)$ with intensity λ has the distribution*

$$P\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \tag{3.5}$$

3. *The characteristic function of a Poisson process $N(t)$ is given by*

$$\mathbb{E}\left[e^{ixN(t)}\right] = \exp\{\lambda t(e^{ix} - 1)\}. \tag{3.6}$$

4. *The Poisson process $N(t)$ has independent increments: If $t \geq s$, $N(t) - N(s)$ is independent of the σ-algebra $\mathbb{F}_s$.*

5. *The Poisson process $N(t)$ has stationary increments: If $t \geq s \geq 0$, then $N(t) - N(s)$ and $N(t-s) - N(0)$ have the same law.*

6. *The Poisson process $N(t)$ has the Markov property, i.e., $\mathbb{E}[N(t)|\mathbb{F}_s] = N(s)$, $\forall t \geq s \geq 0$.*

Corollary 3.2. ***(Shreve, 2004, [56])*** *A Poisson process $N(t)$ with intensity λ satisfies that $\mathbb{E}[N(t)] = \lambda t$ and $Var\,[N(t)] = \lambda t$.*

Definition 3.2. (The Compensated Poisson Process)

Let $N(t)$ be a Poisson process with parameter λ. Then $M(t) = N(t) - \lambda t$ is called a compensated Poisson process.

Theorem 3.3. ***(Shreve, 2004, [56])***

The compensated Poisson process $M(t) = N(t) - \lambda t$ is a martingale.

Definition 3.3. (The Compound Poisson Process)

A compound poisson process is a stochastic process with intensity λ and jump size distribution F defined as

$$Q(t) = \sum_{j=1}^{N(t)} Y_j, \quad t \geq 0, \tag{3.7}$$

where $N(t)$ is Poisson process with intensity λ, and the jump sizes Y_j are independent of one another and also independent of the Poisson process $N(t)$ with distribution F.

A compound Poisson process can be considered as a Poisson process with random jump sizes.

Proposition 3.4. ***(Rama Cont, 2003, [16])***

Let Q(t) be a compound Poisson process. Then, the following conditions are fulfilled:

1. *For any ω, the sample path $t \mapsto Q(t)$ is cádlág (RCLL) piecewise constant.*

2. *The characteristic function of a compound Poisson process $Q(t)$ is given by*

$$\mathbb{E}[e^{ixQ(t)}] = \exp\{\lambda t \int_{\mathbb{R}} (e^{ix} - 1) F(dx)\}. \tag{3.8}$$

3. *The compound Poisson process $Q(t)$ has independent increments: If $t \geq s \geq 0$, then $Q(t) - Q(s)$ is independent of the σ-algebra $\mathbb{F}_s$.*

4. *The compound Poisson process $Q(t)$ has stationary increments: If $t \geq s \geq 0$, then $Q(t) - Q(s)$ and $Q(t-s) - Q(0)$ have the same law.*

5. *Jump sizes $(Y_j)_{j\geq 1}$ are independent and identically distributed random variables (i.i.d.) with law F and the same mean $\mu = \mathbb{E}[Y_j]$.*

Corollary 3.5. ***(Shreve, 2004, [56])***

A compound Poisson process $Q(t)$ with intensity λ satisfies the equation $\mathbb{E}[Q(t)] = \mu\lambda t$.

Theorem 3.6. ***(Shreve, 2004, [56])***

The compensated compound Poisson process $\tilde{Q}(t) = Q(t) - \mu\lambda t$ is a martingale.

Theorem 3.7. ***(Itô-Doeblin formula for jump-diffusion processes) (Shreve, 2004, [56])***

Let $f \in C^2(\mathbb{R})$ and $X(t)$ be a jump diffusion process given as Eqn. (3.2), and Then,

$$f(X(t)) = f(X(0)) + \int_0^t f^{'}(X(s))dX^C(s) + \frac{1}{2}\int_0^t f^{''}(X(s))d[X^C, X^C](s) + \sum_{0<s\leq t}[f(X(s)) - f(X(s-))], \quad t > 0, \tag{3.9}$$

where $X^C(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s)$ according to Eqn. (3.2).

3.2 Infinite Activity Models

In jump-diffusion models, jumps are rare events, and there are only finitely many jumps in every finite time interval. In infinite activity Lévy processes, in every interval there are infinitely many jumps that most of them are very small and there is only a finite number of jumps with absolute value greater than any given positive number. These models do not necessarily involve a Brownian motion and actually move by jumps. As compared with jump-diffusion models, infinite activity models can be constructed by Brownian subordination which gives them additional tractability.

Some examples of Lévy processes are linear drift (simplest Lévy process), Brownian motion (the only non-deterministic continuous Lévy process), Poisson process, compound Poisson process, gamma process (an increasing Lévy process, also called subordinator) and spectrally negative Lévy processes.

Definition 3.4. (Lévy Process)

An adapted, c*á*dl*á*g, real valued stochastic process $(\eta(t))_{t\geq 0}$ defined on a probability measure $(\Omega, \mathbb{F}, \mathbb{P})$ is called a **Lévy process** if it satisfies the following conditions:

1. $P(\eta(0) = 0)$.
2. Independent increments: For $0 \leq s \leq t$, $\eta(t) - \eta(s)$ is independent of $\mathbb{F}_s$.
3. Stationary increments: For $0 \leq s \leq t$, $\eta(t) - \eta(s)$ is equal in distribution to $\eta(t-s)$.
4. Stochastic continuity: $\forall \epsilon > 0$, $\lim_{h\to 0} P(|\eta(t+h) - \eta(t)| \geq \epsilon) = 0$.

Definition 3.5. (Lévy Measure)

Let $\eta(t)$ be a Lévy process on $\mathbb{R}^d$. The measure

$$\nu(A) = \mathbb{E}[M(1, A)] = \mathbb{E}\left[\#\{t \in [0,1] : \Delta\eta(t) \neq 0, \Delta\eta(t) \in A\}\right], A \in \mathbb{B}(\mathbb{R}^d) \tag{3.10}$$

on $\mathbb{R}^d$ is called Lévy Measure of η.

This means that, $\nu(A)$ is the expected number, per unit time, of jumps whose size is in A. Furthermore, $M([0,t], A)$, called as jump measure of η, counts the number of jumps of η up to time t with jump size in the set A, and $M(dt, dx)$ is the differential notation of the $M([0,t], A)$. The compensated jump measure of η is defined by $\tilde{M}(dt, dx) = M(dt, dx) - \nu(dx)$.

Definition 3.6. (Poisson Random Measure) (Rama Cont, 2003, [16])

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, $E \subseteq \mathbb{R}$ and μ a given (positive) Radon measure on (E, ϵ). A Poisson random measure on E with intensity measure μ is an integer

valued random measure such that

$$M : \Omega \times \epsilon \to \mathbb{N}$$
$$(w, A) \mapsto M(w, A),$$

which satisfies the following conditions:

(i) For (almost all) $w \in \Omega$, is an integer-valued Radon measure on E: For any bounded measurable $A \subset E$, $M(A) < \infty$ is an integer valued random variable.

(ii) For each measurable $A \subseteq E$, $M(\cdot, A) = M(A)$ is a Poisson random variable with parameter $\mu(A)$: for all $k \in \mathbb{N}$,

$$P(M(A) = k) = \frac{(\mu(A))^k}{k!} e^{\mu(A)}. \tag{3.11}$$

(iii) For disjoint measurable sets $A_1, ..., A_n \in \epsilon$, the variables $M(A_1), ..., M(A_n)$ are independent.

Theorem 3.8 (**Lévy -Khintchine formula**). ***(Papapantoleon, 2008, [45])***

The probability distribution F_X of a random variable X is infinitely divisible with characteristic exponent

$$\psi(u) = iub - \frac{u^2\sigma}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{|x|<1})\nu(dx) \tag{3.12}$$

if and only if there exists a triplet (b, σ, ν) with $\sigma \in \mathbb{R}$, $b \in \mathbb{R}$ and ν is a measure satisfying $\nu(0) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$, such that

$$\mathbb{E}[e^{iuX}] = e^{\psi(u)}. \tag{3.13}$$

Here, the triplet (b, σ, ν) is called the characteristic or *Lévy triplet* and the characteristic exponent $\psi(u)$ is called the Lévy exponent, $b \in \mathbb{R}$ is drift, $\sigma \in \mathbb{R}_{\geq 0}$ is diffusion coefficient and ν is the Lévy measure.

Theorem 3.9 (**Lévy -Khintchine formula for Lévy processes**). ***(Papapantoleon, 2008, [45])***

Let $b \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure satisfying $\nu(0) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$. Define characteristic exponent $\psi(u) = iub - \frac{u^2\sigma}{2} + \int_{\mathbb{R}}(e^{iux} - 1 - iux1_{|x|<1})\nu(dx)$. Then, there exists a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ on which a Lévy process with characteristic exponent $\psi(u)$ is defined.

Theorem 3.10 (**Lévy -Khintchine Decomposition**). *The Lévy process X_t can be decomposed as:*

$$X_t = B_t + \int_0^t \int_{|z|\leq 1} z\, \tilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{|z|>1} z\, N(\mathrm{d}s, \mathrm{d}z), \tag{3.14}$$

where:

- B_t *is the Brownian motion component.*
- $\tilde{N}(\mathrm{d}s, \mathrm{d}z)$ *is the compensated Poisson random measure.*
- $N(\mathrm{d}s, \mathrm{d}z)$ *is the Poisson random measure of jumps.*

Theorem 3.11 (**Itô Formula with Lévy Measures**). *Using the Lévy-Itô decomposition, the Itô formula for X_t can also be expressed as:*

$$\begin{aligned} f(X_t) = f(X_0) &+ \int_0^t f'(X_{s^-})\, \mathrm{d}B_s + \frac{1}{2}\int_0^t f''(X_{s^-})\, \mathrm{d}[B]_s \\ &+ \int_0^t \int_{\mathbb{R}} \Big[f(X_{s^-} + z) - f(X_{s^-}) - f'(X_{s^-}) z \mathbf{1}_{\{|z|\leq 1\}} \Big] \nu(\mathrm{d}z)\, \mathrm{d}s, \end{aligned} \tag{3.15}$$

where $\nu(\mathrm{d}z)$ is the Lévy measure describing the jump distribution.

Definition 3.7. (Spectrally Negative Lévy Process)

Spectrally Negative Lévy Process is a Lévy process when $\nu(0, \infty) = 0$.

Since Spectrally Negative Lévy Process has no positive jumps, for $\lambda \geq 0$,

$$E\left(\mathrm{e}^{\lambda X_t}\right) = \mathrm{e}^{\psi(\lambda)t}. \tag{3.16}$$

In other words, taking into account the analytical extension of the characteristic exponent, we have $\psi(\lambda) = -\Psi(-\mathrm{i}\lambda)$. In particular

$$\psi(\lambda) = -\mu\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{(-\infty,0)} \left(\mathrm{e}^{\lambda x} - 1 - \lambda x \mathbf{1}_{|x|<1}\right) \nu(\mathrm{d}x).$$

From Hölder inequality and the fact that $\int_{(-\infty,0)} (1 \wedge x^2)\, \nu(dx) < \infty$, Ψ is a convex function. When λ approaches to infinity, Ψ also goes to infinity. This allows us to define $\Phi : [0, \infty) \to [0, \infty)$, the right inverse of ψ so that

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}, \text{and} \quad \psi\left(\Phi(\lambda)\right) = \lambda$$

for all $q \geq 0$.

3.3 Preliminaries for Maximum Drawdown and Maximum Drawup

In this section, the variables associated with maximum drawdown and maximum drawup, which will be used throughout the thesis, are presented.

1. The Drawdown Process of X is defined as

$$\underline{Y_t} = \bar{X}_t - X_t, \quad \bar{X}_t := \sup_{0\leq s\leq t} X_s. \tag{3.17}$$

   Moreover, the drawup process variable is

$$\bar{Y_t} = X_t - \underline{X}_t, \quad \underline{X}_t := \inf_{0\leq s\leq t} X_s. \tag{3.18}$$

2. Maximum Drawdown is defined as

$$M_t^- := \sup_{0\leq u\leq v\leq t} (X_u - X_v) = \sup_{0\leq s\leq t} Y_s. \tag{3.19}$$

   Maximum Drawup is defined as

$$M_t^+ := \sup_{0\leq u\leq v\leq t} (X_v - X_u) = \sup_{0\leq s\leq t} \underline{Y}_s. \tag{3.20}$$

   More generally, in an interval $[p, q]$ the amounts of Maximum Drawdown and Maximum Drawup are defined as, respectively,

$$M_{p,q}^- := \sup_{p\leq u\leq v\leq q} (X_u - X_v) = \sup_{p\leq s\leq q} Y_s, \tag{3.21}$$

$$M_{p,q}^+ := \sup_{p\leq u\leq v\leq q} (X_v - X_u) = \sup_{p\leq s\leq q} \underline{Y}_s. \tag{3.22}$$

3. The first passage times where the process, respectively, goes below and above the level x are indicated as

$$\tau_x^- = \inf\{t \geq 0 : X_t < x\}, \quad \tau_x^+ = \inf\{t \geq 0 : X_t > x\}. \tag{3.23}$$

4. The first time that the drawdown process passes the d level is defined as

$$\alpha_d = \inf\{t \geq 0 : \underline{Y}_t > d\}.$$

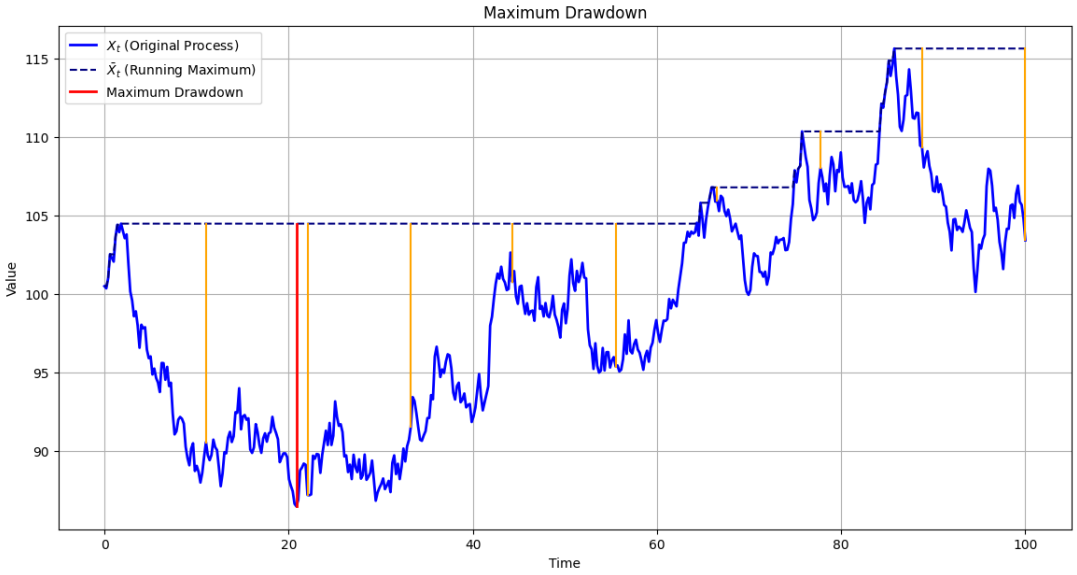


Figure 3.1: Maximum Drawdown Process

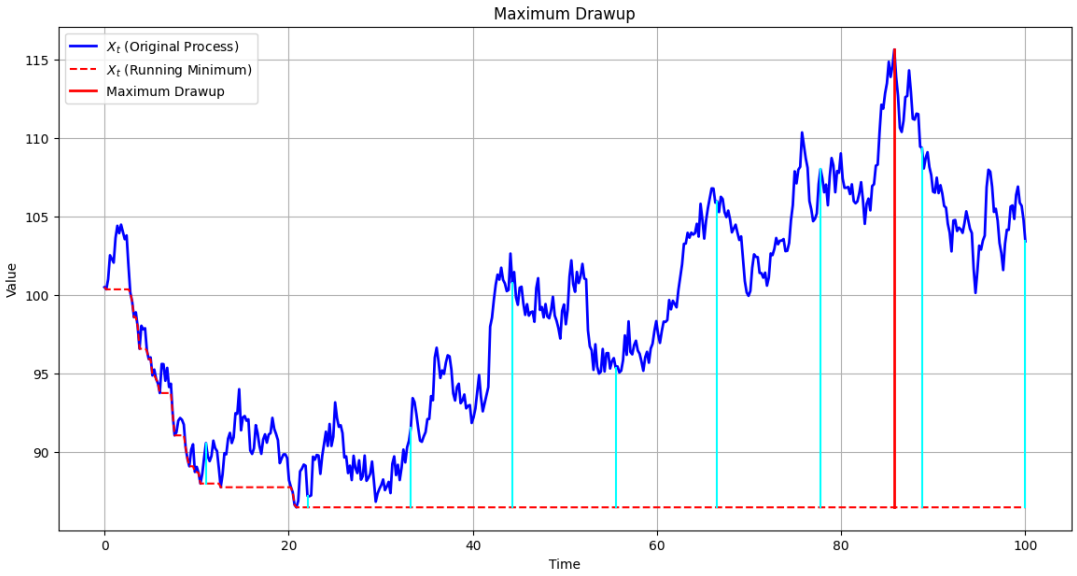


Figure 3.2: Maximum Drawup Process

5. The first time that the drawup process passes the d level is defined as

$$\bar{\alpha}_d = \inf\left\{t \geq 0 : \bar{Y}_t > d\right\}.$$

6. Moreover, since $\bar{X}$ is continuous for spectral negative Lévy process , $\kappa_{\alpha_d} = \inf\left\{0 \leq s \leq \alpha_d : \bar{X}_s = \bar{X}_{\alpha_d}\right\}$ is defined. Thus, the duration of the drawdown just past the d level is expressed as

$$\mathcal{T}^d = \alpha_d - \kappa_{\alpha_d}.$$

7. The supremum and infimum variables up to time T will be expressed using S_T and I_T, respectively,

$$S_T := \bar{X}_T = \sup\left\{X_s : 0 \leq s \leq T\right\}, \quad I_T := \underline{X}_T := \inf\left\{X_s : 0 \leq s \leq T\right\}. \tag{3.24}$$

3.4 Scale Functions

Scale functions are essential for understanding boundary crossing problems and path decompositions within the framework of stochastic processes, including spectrally negative Lévy processes (SNLPs). These functions are key tools for addressing first passage problems, analyzing excursion theory, and describing the probability distributions of extreme events. In this section, we present the definition of scale functions, the existence of scale functions, their fundamental properties, and foundational mathematical tools such as Wiener-Hopf factorization and Doob-h transforms.

3.4.1 Definition of Scale Functions

Definition 3.8. (Kuznetsov, 2003, [28]) Scale functions are defined as a family of continuous and strictly increasing functions, $W^{(q)} : \mathbb{R} \to [0, \infty)$, indexed by $q \geq 0$ for a spectrally negative Lévy process X with Laplace exponent ψ which satisfies:

- $W^{(q)}(x) = 0$ for $x < 0$,

- $W^{(q)}(x)$ is uniquely determined on $[0, \infty)$ by its Laplace transform:

$$\int_0^\infty \mathrm{e}^{-\beta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\beta) - q}, \quad \beta > \Phi(q). \tag{3.25}$$

  where $\Phi(q)$ represents the root of $\psi(\beta) - q = 0$.

Associated with $W^{(q)}(x)$, the second scale function is defined as:

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(\beta)\mathrm{d}\beta. \tag{3.26}$$

These functions serve as critical tools for analyzing first passage problems and path decompositions in Lévy processes. Before exploring their relationship with first passage problems, we first demonstrate the existence of scale functions

3.4.2 Existence of scale functions

In this subsection, we demonstrate the existence of scale functions. Specifically, we aim to show that, for Definition 3.8 to be valid, the function $(\psi(\beta) - q)^{-1}$ is indeed a genuine Laplace transform in β. This will primarily be achieved by first establishing the Exponential Change of Measure and the Wiener-Hopf Factorization, which will then be employed in conjunction to proceed with the proof.

3.4.2.1 Exponential change of measure

The Laplace exponent serves as a fundamental tool for constructing an "*exponential change of measure*" on X, inspired by the classical Cameron-Martin-Girsanov transformation. This transformation plays a pivotal role throughout this text. By leveraging the equality 3.16, we define a Girsanov-type change of measure as follows:

$$\left.\frac{\mathrm{d}P_x^c}{\mathrm{d}P_x}\right|_{\mathbb{F}_t} = \mathrm{e}^{c(X_t - x) - \psi(c)t}, \quad t \geq 0, \tag{3.27}$$

for any c such that $|\psi(c)| < \infty$. It can be verified that the right-hand side above forms a martingale, which is a consequence of the stationary and independent increments property of X alongside 3.16.

This change of measure ensures absolute continuity between the original and transformed measures. As a result, the process X retains its classification as a spectrally negative Lévy process under the new measure. Moreover, the Laplace exponent of X under the transformed measure P_x^c is given by:

$$\psi_c(\theta) = \psi(\theta + c) - \psi(c), \quad \theta \geq -c. \tag{3.28}$$

If ν_c represents the Lévy measure under P^c, the relationship between the original and transformed measures is expressed as:

$$\nu_c(\,\mathrm{d}x) = \mathrm{e}^{cx}\nu(\mathrm{d}x), \quad x < 0. \tag{3.29}$$

This transformation provides a powerful mechanism for analyzing spectrally negative processes by modifying their probabilistic structure while preserving their essential characteristics.

3.4.2.2 The Wiener-Hopf Factorization

The Wiener-Hopf factorization is a fundamental result that provides insight into the probabilistic structure of Lévy processes. It separates the characteristic exponent $\psi(\beta)$ into positive and negative components, facilitating the analysis of the process. For spectrally negative Lévy processes, the Wiener-Hopf factorization plays a crucial role in deriving results related to path decomposition and boundary-crossing problems.

Let $\underline{X}_t := \inf_{s \leq t} X_s$ denote the running infimum and $\mathbf{e}_q$ be an independent and exponentially distributed random variable with rate $q > 0$. The key property of this factorization is that the variables $X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q}$ and $-\underline{X}_{\mathbf{e}_q}$ are independent. This independence is a direct consequence of the stationary increments property and implies that the process $\{X_s : s \leq t\}$ is equal in law to $\{X_t - X_{t-s} : s \leq t\}$. Consequently, we have:

$$X_t - \underline{X}_t \stackrel{d}{=} \bar{X}_t := \sup_{s \leq t} X_s. \tag{3.30}$$

It follows that for all $\theta \in \mathbb{R}$,

$$E\left(e^{i\theta X_{\mathbf{e}_q}}\right) = E\left(e^{i\theta \bar{X}_{\mathbf{e}_q}}\right) E\left(e^{i\theta \underline{X}_{\mathbf{e}_q}}\right). \tag{3.31}$$

This identity, known as the Wiener-Hopf factorization, holds for general Lévy processes.

In the case of spectrally negative Lévy processes, the factorization simplifies further. Let $a > 0$. Since $t \wedge \tau_a^+$ is a bounded stopping time and $X_{t\wedge\tau_a^+} \leq a$, Doob's Optional Stopping Theorem applied to the exponential martingale yields:

$$E\left(e^{\Phi(q)X_{t\wedge\tau_a^+} - q\left(t\wedge\tau_a^+\right)}\right) = 1. \tag{3.32}$$

By dominated convergence and the fact that $X_{\tau_a^+} = a$ on $\tau_a^+ < \infty$, we obtain:

$$\begin{aligned} P\left(\bar{X}_{\mathbf{e}_q} > a\right) &= P\left(\tau_a^+ < \mathbf{e}_q\right) \\ &= E\left(e^{-q\tau_a^+}\mathbf{1}_{\left(\tau_a^+<\infty\right)}\right) \\ &= e^{-\Phi(q)a}. \end{aligned} \tag{3.33}$$

The conclusion is that the quantity $\bar{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$. Therefore, for all $\theta \in \mathbb{R}$,

$$E\left(e^{i\theta\bar{X}_{\mathbf{e}_q}}\right) = \frac{\Phi(q)}{\Phi(q) - i\theta}. \tag{3.34}$$

From the Wiener-Hopf factorization, it follows that:

$$E\left(e^{i\theta\underline{X}_{\mathbf{e}_q}}\right) = \frac{q}{\Phi(q)}\frac{\Phi(q) - i\theta}{q + \psi(\theta)}. \tag{3.35}$$

This factorization is a cornerstone for studying path decompositions, scale functions, and their applications in stochastic processes.

After demonstrating the exponential change of measure and the Wiener-Hopf factorization as the necessary foundations, we are now ready to show the existence of scale functions with the theorem below.

Theorem 3.12 (**Existence of Scale Functions**). ***(Kuznetsov, 2003, [28])*** *For any spectrally negative Lévy process, q-scale functions exist for all $q \geq 0$.*

Proof. We begin by assuming that $\psi'(0+) > 0$. Define the function

$$W(x) = \frac{1}{\psi'(0+)}P_x\left(\underline{X}_\infty \geq 0\right), \tag{3.36}$$

where $W(x)$ is equal to zero for $x < 0$ and is right-continuous as it corresponds to the cumulative distribution function $P\left(-\underline{X}_\infty \leq x\right)$. By integrating by parts, we can express $W(x)$ as follows:

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\beta x} W(x)\mathrm{d}x &= \tfrac{1}{\psi'(0+)} \int_0^\infty \mathrm{e}^{-\beta x} P\left(-\underline{X}_\infty \leq x\right) \mathrm{d}x, \\ &= \tfrac{1}{\psi'(0+)\beta} \int_{[0,\infty)} \mathrm{e}^{-\beta x} P\left(-\underline{X}_\infty \in \mathrm{d}x\right), \\ &= \tfrac{1}{\psi'(0+)\beta} E\left(\mathrm{e}^{\beta \underline{X}_\infty}\right). \end{aligned} \tag{3.37}$$

Using the Wiener-Hopf factorization, as $q \to 0$, we find

$$E\left(\mathrm{e}^{\mathrm{i}\theta \underline{X}_\infty}\right) = -\psi'(0+)\frac{\mathrm{i}\theta}{\psi(\theta)}, \tag{3.38}$$

which relies on the fact that $\Phi(0) = 0$. Consequently, we have

$$\lim_{q \downarrow 0} \frac{q}{\Phi(q)} = \lim_{q \downarrow 0} \frac{\psi(\Phi(q))}{\Phi(q)} = \lim_{\theta \downarrow 0} \frac{\psi(\theta)}{\theta} = \psi'(0+). \tag{3.39}$$

Extending this analytically, we obtain

$$E\left(\mathrm{e}^{\beta \underline{X}_\infty}\right) = \frac{\psi'(0+)\beta}{\psi(\beta)}. \tag{3.40}$$

This confirms that $W(x)$ satisfies the conditions for $q = 0$ when $\psi'(0+) > 0$.

Next, consider the case where $q > 0$ or $q = 0$ with $\psi'(0+) < 0$. Define the function

$$W^{(q)}(x) = \mathrm{e}^{\Phi(q)x} W_{\Phi(q)}(x), \tag{3.41}$$

where $W_{\Phi(q)}(x)$ represents the scale function for the modified process $\left(X, P^{\Phi(q)}\right)$. For this process, the Laplace exponent is

$$\psi_{\Phi(q)}(\theta) = \psi(\theta + \Phi(q)) - q, \quad \theta \geq 0, \tag{3.42}$$

with $\psi'_{\Phi(q)}(0+) = \psi'(\Phi(q)) > 0$. The Laplace transform of $W^{(q)}(x)$, valid for $\beta > \Phi(q)$, is given by:

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\beta x} W^{(q)}(x)\mathrm{d}x &= \int_0^\infty \mathrm{e}^{-(\beta - \Phi(q))x} W_{\Phi(q)}(x)\mathrm{d}x \\ &= \tfrac{1}{\psi_{\Phi(q)}(\beta - \Phi(q))} = \tfrac{1}{\psi(\beta) - q}. \end{aligned} \tag{3.43}$$

This establishes the existence of $W^{(q)}(x)$ for $q > 0$ or $q = 0$ with $\psi'(0+) < 0$.

Finally, in the case where $q = 0$ and $\psi'(0+) = 0$, the continuity theorem for Laplace transforms guarantees the existence of $W(x)$, completing the proof. □

3.4.3 Scale Functions Applications in First Passage Problems

Scale functions are fundamental tools for analyzing first passage problems in spectrally negative Lévy processes (SNLPs). They enable precise calculations of probabilities and expectations related to boundary crossings.

First passage times, crucial in stochastic process studies, represent the moments when a process first crosses specific levels. Formally, they are defined as:

$$\tau_a^+ = \inf\{t > 0 : X_t > a\}, \quad \tau_0^- = \inf\{t > 0 : X_t < 0\}, \tag{3.44}$$

where τ_a^+ is the first time X_t exceeds a, and τ_0^- is the first time it drops below zero.

Using scale functions, these first passage probabilities can be expressed compactly, reducing complex boundary-crossing problems to computations involving $W^{(q)}$ and $Z^{(q)}$, as shown in the following theorems.

In Lévy processes, analyzing first passage problems involves decomposing the characteristic exponent $\psi(\lambda)$ of the killed process $\psi_\delta(\lambda) := \psi(\lambda) - \delta$ using Wiener-Hopf factorization. For SNLPs, this yields a unique root:

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) - q = 0\}, \quad q \geq 0. \tag{3.45}$$

Next, we introduce two key theorems used for path decomposition and joint probability calculations of maximum drawdown, maximum drawup, and their durations. Before diving into the theorems and proofs, we first explore two foundational concepts: Lévy processes conditioned to stay positive and creeping.

3.4.3.1 Lévy Process Conditioned to Stay Positive

Understanding the behavior of spectrally negative Lévy processes (SNLPs) under the condition that they remain positive is crucial, particularly as they approach a half-line boundary. Let X be a spectrally negative Lévy process, and recall the definition of the first passage time τ_0^- (see Equation 3.23). Assume $\psi'(0+) \geq 0$, where ψ is the Laplace exponent of the process.

For all $t \geq 0$ and $x, y > 0$, the probability of X_t taking the value y under the condition

that the process has not crossed zero up to time t, can be expressed as:

$$P_x^{\uparrow}\left(X_t \in \mathrm{d}y\right) = \lim_{q \to 0} P_x\left(X_t \in \mathrm{d}y, t < \mathbf{e}/q \mid \tau_0^- > \mathbf{e}/q\right), \tag{3.46}$$

where $\mathbf{e}$ is an independent exponential random variable with unit mean. This probability distribution describes a conservative strong Markov process, often referred to as the spectrally negative Lévy process conditioned to stay positive.

The laws governing such conditioned processes, denoted by $\{P_x^{\uparrow} : x > 0\}$, can be reformulated in terms of the original laws $\{P_x : x > 0\}$ through a classical Doob h-transform. Specifically, for any event $A \in \mathbb{F}_t$ and $x > 0$, the conditional probability is given by:

$$P_x^{\uparrow}(A) = E_x\left(\frac{W\left(X_t\right)}{W(x)} \mathbf{1}_{\{A, \tau_0^- > t\}}\right), \tag{3.47}$$

where $W(x)$ is the scale function associated with the spectrally negative Lévy process. This formulation highlights how the process's paths can be transformed and analyzed under the condition of staying positive.

3.4.3.2 Creeping

When a spectrally negative Lévy process (SNLP) starts at $x > 0$ and enters $(-\infty, 0)$ for the first time, it may do so either by jumping directly below zero or by reaching zero continuously. The latter scenario, where $\{\tau_0^- < \infty\}$ and $X_{\tau_0^-} = 0$, is referred to as "*creeping downwards*" and occurs with positive probability for certain SNLPs.

The occurrence of creeping is not tied to a specific initial value. If creeping is possible at any $y > 0$, then it is also possible for all $x > 0$. Importantly, only SNLPs with a Gaussian component can exhibit this behavior.

Additionally, due to the absence of positive jumps in SNLPs, the process crosses any level above its starting value continuously if it ever reaches that level. Specifically, for $a \geq x$,

$$P_x\left(X_{\tau_a^+} = a,\ \tau_a^+ < \infty\right) = P_x\left(\tau_a^+ < \infty\right), \tag{3.48}$$

where τ_a^+, the first passage time above a, is defined as:

$$\tau_a^+ = \inf\{t \geq 0 : X_t > a\}. \tag{3.49}$$

The "first passage" problems for SNLPs are closely linked to scale functions. Defined via Laplace transforms, these functions take positive values for $x \geq 0$ and are zero for $x < 0$. Their importance in boundary crossing problems is illustrated in the following theorem.

Theorem 3.13 (**First Passage Formulations**). ***(Kuznetsov, 2003, [28])*** *For all $q \geq 0$ and $x < a$, the following hold:*

$$E_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{3.50}$$

$$E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right) = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{3.51}$$

where τ_a^+ and τ_0^- are the first passage times defined in (3.44)*.*

Proof. We will prove the theorem by considering three cases based on the values of q and $\psi'(0+)$.

Case 1: $q = 0$ **and** $\psi'(0+) > 0$

In this case, we utilize the relationship $W(x) = \frac{1}{\psi'(0+)}P_x(\underline{X}_\infty \geq 0)$. Using the law of total probability and the Strong Markov Property, we find for $x \in [0, a]$:

$$\begin{aligned} P_x(\underline{X}_\infty \geq 0) &= E_x\left(P_x(\underline{X}_\infty \geq 0 \mid \mathbb{F}_{\tau_a^+})\right) \\ &= E_x\left(\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}P_a(\underline{X}_\infty \geq 0)\right) + E_x\left(\mathbf{1}_{\{\tau_a^+>\tau_0^-\}}P_{X_{\tau_0^-}}(\underline{X}_\infty \geq 0)\right). \end{aligned} \tag{3.52}$$

For the first term, we observe:

$$E_x\left(\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}P_a(\underline{X}_\infty \geq 0)\right) = P_a(\underline{X}_\infty \geq 0)P_x(\tau_a^+ < \tau_0^-). \tag{3.53}$$

The second term simplifies to zero due to the following reasoning: 1. If X has no Gaussian component, it cannot creep downwards, and $X_{\tau_0^-} < 0$. Consequently, $P_x(\underline{X}_\infty \geq 0) = 0$ for $x < 0$. 2. If X has a Gaussian component, regularity of $(-\infty, 0)$ for 0 ensures that $P(\underline{X}_\infty \geq 0) = 0$.

Thus, we conclude:

$$P_x(\tau_a^+ < \tau_0^-) = \frac{W(x)}{W(a)}. \tag{3.54}$$

The same equality holds for $x < 0$, as both sides are identically zero.

Case 2: $q > 0$

Here, we leverage the definition of the scale function and the fact that $X_{\tau_a^+} = a$. From the Strong Markov Property:

$$\begin{aligned} E_x\left(e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right) &= E_x\left(e^{\Phi(q)(X_{\tau_a^+}-x)-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right)e^{-\Phi(q)(a-x)} \\ &= e^{-\Phi(q)(a-x)}P_x^{\Phi(q)}(\tau_a^+ < \tau_0^-) \\ &= e^{-\Phi(q)(a-x)}\frac{W_{\Phi(q)}(x)}{W_{\Phi(q)}(a)} \\ &= \frac{W^{(q)}(x)}{W^{(q)}(a)}. \end{aligned} \tag{3.55}$$

Case 3: $q = 0$ **and** $\psi'(0+) \leq 0$

For this scenario, we take the limit as $q \to 0$. Using the Continuity Theorem for Laplace transforms and monotone convergence, we obtain:

$$\lim_{q\to 0} E_x\left(e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right) = \frac{W(x)}{W(a)}. \tag{3.56}$$

This completes the proof. □

Theorem 3.14 (**One and Two-Sided Exit Formulae**). ***(Kuznetsov, 2003, [28])*** *For spectrally negative Lévy processes (SNLPs), the following identities hold:*

1. ***One-Sided Exit Formula:*** *For any $x \in \mathbb{R}$ and $q \geq 0$,*

$$E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right) = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x), \tag{3.57}$$

where $q/\Phi(q)$ is interpreted in the limiting sense when $q = 0$. Specifically:

$$P_x\left(\tau_0^- < \infty\right) = \begin{cases} 1 - \psi'(0+)W(x), & \text{if } \psi'(0+) > 0, \\ 1, & \text{if } \psi'(0+) \leq 0. \end{cases} \tag{3.58}$$

2. ***Two-Sided Exit Formula:*** *For $x \leq a$ and $q \geq 0$,*

$$E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right) = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)}. \tag{3.59}$$

Sketch of Proof. We provide a brief outline of the proof for the one- and two-sided exit formulae:

1. For $q > 0$ and $x \geq 0$, start with the relation:

$$E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right) = P_x\left(\mathbf{e}_q > \tau_0^-\right), \tag{3.60}$$

where $\mathbf{e}_q$ is an independent exponentially distributed random variable with rate q. Rewriting this probability in terms of $\underline{X}_{\mathbf{e}_q}$ gives:

$$\begin{aligned} E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right) &= P_x\left(\underline{X}_{\mathbf{e}_q} < 0\right) \\ &= P\left(-\underline{X}_{\mathbf{e}_q} > x\right) \\ &= 1 - P\left(-\underline{X}_{\mathbf{e}_q} \leq x\right). \end{aligned} \tag{3.61}$$

Substituting the expression for the Laplace transform of $W^{(q)}(x)$ yields:

$$E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right) = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x). \tag{3.62}$$

The case $q = 0$ follows taking the limits as $q \to 0$.

2. For $q > 0$ and $x \leq a$, consider the two-sided exit probability:

$$E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right) = E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right) - E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right). \tag{3.63}$$

By the Strong Markov Property at τ_a^+ and the fact that X creeps upwards, we have:

$$E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right) = E_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right) E_a\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right). \tag{3.64}$$

Combining these results gives:

$$E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}\right) = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)}. \tag{3.65}$$

The case $q = 0$ follows by again taking the limits as $q \to 0$ on both sides of the equation. □

3.4.4 Scale Functions and Excursion Theory

Excursion theory provides a structured framework for analyzing the behavior of spectrally negative Lévy processes (SNLPs) as they deviate from their running maximum. It focuses on the segments of the process that leave and return to the maximum, offering insights into boundary-crossing problems and their connection to scale functions.

For SNLPs, the running maximum, denoted $L = \{\bar{X}_t : t \geq 0\}$, increases only when X reaches a new maximum. The inverse local time, $L_t^{-1} = \inf\{s > 0 : L_s > t\}$, identifies the real time when X first exceeds level t. This is equivalent to the first passage time, $L_t^{-1} = \tau_t^+$.

Excursions occur whenever $\Delta L_t^{-1} > 0$, marking intervals where X departs from and returns to the maximum. These excursions, represented by ε_t, form a Poisson point process $\{(t, \varepsilon_t)\}$ with the intensity measure $\mathrm{d}t \times n$, where n is the excursion measure. Each excursion captures the path of X during a departure from its maximum until it returns.

The connection between scale functions and excursion theory is established through the probability of remaining above zero. Let ζ denote the lifetime and $\bar{\varepsilon}$ the height of an excursion ε. For $a > x \geq 0$, the probability that $\underline{X}_{\tau_a^+} \geq 0$ is expressed in terms of the scale function W as:

$$\frac{W(x)}{W(a)} = P_x(\underline{X}_{\tau_a^+} \geq 0) = \exp\left(-\int_x^a n(\bar{\varepsilon} > t)\,\mathrm{d}t\right), \tag{3.66}$$

where $n(\bar{\varepsilon} > t)$ represents the measure of excursions that exceed height t. This relationship demonstrates how scale functions encapsulate the properties of excursions, highlighting their significance in the analysis of SNLPs and their boundary behaviors.

3.4.5 Doob-*h* Transforms

The Doob-h transform is a method to construct a new probability measure P^h for a Markov process X with an initial value $X_0 = x$, based on a h-harmonic function. This new measure is defined at time t as:

$$P_x^h(A) := e^{-\alpha t} E_x\left(\frac{h(X_t)}{h(x)}; A\right), \tag{3.67}$$

where h is the harmonic function associated with X, and α is a constant. The resulting process, called X^h, is the Doob-h transform of X. Under P^h, the future path of X is adjusted according to the weighting provided by h.

The probability semigroup P_t^h of X^h is given by:

$$P_t^h f(x) := E_x^h(f(X_t)) = e^{-\alpha t} E_x\left(f(X_t)\frac{h(X_t)}{h(x)}\right), \tag{3.68}$$

for bounded and real-valued functions f. In terms of the density function $p(t; x, y)$ of the original process and its speed measure m, which represents the relative velocity of the process across states, this can be written as:

$$P_t^h f(x) = \int e^{-\alpha t} p(t; x, y) f(y) \frac{h(y)}{h(x)} \, m(\mathrm{d}y). \tag{3.69}$$

The function h depends on the information available about the process up to time t, effectively conditioning the process at that moment. This transform allows for the adjustment of the behavior of X by re-weighting its probabilities, providing a powerful tool in the analysis of Markov processes.

CHAPTER 4

METHODOLOGY

In this chapter, the main approaches and technical tools are designed and explained to solve the targeted problems, and the mathematical methods to be used are described in more detail.

4.1 The Method to be Used for Determining the Joint Distribution of the Maximum Drawdown and Maximum Gain

As explained in the Introduction and Literature Review chapters, the distribution of the path prior to the infimum has been an open question in the literature [61]. Since the trajectory of a Lévy process is not continuous like that of Brownian motion but involves jumps, applying the methods used for Brownian motion becomes challenging, and finding the distribution requires new approaches. In this thesis, to achieve the goal of determining the joint distribution of the maximum drawdown and the maximum gain, this part will be analyzed using the Doob-h transform method. By doing so, the main methodological steps previously successfully applied to the Brownian motion by [52] will be followed. The methods and results of [58], [4], [5], [8], [61], [51] are also used. In summary,

- The distribution of the path prior to the infimum value, given the condition that the infimum occurs before the supremum, is determined in Chapter 5.
- Under the condition that the supremum occurs before the infimum, the distributions of the path before the supremum value, the part between the supremum

and infimum values, and the path after the infimum value are found in Chapter 5.

- The maximum drawdowns for the parts pre-infimum, intermediate and post-supremum under the condition that the infimum occurs before the supremum are found in Chapter 6.
- The maximum drawdups for the parts pre-supremum, intermediate and post-infimum under the condition that the supremum occurs before the infimum are found in Chapter 6.
- The joint distribution is calculated in Chapter 7 by determining and summing the probabilities of events that define the joint values of the maximum drawdown and maximum gain.

These steps, along with the probability methods to be used, are explained in more detail below:

4.1.1 Finding the distributions of the paths given the condition that the infimum occurs before the supremum

The case where the infimum occurs before the supremum: $H_I < H_S$

The path of the Lévy process will be divided into three parts based on the infimum and supremum formed up to the random variable T, which follows an exponential distribution. The conditional independence of each part from the others will be demonstrated, and their distributions will be determined.

Using the W-scale and Z-scale functions, [61] derived the expressions provided in Equations 2.18 and 2.19:

$$P\{M^-_{H_I,H_S} < d \mid H_I < H_S, I_T = a, S_T = b\}$$

and

$$P\{M^-_{H_S,T} < d \mid H_I < H_S, I_T = a, S_T = b\}.$$

However, regarding the process prior to the infimum, the probability

$$P\{M^-_{0,H_I} < d \mid H_I < H_S, I_T = a, S_T = b\} \tag{4.1}$$

remained an open problem. In our study, to address this, we derive the probability $P\{M^-_{0,H_I} < d \mid I_T = a\}$ using Doob-h transform by considering the hitting time rather than the passage time of Lévy process. It is possible to characterize the distribution of this process, given the event $\{I_T = a\}$, using probabilistic investigations [31].

The hitting time of the process at specific levels can be obtained using the W- and Z-scale functions.

The results obtained to this point correspond to the generalization of the results in ([52][Thm. 3.5, Thm. 4.1]) to the Lévy process. Based on the conditional results, the probability $P\{H_I < H_S, I_T \leq a, S_T \leq b\}$, that is, the intersection of the event $\{H_I < H_S\}$ with the joint distribution of the infimum and supremum, will be determined in Chapter 7. This is a computation related to the condition, and it will be used below to eliminate the conditional probability.

4.1.2 Finding the distributions of the paths given the condition that the supremum occurs before the infimum

The case where the supremum occurs before the infimum: $H_S < H_I$

In this case, the work in subsection 4.1.1 is repeated to obtain the probabilities

$$P\{M^-_{H_S,H_I} < d \mid H_S < H_I, I_T = a, S_T = b\},$$

$$P\{M^-_{H_I,T} < d \mid H_S < H_I, I_T = a, S_T = b\},$$

$$P\{M^-_{0,H_S} < d \mid H_S < H_I, I_T = a, S_T = b\}.$$

Here, since the symmetry present in driftless Brownian motion does not exist for the Lévy process, each part must be analyzed individually. However, the methods used in part 1 and in [61] will still apply here.

One of the most important methods used to determine the distribution of each part is the Doob-h transform. For example, the h function in Eqn. 2.12, $\kappa_{a,d}$, determines the distribution of the trajectory segment after a and is used to prove the Laplace transform stated in Proposition 2.1, i.e., to characterize the distribution of the drop

duration T_d. Similarly, to determine the above probabilities, the corresponding h-transforms are first derived as in [61].

4.1.3 Finding the joint distribution of the maximum drawdown and the maximum gain.

The expression that determines this distribution will be calculated by evaluating and summing the probabilities of distinctively expressible events. First, the probability

$$P\{M_T^- < \alpha, M_T^+ < \beta\}, \tag{4.2}$$

where $\alpha, \beta \in \mathbb{R}$, which defines the joint distribution, can be written as the sum of the probabilities of two distinct events:

$$P\{M_T^- < \alpha, M_T^+ < \beta, H_I < H_S\} + P\{M_T^- < \alpha, M_T^+ < \beta, H_S < H_I\}. \tag{4.3}$$

The calculations of these probabilities will be performed separately for Lévy processes due to the lack of symmetry. The first probability, in the case where $\{H_I < H_S\}$, corresponds to the joint distributions of M_T^- and M_T^+ and can be characterized using a bounded function ϕ as follows:

$$E[\phi(M_T^+); M_T^- < \alpha, H_I < H_S] = E[\phi(S_T - I_T); M_{0,H_I}^- < \alpha, \\ M_{H_I,H_S}^- < \alpha, M_{H_S,T}^- < \alpha, H_I < H_S]. \tag{4.4}$$

Here, the equality $M_T^+ = S_T - I_T$ is considered when $H_I < H_S$. Due to the conditional independence shown in subsection 4.1.1, given I_T, S_T, and $H_I < H_S$, the variables M_{0,H_I}^-, M_{H_I,H_S}^-, and $M_{H_S,T}^-$ are independent. Therefore, the above expected value can be computed as a product, and the other probability in the sum can similarly be decomposed and analyzed.

Finally, by demonstrating the equivalence

$$H_I < H_S \iff M_T^+ \geq M_T^-, \tag{4.5}$$

the distribution in Eqn. 4.2 can be determined. The result will be obtained by considering the orderings $\alpha \leq \beta$ and $\beta \leq \alpha$ ([52], Proposition 4.5.) For example, starting from the expected value

$$E[\phi(M_T^+); M_T^- < \alpha, H_I < H_S], \tag{4.6}$$

and expressing the relevant probability containing the event $\{M_T^+ < \beta\}$, the transition to Eqn. 4.2 can be made using the equality

$$P\{M_T^+ < \beta, M_T^- < \alpha, H_I < H_S\} = P\{M_T^+ < \beta, M_T^- < \alpha, M_T^+ \geq M_T^-\}. \quad (4.7)$$

Using the joint distribution, the correlation of the variables M_T^+ and M_T^- will be determined.

4.2 Methodologies for the Planned Future Works

This section outlines the methodological approaches intended to be employed in the continuation of this research, addressing extensions and open problems identified throughout the thesis.

4.2.1 The Method to be Used for Determining the Joint Distribution of the Maximum Drawdown, Maximum Gain and Their Durations

To reach the result related to the drawdown duration corresponding to the maximum drawdown, the trajectory decomposition method will be applied once more. By using the conditional independence of the parts, the largest of the three drawdowns will be analyzed, and results regarding the maximum drawdown are obtained. Before this, the joint distribution of the drawdown and its duration within each part will be determined, similar to the joint distributions given by [4] , and considering the time G_t defined in [35]. Using the joint distribution,

- The joint distributions of the maximum drawdown duration and M_T^-,
- The joint distributions of the maximum gain duration and M_T^+,

will be determined.

4.2.2 The Methodology for Financial Application

As part of future work, the theoretical findings of this thesis are planned to be applied in a financial context. The following outlines the tasks and corresponding method-

ologies.

4.2.2.1 Data Selection

The implementation of the results obtained in this project will be carried out by selecting stocks traded on the BIST-30 index, such as İş-C, Ereğli Demir Çelik, and Anadolu Efes Holding. These stocks are chosen because they are among the most actively traded stocks in Turkey and are less affected by manipulative movements.

In short-term investments, volatility or Value-at-Risk (VaR) is generally used as a risk measure. However, for long-term investors, the maximum drawdown and the duration of the maximum drawdown are frequently used risk measures. Stocks traded on the BIST-30 index typically attract long-term investors who prioritize risk management, hedging, and understanding the duration of risks over short-term speculations.

For these reasons and because our aim is to contribute to our country by conducting research with scientific methods, we plan to analyze stocks with high trading volumes. By examining historical BIST-30 data, we will use a dataset that satisfies the properties of Lévy processes, which is the subject of our research.

We will prioritize data with high frequency, high trading volume, and a long-term time horizon. In the long-term interval, the information on maximum drawdown, maximum gain, and their durations not only assists in determining optimal trading times but also holds significance in financial concepts such as supercycles and business cycles.

4.2.2.2 Verifying that the data set satisfies the properties of Lévy processes

Financial data sets have been observed to exhibit sudden negative jumps due to events such as the 2008 global economic crisis, sudden fluctuations in exchange rates, sharp declines caused by COVID-19, and economic conditions. For this reason, it is well known that the Lévy process model is a much more general model for explaining the logarithms of financial data compared to diffusion processes and Brownian motion processes. In other words, financial data sets tend to behave according to the Lévy

model when their logarithms are taken.

To identify which datasets align well with the proposed model, we will check whether the data meet the model's assumptions, as outlined below:

- **Independent Increments Property:** To test the independence of increments, serial correlation diagrams will be used using the Partial Autocorrelation Function (PACF) and Box-Pierce tests. The data set with independent increments or those closest to independence will be selected for analysis.

- **Stationarity Property:** To test for stationarity, the commonly used KPSS test and ADF test will be applied to the logarithmic returns. It is well-known that the logarithmic (sequential interval) returns of asset prices always exhibit stationarity. However, it is also required that the logarithms of asset prices themselves satisfy the stationarity property. Different intervals where the dataset exhibits stationarity will be analyzed, and in the long term, a high-frequency and high-volume dataset that satisfies this assumption will be used.

- **Jump Test:** In the literature, there are tests designed to determine whether an observed increment in the process is a jump. Mancini (2004) introduced a threshold test that identifies a movement as a jump if it exceeds a certain threshold. Ait-Sahalia and Jacod (2007) proposed a non-parametric test to determine whether increments in the process are jumps. Ait-Sahalia and Jacod (2011) further introduced a test to distinguish between finite-activity jumps and infinite-activity jumps. Ren (2013) contributed a jump test to the literature that not only includes the frequency and size distribution of jumps but also classifies increments and provides a method for parameter estimation for Lévy processes.

  In this method, the following alternative expression of the Lévy characteristic exponent is used to estimate the parameter b and the jump function G:

$$\gamma(u) = ibu + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1}{1+x^2} x^2 G(dx). \tag{4.8}$$

4.2.2.3 Parameter Estimation

In Lévy models, the parameters to be estimated include the instantaneous expected return, volatility, the expected value of the jump size, the standard deviation of the jump size, and the jump intensity. To estimate these parameters, methods such as Maximum Likelihood Estimation, the Method of Moments, and Bayesian approaches are commonly used. We will select the most suitable method among these and perform the specified parameter estimations.

4.2.2.4 Validating the Model

To validate the proposed model, we will select financial data from high trading volume datasets such as the Ornstein-Uhlenbeck process, Non-Homogeneous Geometric Brownian Motion, the Peral-Verhulst process, the Schwartz process, the Cox-Ingersoll-Ross process, Jump-Diffusion models, Geometric Brownian Motion, Infinite-Activity Lévy Processes, Fractional Brownian Motion, and Spectrally Negative Lévy Processes. Using these datasets, parameter estimations will be performed according to the methodologies outlined above.

To ensure and demonstrate that the proposed model is appropriate and effective, we will generate data using the proposed model and alternative jump-diffusion and Brownian motion models commonly used in the literature. The generated data will then be compared with market prices by calculating the mean squared Monte Carlo errors. This approach will enable a comparison of the models, and we plan to show that the model with the smallest error, for the selected dataset, is the Spectrally Negative Lévy model.

In this thesis, the validation steps for the application phase will again follow similar steps, working with a suitable dataset for Lévy processes.

Time series models are also frequently used to model financial data [23], [17], [27], [1]. Time-series models provide good predictions for short- and medium-term forecasts. However, the variables of interest in this study are the maximum drawdown/-maximum gain values and their durations. Since it is not possible to detect when

these variables will occur or be completed using time series methods, and because these variables are related to analyzing the probabilistic properties of the trajectories of the models, stochastic processes are preferred over time series methods to explicitly define them.

These variables can be modeled using different stochastic processes [26], [41], [48]. Since the theoretical results we will obtain in this project are specific to Spectrally Negative Lévy Processes, we aim to use high-frequency datasets that are most suitable for this model.

4.2.2.5 Application for Determining Buy and Sell Times

The theoretical results we aim to achieve in this study to determine buy and sell times include:

- The joint distributions and correlation coefficient of the variables M_T^+ and M_T^-,
- The joint distributions of the maximum drawdown duration and M_T^-,
- The joint distributions of the maximum gain duration and M_T^+.

The theoretical determination of the joint distributions of these variables, along with the calculation of their correlation coefficient, explicitly characterizes their joint behavior. For example, all joint distributions can help address practical questions such as whether, on average, the maximum drawdown duration is shorter than the maximum gain duration.

Additionally, the characterization of the joint distribution allows for the calculation of the conditional expected value or other probabilistic computations when the value of one variable is known. For instance, it helps answer questions such as: "How long does the maximum drawdown take on average if its magnitude is known?" or "What is the expected magnitude of the maximum drawdown during a specific investment period?"

4.2.2.6 Application of Theoretical Results in the Implementation Phase

We will use these theoretical results during the application phase as described below:

1. First, for the dataset modeled by the Spectrally Negative Lévy process, we will determine the maximum drawdown and maximum gain values that occurred during that period, along with their respective durations.

2. Next, using our theoretical results, we will calculate the conditional expected maximum gain corresponding to the maximum drawdown observed in the data.

3. Then, we will compare the observed maximum gain from the dataset with the theoretically calculated conditional expected value. Based on this comparison, we can decide whether the maximum gain has been completed or is still ongoing. If the observed gain value is greater than or equal to the theoretically calculated conditional expected value, this provides additional information to the investor for determining the sell time.

4. Finally, we will apply the first three steps to compare the maximum drawdown observed in the dataset with the theoretically calculated conditional expected maximum drawdown. This comparison will provide useful information for determining the buy time.

5. Using the joint distributions of the maximum drawdown duration and M_T^-, as well as the maximum gain duration and M_T^+, theoretical results and observed historical values from the dataset will be used to calculate the conditional theoretical expected values of other variables. For example, the conditional expected duration corresponding to the observed maximum drawdown value in the dataset will be calculated. These results will be compared with observations from real data and used to determine buy/sell times.

4.2.3 Application on Fair Pricing of Insurance Contracts

In the study [43], a new insurance contract providing protection during the drawdown period is defined as follows:

$$f(y,p) = E\left[y - \int_0^{\alpha_d} e^{-rt} p\, dt + \kappa e^{-r\alpha_d}\right]. \tag{4.9}$$

Such an insurance contract is one in which the buyer continuously pays a fixed premium $p \geq 0$ until a drawdown of size $d > 0$ in the logarithm of the asset price occurs. Upon purchasing this contract, the insured receives an amount $\kappa \geq 0$ during the drawdown period, thereby being protected against the drawdown. The fair price of the contract today, considering a risk-free interest rate $r \geq 0$, is equal to the present value of the future cash flows.

In this contract, the insured considers that even if the price increases after the first drawdown of size e^d, it would not yield sufficient profit and thus agrees to enter such a contract to mitigate the loss by receiving $\kappa > 0$ during the drawdown period. In the above-defined insurance contract, the buyer starts the contract under the condition of the drawdown variable $Y_0 = y, 0 < y < d$, and the contract function can also be expressed as:

$$f(y,p) = \left(\frac{p}{r} + \kappa\right)\xi(y) - \frac{p}{r}, \tag{4.10}$$

where $\xi(y) := E_{|y}[e^{-r\alpha_d}]$ is the Laplace transform of α_d under the condition $Y_0 = y, 0 < y < d$.

The fair price, or fair premium payment p^*, of such a contract for both the insurance company and the buyer can be determined by solving $f(y,p^*) = 0$ at the time $\tau_D^+(a)$, as seen from the formula. To find the fair premium p^*, the conditional Laplace transform $\xi(y)$ must be expressed in terms of scale functions.

Now, note that the required conditional Laplace transform (drawdown duration under the condition of drawdown magnitude) can only be obtained through the joint distribution of the drawdown magnitude and duration.

In our future studies, as an application of the joint distributions determined during the thesis, we will define new insurance contracts inspired by the definition in Equation 4.9 that aim to protect the individual against the largest possible loss. These con-

tracts will be defined in the following different forms, and we will derive the explicit formulas for the fair premiums for each:

$$f(m,p) := E_{|m_d}\left[-\int_0^T e^{-rt}p\,dt + \alpha e^{-rT}\right], \tag{4.11}$$

$$f(m,p) := E_{|m_d}\left[-\int_0^T e^{-rt}p\,dt + \alpha(T) e^{-rT}\right], \tag{4.12}$$

$$f(m,p) := E_{|m_d}\left[-\int_0^T e^{-rt}p\,dt + \alpha(M_T^+) e^{-rT}\right]. \tag{4.13}$$

In the literature, for all these newly defined insurance contracts, the condition $M_T^- = m$ can be taken as the largest drawdown value anticipated by the contract buyer. Let time 0 denote the contract's starting time, and T represent the duration during which the specified largest drawdown occurs.

During this time:

- The first contract provides protection/compensation at a fixed amount α,
- The second contract provides protection as a function of the value of the largest drawdown at the end of the duration, $\alpha(M_T^-)$,
- The final contract provides protection as a function of the value of the largest gain at the end of the duration, $\alpha(M_T^+)$.

As in the example above, for all these contracts, the fair premium p^* can be obtained by solving $f(m,p^*) = 0$ at time T. To achieve this solution:

- For the first two contracts, the Laplace transform and conditional expected value of the maximum drawdown duration, given the condition of the maximum drawdown value, are required.
- For the proposed final contract, the conditional expected values of the maximum gain and the maximum drawdown duration, given the maximum drawdown value, must be calculated.

Using the theoretical joint distributions obtained in this thesis, we will determine the fair premium p^* for these contracts. (Note that insurance contracts defined for risk management purposes can be diversified.)

Subsequently, by substituting the parameter estimates obtained from the dataset modeled as a Spectrally Negative Lévy process for the logarithms into the derived expressions, we will obtain concrete values for p^*.

CHAPTER 5

PATH DECOMPOSITION AT EXTREMA

In this chapter, we revisit the study of [61], which provides a detailed characterization of the post-infimum, pre-supremum, and post-supremum segments of a spectrally negative Lévy process up to an independent exponential time T. Their analysis also includes the intermediate and post-supremum processes under the condition that the infimum occurs before the supremum. However, their framework does not address the distribution of the pre-infimum path. Building upon their foundation, we fill this gap by characterizing the pre-infimum segment under the same condition. Furthermore, we extend the decomposition by considering the complementary case where the supremum occurs before the infimum. In this setting, we derive the characterizations of the pre-supremum, intermediate, and post-infimum processes. As a result, we provide a complete description of the law of all path parts under both possible orderings of the extrema.

In Section 5.1, we provide, for the first time, a complete characterization of the pre-infimum part of a spectrally negative Lévy process. Furthermore, we derive novel distributional results regarding the extremal behavior of the process: (i) the distribution of the infimum up to the time of the supremum, (ii) the distribution of the infimum after the supremum, and (iii) the distribution of the supremum up to the time of the infimum. These results will be used in the computation of the joint distribution of the maximum drawdown and drawup in the following sections.

In Section 5.2, we provide a solution to the open problem concerning the characterization of the pre-infimum part, under the condition that the infimum occurs before the supremum.

In Section 5.3, under the condition that the supremum occurs before the infimum, we provide—for the first time—the characterizations of the pre-supremum, intermediate, and post-infimum parts. In contrast to the Brownian motion case, where symmetry eliminates the need for such detailed computations, we must obtain them explicitly in the Lévy setting to compute the joint distribution.

5.1 Path Decomposition at Extrema; Distributions of Pre-/Post-Supremum and Pre-/Post-Infimum Processes

In this section, we investigated the study of of C. Vardar-Acar et. al. [61], *Maximum Drawdown and Drawdown Duration of Spectrally Negative Lévy Processes Decomposed at Extremes*, in which the distributions of the pre-/post-supremum of the post-infimum process and the maximum drawdown of the pre-/post-supremum, post-infimum processes and the intermediate processes were obtained together with the law of drawdown durations. Path decomposition is performed to characterize the law of the pre-/post-supremum, post-infimum and the intermediate processes of a spectrally negative Lévy process taken up to an independent exponential time T. In their study, mainly the distributions of the supremum of the post-infimum process and the maximum drawdown of the pre/post-supremum, post-infimum processes and the intermediate processes are obtained together with the law of drawdown durations. We analyzed the theorems, lemmas and proofs in their study. We extend their study by providing the characterization of the pre-infimum part.

The pre-/post- H_S and pre-/post- H_I processes are defined respectively by

$$\begin{cases} X_u : 0 \leq u \leq H_S \\ X_{H_S+u} : 0 \leq u \leq T - H_S \\ X_u : 0 \leq u \leq H_I \\ X_{H_I+u} : 0 \leq u \leq T - H_I \end{cases}$$

In Lemma 5.1, we investigated the distributions of the pre-supremum process $\{X_u : 0 \leq u \leq H_S\}$ and post- supremum process $\{X_{H_S+u} : 0 \leq u \leq T - H_S\}$. Conditioning on the supremum value, we obtain the distribution of the infimum before and after the time of the supremum in addition to the results of [61].

Lemma 5.1. ***(C. Vardar-Acar, 2021, [61])*** *Conditionally on $S_T = b$, it holds that:*

a) The pre-H_S process and the post-H_S process are independent.

b) The law of pre-H_S process is an h-transform of the law of the spectrally negative Lévy process killed at $T \wedge \tau_b^+$, with

$$h(x) = e^{-\Phi(\gamma)(b-x)}, \quad x \leq b.$$

More precisely, it is a spectrally negative Lévy process killed when it hits b, and governed by an Esscher exponential change of measure $\mathbb{P}^{\Phi(\gamma)}$ of the original measure, with

$$\left.\frac{d\mathbb{P}^{\Phi(\gamma)}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\Phi(\gamma)X_t - \gamma t} \tag{5.1}$$

and with Laplace exponent

$$\widetilde{\psi}(\lambda) = \psi_{\Phi(\gamma)}(\lambda) := \psi(\lambda + \Phi(\gamma)) - \gamma, \quad \lambda \geq -\Phi(\gamma).$$

Moreover, the distribution of the infimum of the pre-supremum process, I_{0,H_S}, when $S_T = b$ is given by

$$\mathbb{P}\left(I_{0,H_S} \leq a \mid S_T = b\right) = 1 - \frac{W^{\Phi(\gamma)}(-a)}{W^{\Phi(\gamma}(b-a))} \tag{5.2}$$

for $b > a$.

c) The law of the post-supremum process is a h-transform of the spectrally negative Lévy process killed at $T \wedge \tau_b^+$ with

$$h(x) = 1 - e^{-\Phi(\gamma)(b-x)}, \quad x \leq b.$$

Equivalently, the post- H_S process is equal in distribution to $b + \Gamma$, where the law of Γ is the h-transform of the law of a spectrally negative Lévy process conditioned to stay negative and killed at $T \wedge \tau_0^+$.

Moreover, the distribution of the infimum of the post-supremum process, $I_{H_S,T}$, when $S_T = b$ is given by

$$\mathbb{P}\left(I_{H_S,T} \leq a \mid S_T = b\right) = -\frac{\gamma}{\Phi(\gamma)} W^{(\gamma)}(b) + Z^{(\gamma)}(a) \frac{W^{(\gamma)}(b)}{W^{(\gamma)}(a)} \tag{5.3}$$

for $b > a$.

Proof. a) Note that $X_{H_S-} = S_T$ as X is spectrally negative. Since we have assumed that X is of unbounded variation, which in turn is equivalent to regularity of 0 for $(-\infty, 0)$. For Markov processes yields the independence conditionally on S_T.

b) The pre-H_S process evolves as a spectrally negative Lévy process that stays below b before an exponential time T, given that the supremum is reached during the exponential horizon. Its transition function P_t can be identified by

$$P_t(x,\ \mathrm{d}y) = \mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_b^+ \wedge T \mid \tau_b^+ < T\right\}, y \leq b \tag{5.4}$$

which is equal to

$$\begin{aligned}
&\frac{\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+, \tau_b^+ < T\right\}}{\mathbb{P}_x\left\{\tau_b^+ < T\right\}} \\
&= \frac{\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+\right\}\mathbb{P}_x\left\{\tau_b^+ < T \mid X_t = y, t < T \wedge \tau_b^+\right\}}{\mathbb{P}_x\left\{\tau_b^+ < T\right\}} \\
&= \frac{\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+\right\}\mathbb{P}_y\left\{T \circ \theta_t > \tau_b^+ \circ \theta_t\right\}}{\mathbb{P}_x\left\{T > \tau_b^+\right\}} \\
&= \frac{\mathbb{P}_y\left\{T > \tau_b^+\right\}}{\mathbb{P}_x\left\{T > \tau_b^+\right\}}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+\right\}
\end{aligned}$$

by Markov property and the fact that T is memoryless, where θ is the shift operator. From 5.4, we have

$$P_t(x,\ \mathrm{d}y) = \frac{h(y)}{h(x)}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+\right\} \tag{5.5}$$

where $h(z) = \mathbb{P}_z\left\{T > \tau_b^+\right\} = e^{-\Phi(\gamma)(b-z)}$ by [29]. Now, note that the entrance law for the pre-supremum process is obtained as $x \to 0$ in 5.5 and is equivalent to the change of measure 5.1 because

$$\begin{aligned}
\lim_{x\to 0} P_t(x,\ \mathrm{d}y) &= \lim_{x\to 0}\frac{e^{-\Phi(\gamma)(b-y)}}{e^{-\Phi(\gamma)(b-x)}}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+\right\} \\
&= e^{\Phi(\gamma)y}\mathbb{P}\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+\right\} \\
&= e^{\Phi(\gamma)y-\gamma t}\mathbb{P}\left\{X_t \in \mathrm{d}y, t < \tau_b^+\right\}
\end{aligned}$$

which is indeed the Esscher exponential change of measure given in 5.1.

$$\mathbb{P}\left(I_{0,H_S} \leq a \mid S_T = b\right) = \mathbb{P}_0^{\Phi(\gamma)}\left(I_{\tau_b^+} \leq a\right) = 1 - \frac{W^{\Phi(\gamma)}(-a)}{W^{\Phi(\gamma)}(-a+b)} \tag{5.6}$$

c) The law of the post- H_S process can be characterized by noting that the post- H_S process evolves as a spectrally negative Lévy process which is conditioned to stay

below b and killed at an exponential time T. That is, we have

$$\begin{aligned}
&\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+ \mid T < \tau_b^+\right\} \\
&= \frac{\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+\right\}\mathbb{P}_x\left\{T < \tau_b^+ \mid X_t = y, t < T \wedge \tau_b^+\right\}}{\mathbb{P}_x\left\{T < \tau_b^+\right\}} \\
&= \frac{\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+\right\}\mathbb{P}_y\left\{T \circ \theta_t < \tau_b^+ \circ \theta_t\right\}}{\mathbb{P}_x\left\{T < \tau_b^+\right\}} \\
&= \frac{\mathbb{P}_y\left\{T < \tau_b^+\right\}}{\mathbb{P}_x\left\{T < \tau_b^+\right\}}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_b^+\right\}
\end{aligned}$$

by Markov property and the fact that T is memoryless. This is the h-transform of the law of a spectrally negative Lévy process killed at the minimum of an exponential time and passage time above b, with

$$\begin{aligned}
h(z) = \mathbb{P}_z\left\{T < \tau_b^+\right\} &= \mathbb{P}\left\{T < \tau_{b-z}^+\right\} \\
&= 1 - \mathbb{E}\left[e^{-\gamma\tau_{b-z}^+}1_{\left\{\tau_{b-z}^+ < \infty\right\}}\right] \\
&= 1 - e^{-\Phi(\gamma)(b-z)}
\end{aligned} \tag{5.7}$$

for $z < b$ [29]. That is, when $S_T = b$, the transition semigroup of the post- H_S process is given by

$$P_t(x,\ \mathrm{d}y) = \frac{h(y)}{h(x)}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T\right\}$$

for $x, y < b$, and the entrance law is obtained as $x \to b-$. Then, it follows that by applying L'Hôpital's Rule.

$$\begin{gathered}
\lim_{x\to b}\mathbb{P}_x\left(I_{H_S,T} > a \mid S_T = b\right) = \lim_{x\to b}\mathbb{P}_x\left(I_{\tau_b^+,T} > a \mid S_T = b\right) = \qquad (5.8) \\
\lim_{x\to b}\mathbb{P}_x^h\left(\tau_b^+ < \tau_a^-, T < \tau_a^-\right) = \lim_{x\to b}\tfrac{1}{h(x)}\mathbb{E}\left(h(X_{\tau_b^+}); \tau_b^+ < \tau_a^-, T < \tau_a^-\right) = \\
\lim_{x\to b}\tfrac{h(b)}{h(x)}\left(\mathbb{E}_x\left[\tau_b^+ < \tau_a^-\right] - \mathbb{E}_x\left[\tau_b^+ < \tau_a^-, \tau_a^- < T\right]\right) = \\
\lim_{x\to b}\tfrac{h(b)}{h(x)}\left[\tfrac{W^{(\gamma)}(x+a)}{W^{(\gamma)}(a+b)} - \mathbb{E}_x\left[\tau_a^- < T\right] + \mathbb{E}_x\left[\tau_a^- < \tau_b^+, \tau_a^- < T\right]\right] = \\
\lim_{x\to b}\tfrac{h(b)}{h(x)}\left[\tfrac{W^{(\gamma)}(x+a)}{W^{(\gamma)}(a+b)} - Z^{(\gamma)}(x) + \tfrac{\gamma}{\Phi(\gamma)}W^{(\gamma)}(x) + Z^{(\gamma)}(x) - Z^{(\gamma)}(a)\tfrac{W^{(\gamma)}(x)}{W^{(\gamma)}(a)}\right] \\
= 1 + \tfrac{\gamma}{\Phi(\gamma)}W^{(\gamma)}(b) - Z^{(\gamma)}(a)\tfrac{W^{(\gamma)}(b)}{W^{(\gamma)}(a)}
\end{gathered}$$

□

The following proposition identifies the law of the pre-infimum and post-infimum process and the distribution of the supremum within these parts of the path given the infimum.

Proposition 5.2. *Conditionally on $I_T = a$, the pre-H_I and post-H_I processes are independent.*

i. The law of the pre-H_I process is an h-transform of the law of the spectrally negative Levy process killed at $\tau_a \wedge T$ with

$$h(x) = \frac{\gamma}{\Phi(\gamma)} W^{(\gamma)\prime}(x-a) - \gamma W^{(\gamma)}(x-a)$$

Moreover, the distribution of the supremum of the pre-infimum process, S_{H_I}, when $I_T = a$ is given by

$$\mathbb{P}_0\left(S_{H_I} > b \mid I_T = a\right) = \frac{\left(W^{(\gamma)\prime}(b-a) - \Phi(\gamma) W^{(\gamma)}(b-a)\right) W^{(\gamma)}(x-a)}{\left(W^{(\gamma)\prime}(x-a) - \Phi(\gamma) W^{(\gamma)}(x-a)\right) W^{(\gamma)}(b-a)}$$

for $b > a$.

ii. The law of the post-H_I process is the h-transform of the law of the spectrally negative Lévy process killed at $T \wedge \tau_a^-$ with

$$h(x) = 1 - Z^{(\gamma)}(x-a) + \frac{\gamma}{\Phi(\gamma)} W^{(\gamma)}(x-a).$$

Moreover, the distribution of the supremum of the post-infimum process, $S_{H_I,T}$, when $I_T = a$ is given by

$$\mathbb{P}\left(S_{H_I,T} > b \mid I_T = a\right) = 1 - \frac{\Phi(\gamma)}{W^{(\gamma)}(b-a)} \int_0^{b-a} W^{(\gamma)}(z) \mathrm{d}z \tag{5.9}$$

for $b > a$.

Proof. i. For $a < 0$ and $x, y > a$ we have;

$$\begin{aligned}
&\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a \mid I_T = a\right\} = \frac{\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_x\left\{X_t \in dy, t < H_I, I_T \leq a\right\}}{\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_x\left\{I_T \leqslant a\right\}} \\
= &\frac{\frac{d}{da}\mathbb{P}_x\left\{I_T \leqslant a \mid X_t \in dy, t < \tau_a \wedge T\right\}\mathbb{P}_x\left\{X_t \in dy, t < \tau_a \wedge T\right\}}{\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_x\left\{I_T \leqslant a\right\}} \\
= &\frac{\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_y\left\{I_T \leqslant a\right\}\mathbb{P}_x\left\{X_t \in dy, t < \tau_a \wedge T\right\}}{\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_x\left\{I_T \leqslant a\right\}} \\
= &\frac{h(y)}{h(x)}\,\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a \wedge T\right\}
\end{aligned}$$

where under $\mathbb{P}_x$, $a \leq x$,

$$\begin{aligned} h(x) &:= \frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_x\{I_T \leq a\} = \frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_{-x}\{-I_T \leq -a\} = \frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}\{-I_T \leq x-a\} = \\ &= \frac{\mathrm{d}}{\mathrm{d}a}\int_{x-a}^{\infty} \frac{\gamma}{\Phi(\gamma)} W^{(\gamma)}(dy) - \gamma W^{(\gamma)}(y)\,\mathrm{dy} = \\ &= \frac{\mathrm{d}}{\mathrm{d}a}\int_{x-a}^{\infty} \frac{\gamma}{\Phi(\gamma)} W^{(\gamma)\prime}(y)dy - \gamma W^{(\gamma)}(y)\,\mathrm{dy} = \\ &= \frac{\gamma}{\Phi(\gamma)} W^{(\gamma)\prime}(x-a) - \gamma W^{(\gamma)}(x-a) \end{aligned}$$

by [29] pg. 239 Eqn. 8.24 assuming that Levy process X is of unbounded variation so that $W^{(\gamma)}$ is differentiable by Lemma 2.4 of [28].

Therefore, the law of the pre-H_I process is an h-transform of the law of the spectrally negative Levy process killed at $\tau_a \wedge T$ with

$$h(x) = \frac{\gamma}{\Phi(\gamma)} W^{(\gamma)\prime}(x-a) - \gamma W^{(\gamma)}(x-a).$$

Recall that by *Theorem 3.1 of [40]*: Let T be a random time, let $I_T = \inf_{0\leq t\leq T} X_t$, and let M be the point in $[0,T]$ where this minimum is achieved. Assume that $P\{M < T\} > 0$. Then,

$$i) P\{X_M = I_T | M < T\} = 1 \quad \text{if 0 is regular for } (0,\infty)$$

$$ii) P\{X_M > I_T | M < T\} = 1 \quad \text{if 0 is not regular for } (0,\infty)$$

Since 0 is regular for $(0,\infty)$ for a spectrally negative Levy process and $P\{H_I < T\} > 0$, it follows from Theorem 3.1 of [40] that

$$X_{H_I} = I_T \tag{5.10}$$

with probability 1, where $H_I = \inf\{t < T : X_t = \underline{X}_t\}$.

Remark 5.1. This can happen in two ways: the process X either jumps into I_T or X is continuous at I_T. If 0 is regular also for $(-\infty, 0)$, which happens when X is of unbounded variation, then X is continuous at I_T ([40], pg.370, Remark after Prop.2.4).

Recall $\tau_a := \inf\{t \geq 0 : X_t = a\}$. Given $\{I_T = a\}$, $X_{H_I} = a$ by (5.10) and also because the minimum before T occurs at a single point in time almost surely by [[40], Prop.2.2]. This justifies the use of τ_a in the previous page. Also note that $\tau_a < T$ when 0 is regular for both $(0,\infty)$ and $(-\infty, 0)$ by [[40], Prop.2.1].

Then, it follows that

$$\begin{aligned}\mathbb{P}_0\left(S_{H_I} > b \mid I_T = a\right) &= \tfrac{1}{h(0)}\mathbb{E}_0\left[h\left(X_{\tau_b^+}\right); \tau_b^+ < \tau_a^-, \tau_b^+ < T\right] \\ &= \frac{h(b)}{h(0)}\mathbb{P}_0^h\left(\tau_b^+ < \tau_a^-, \tau_b^+ < T\right) = \frac{h(b)}{h(0)}\frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b-a)} \\ &= \frac{\frac{\gamma}{\Phi(\gamma)}W^{(\gamma)\prime}(b-a) - \gamma W^{(\gamma)}(b-a)}{\frac{\gamma}{\Phi(\gamma)}W^{(\gamma)\prime}(-a) - \gamma W^{(\gamma)}(-a)}\frac{W^{(\gamma)}(-a)}{W^{(\gamma)}(b-a)}\end{aligned}$$

More generally, we have

$$\begin{aligned}\mathbb{P}_x\left(S_{H_I} > b \mid I_T = a\right) &= \tfrac{1}{h(x)}\mathbb{E}_x\left[h\left(X_{\tau_b^+}\right); \tau_b^+ < \tau_a^-, \tau_b^+ < T\right] \\ &= \frac{h(b)}{h(x)}\mathbb{P}_0^h\left(\tau_b^+ < \tau_a^-, \tau_b^+ < T\right) = \frac{h(b)}{h(x)}\frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)} \\ &= \frac{\frac{\gamma}{\Phi(\gamma)}W^{(\gamma)\prime}(b-a) - \gamma W^{(\gamma)}(b-a)}{\frac{\gamma}{\Phi(\gamma)}W^{(\gamma)\prime}(x-a) - \gamma W^{(\gamma)}(x-a)}\frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)} \\ &= \frac{\left(W^{(\gamma)\prime}(b-a) - \Phi(\gamma)W^{(\gamma)}(b-a)\right)W^{(\gamma)}(x-a)}{\left(W^{(\gamma)\prime}(x-a) - \Phi(\gamma)W^{(\gamma)}(x-a)\right)W^{(\gamma)}(b-a)}.\end{aligned}$$

ii. The post-H_I process is a Markov process with stationary transitions, which depend on the value of the infimum, I_T [40]. The post-H_I process evolves as a spectrally negative Lévy process which is conditioned to stay above I_T and killed at the minimum of the exponential time T and τ_a^-. We can identify its transition function P_t as

$$P_t(x,\ \mathrm{d}y) = \mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_a^- \mid T < \tau_a^-\right\}, \tag{5.11}$$

which is equal to

$$\begin{aligned}&\frac{\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_a^-, T < \tau_a^-\right\}}{\mathbb{P}_x\left\{T < \tau_a^-\right\}} \\ &= \frac{\mathbb{P}_x\left\{T < \tau_a^- \mid X_t = y, t < T \wedge \tau_a^-\right\}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_a^-\right\}}{\mathbb{P}_x\left\{T < \tau_a^-\right\}} \\ &= \frac{\mathbb{P}_y\left\{T \circ \theta_t < \tau_a^- \circ \theta_t\right\}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_a^-\right\}}{\mathbb{P}_x\left\{T < \tau_a^-\right\}} \\ &= \frac{\mathbb{P}_y\left\{T < \tau_a^-\right\}}{\mathbb{P}_x\left\{T < \tau_a^-\right\}}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_a^-\right\}\end{aligned}$$

by Markov property and the fact that T is memoryless, where θ is the shift operator. Let $h(x) = \mathbb{P}_x\left\{T < \tau_a^-\right\}$, which can be evaluated as

$$\begin{aligned}h(x) &= \mathbb{E}_x\left[\mathbb{P}_x\left\{T < \tau_a^- \mid \tau_a^-\right\}\right] = \mathbb{E}_x\left[1 - e^{-\gamma\tau_a^-}1_{\left\{\tau_a^- < \infty\right\}}\right] \\ &= 1 - \mathbb{E}_x\left[e^{-\gamma\tau_a^-}1_{\left\{\tau_a^- < \infty\right\}}\right] = 1 - \mathbb{E}_{x-a}\left[e^{-\gamma\tau_0^-}1_{\left\{\tau_0^- < \infty\right\}}\right] \\ &= 1 - Z^{(\gamma)}(x-a) + \frac{\gamma}{\Phi(\gamma)}W^{(\gamma)}(x-a).\end{aligned}$$

Note that the transition density of the post-infimum process can be written as

$$P_t(x,\ \mathrm{d}y) = \frac{h(y)}{h(x)}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < T \wedge \tau_a^-\right\}$$

for $x, y > a$. Then, it follows that

$$\begin{aligned}
\mathbb{P}\left(S_{H_I,T} > b \mid I_T = a\right) &= \lim_{x\to a}\frac{1}{h(x)}\mathbb{E}_x\left[h\left(X_{\tau_b^+}\right); \tau_b^+ < \tau_a^-, \tau_b^+ < T\right] \\
&= \lim_{x\to a}\frac{h(b)}{h(x)}\frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)} \\
&= \lim_{x\to a}\frac{\frac{\gamma W^{(\gamma)}(b-a)}{\Phi(\gamma)} - \gamma\int_0^{b-a} W^{(\gamma)}(z)\mathrm{d}z}{\frac{\gamma W^{(\gamma)}(x-a)}{\Phi(\gamma)} - \gamma\int_0^{x-a} W^{(\gamma)}(z)\mathrm{d}z}\frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)} \\
&= 1 - \frac{\Phi(\gamma)}{W^{(\gamma)}(b-a)}\int_0^{b-a} W^{(\gamma)}(z)\mathrm{d}z,
\end{aligned}$$

□

Remark 5.2 (Comparison of the pre-H_I process with Salminen & Vallois ([52], Thm 3.2)). In Proposition 5.2, conditioned on $I_T = a$, the pre-H_I process is the Doob-h transform of the spectrally negative Lévy process killed at $\tau_a \wedge T$, with

$$h(x) = \frac{\gamma}{\Phi(\gamma)}W^{(\gamma)\prime}(x-a) - \gamma W^{(\gamma)}(x-a).$$

For the Brownian motion case ($\mu = 0$), the scale functions are given by $W^{(\gamma)}(x) = \frac{2}{\sqrt{2\gamma}}\sinh(\sqrt{2\gamma}x)$ and $W^{(\gamma)\prime}(x) = 2\cosh(\sqrt{2\gamma}x)$. Substituting these into h gives

$$h(x) = \sqrt{2\gamma}\big(\cosh(\sqrt{2\gamma}(x-a)) - \sinh(\sqrt{2\gamma}(x-a))\big) = \sqrt{2\gamma}\,e^{-\sqrt{2\gamma}(x-a)}.$$

The Doob-h transform with $h(x) = e^{-\sqrt{2\gamma}(x-a)}$ (the positive constant $\sqrt{2\gamma}$ does not affect the transform) changes the generator of the Brownian motion from $\mathcal{L}f = \frac{1}{2}f''$ to

$$\mathcal{L}^h f = \frac{1}{h}\mathcal{L}(hf) = \tfrac{1}{2}f'' - \sqrt{2\gamma}f',$$

which is the generator of a Brownian motion with drift $-\sqrt{2\gamma}$ killed upon hitting a. This coincides exactly with the result of Salminen & Vallois ([52], Theorem 3.2).

5.2 Path Decomposition at Extrema Given the Condition that the Infimum Occurs Before the Supremum

We now identify the conditional law of the decomposed paths pre-infimum, between the infimum and the supremum, and post-supremum in the following proposition. We

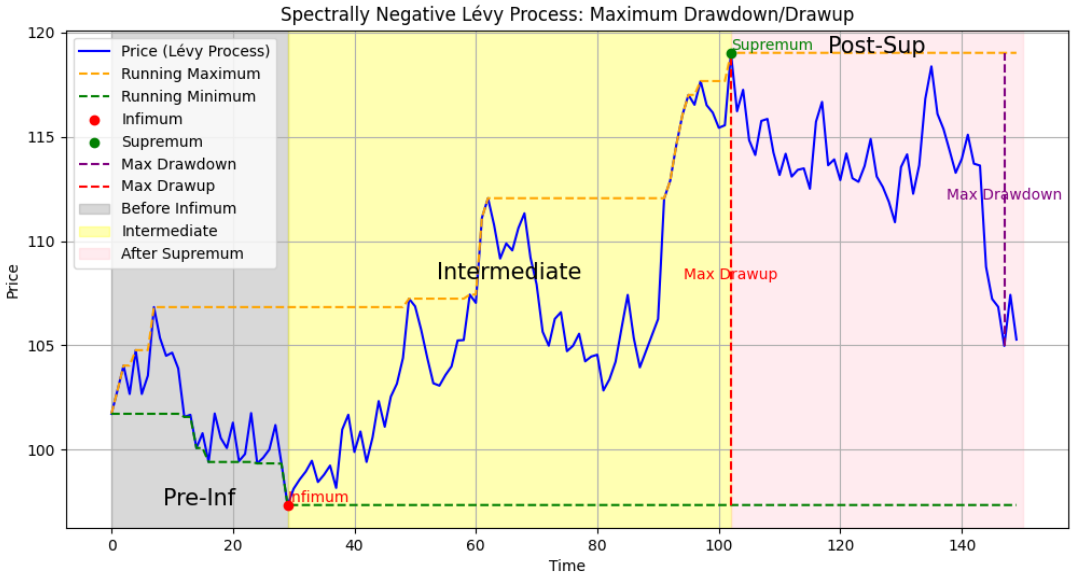


Figure 5.1: Path Decomposition at Extrema Given the Condition that the Infimum Occurs Before the Supremum

call the process $\{X_{H_I+u} : 0 \leq u \leq H_S - H_I\}$ as the intermediate process.

Proposition 5.3. *Conditionally on $H_I < H_S, I_T = a, S_T = b$, the pre-$H_I$, the intermediate process and the post-H_S process are independent and have the following distributions:*

i. the law of the pre-infimum process is the h-transform of the law of the spectrally negative Lévy process with Laplace exponent $\widetilde{\psi}$ killed at the first hitting time of a before the exponential time T where

$$h(x) = e^{-\Phi(\gamma)(b-x)} \frac{W'^{\Phi(\gamma)}(x-a)W^{\Phi(\gamma)}(b-a) - W'^{\Phi(\gamma)}(b-a)W^{\Phi(\gamma)}(x-a)}{(W^{\Phi(\gamma)}(b-a))^2}.$$

ii. the law of the intermediate process is that of $a + \Gamma$, where the law of Γ is the h-transform of the spectrally negative Lévy process killed at τ^+_{b-a}, with

$$h(x) = e^{-\Phi(\gamma)(b-x)} \frac{W^{(\gamma)}(x)}{W^{(\gamma)}(b)}$$

iii. the law of the post-H_S process is the h-transform of the law of the spectrally negative Lévy process killed at $T \wedge \tau^+_b \wedge \tau^-_a$ with

$$h(x) = 1 - Z^{(\gamma)}(x-a) + \left(Z^{(\gamma)}(b-a) - 1\right) \frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)}.$$

Proof. i. By Lemma 5.1, the pre-H_S process is a Lévy process with Laplace exponent $\widetilde{\psi}$. Let $P^{\Phi(\gamma)}$ denote the probability corresponding to this change of measure. The

infimum of the presupremum process is also equal to a. Under the given conditions, note that the intermediate process is independent from the preinfimum process and both governed by $P^{\Phi(\gamma)}$. Therefore, the preinfimum process starts at 0 and evolves independently from the other parts as a process killed at the first hitting time at a before the exponential time T under the condition that it stays below b, and hits a before going above b.

For $a < x, y < b$, we have

$$\begin{aligned}
P_t(x, dy) &= P_x\{X_t \in dy, t < H_I \,|\, H_I < H_S, I_T = a, S_T = b\} \\
&= \frac{\frac{d}{da}P_x\{X_t \in dy, t < H_I, H_I < H_S, I_T \leq a \,|\, S_T = b\}}{\frac{d}{da}P_x\{H_I < H_S, I_T \leq a \,|\, S_T = b\}} \\
&= \frac{\frac{d}{da}P_x^{\Phi(\gamma)}\{I_{\tau_b^+} \leq a \,|X_t \in dy, t < \tau_a\}P_x^{\Phi(\gamma)}\{X_t \in dy, t < \tau_a\}}{\frac{d}{da}P_x^{\Phi(\gamma)}\{I_{\tau_b^+} \leq a\}} \\
&= \frac{\frac{d}{da}P_y^{\Phi(\gamma)}\{I_{\tau_b^+} \leq a\}}{\frac{d}{da}P_x^{\Phi(\gamma)}\{I_{\tau_b^+} \leq a\}}P_x^{\Phi(\gamma)}\{X_t \in dy, t < \tau_a\} \\
&= \frac{\frac{d}{da}P_y^{\Phi(\gamma)}\{I_{\tau_b^+} \leq a\}}{\frac{d}{da}P_x^{\Phi(\gamma)}\{I_{\tau_b^+} \leq a\}}\frac{e^{-\Phi(\gamma)(b-y)}}{e^{-\Phi(\gamma)(b-x)}}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a\right\}.
\end{aligned}$$

where

$$\frac{d}{da}P_x^{\Phi(\gamma)}\{I_{\tau_b^+} \leq a\} = \frac{d}{da}P_x\{H_I < H_S, I_T \leq a \,|\, S_T = b\}.$$

By [60], the result given in the proof of [Theorem 1, ii.)] we have

$$1 - \mathbb{P}^{\Phi(\gamma)}\left(I_{\tau_b^+} \leq a\right) = \frac{W^{\Phi(\gamma)}(-a)}{W^{\Phi(\gamma)}(-a+b)}.$$

Therefore;

$$\frac{d}{da}\mathbb{P}^{\Phi(\gamma)}\left(I_{\tau_b^+} \leq a\right) = \frac{W'^{\Phi(\gamma)}(-a)W^{\Phi(\gamma)}(b-a) - W'^{\Phi(\gamma)}(b-a)W^{\Phi(\gamma)}(-a)}{(W^{\Phi(\gamma)}(b-a))^2}.$$

$$\mathbb{P}_x^{\Phi(\gamma)}\left(I_{\tau_b^+} \leq a\right) = \mathbb{P}_0^{\Phi(\gamma)}\left(I_{\tau_{b-x}^+} \leq a-x\right) = 1 - \frac{W^{\Phi(\gamma)}(x-a)}{W^{\Phi(\gamma)}(b-a)}.$$

$$\frac{d}{da}\mathbb{P}_x^{\Phi(\gamma)}\left(I_{\tau_b^+} \leq a\right) = \frac{W'^{\Phi(\gamma)}(x-a)W^{\Phi(\gamma)}(b-a) - W'^{\Phi(\gamma)}(b-a)W^{\Phi(\gamma)}(x-a)}{(W^{\Phi(\gamma)}(b-a))^2}.$$

ii. By Lemma 5.1, the pre-H_S process is a Lévy process with Laplace exponent $\widetilde{\psi}$. Let $\mathbb{P}^{\Phi(\gamma)}$ denote the probability corresponding to this change of measure. Now, first note that the intermediate process being before the supremum is clearly governed by $\mathbb{P}^{\Phi(\gamma)}$.

Next, the infimum of the pre-supremum process is also equal to a under the given conditions. Therefore, the intermediate process starts at a and evolves independently from the past before the infimum as a process conditioned to stay above a and killed at the first passage time above b before the exponential time T. This implies that we can write the probability transition function of the intermediate process as

$$P_t(x,\ \mathrm{d}y) = \mathbb{P}_x^{\Phi(\gamma)}\left\{X_t \in \mathrm{d}y, t < \tau_b^+ \mid \tau_b^+ < \tau_a^- \wedge T\right\} \tag{5.12}$$

In view of Lemma 5.1, in particular 5.5 in its proof, we get

$$P_t(x,\ \mathrm{d}y) = \frac{e^{-\Phi(\gamma)(b-y)}}{e^{-\Phi(\gamma)(b-x)}}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_b^+ \wedge T \mid \tau_b^+ < \tau_a^- \wedge T\right\} \tag{5.13}$$

We evaluate the conditional distribution on the right-hand side of 5.13 as

$$\begin{aligned}
&\frac{\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_b^+, \tau_b^+ < \tau_a^- \wedge T\right\}}{\mathbb{P}_x\left\{\tau_b^+ < \tau_a^- \wedge T\right\}} \\
&= \frac{\mathbb{P}_x\left\{\tau_b^+ < \tau_a^- \wedge T \mid X_t = y, t < \tau_b^+\right\}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_b^+\right\}}{\mathbb{P}_x\left\{\tau_b^+ < \tau_a^- \wedge T\right\}} \\
&= \frac{\mathbb{P}_y\left\{\tau_b^+ \circ \theta_t < \tau_a^- \circ \theta_t \wedge T \circ \theta_t\right\}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_b^+\right\}}{\mathbb{P}_x\left\{\tau_b^+ < \tau_a^- \wedge T\right\}} \\
&= \frac{\mathbb{P}_y\left\{\tau_b^+ < \tau_a^- \wedge T\right\}}{\mathbb{P}_x\left\{\tau_b^+ < \tau_a^- \wedge T\right\}}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_b^+\right\}
\end{aligned}$$

by Markov property. Plugging this expression in 5.13, we have

$$P_t(x,\ \mathrm{d}y) = \frac{e^{-\Phi(\gamma)(b-y)}\mathbb{P}_y\left\{\tau_b^+ < \tau_a^- \wedge T\right\}}{e^{-\Phi(\gamma)(b-x)}\mathbb{P}_x\left\{\tau_b^+ < \tau_a^- \wedge T\right\}}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_b^+\right\}.$$

As a result, the distribution of the intermediate process is Doob h-transform of the law of the original Lévy process killed at $\tau_b^+$with

$$\begin{aligned}
h(x) :&= e^{-\Phi(\gamma)(b-x)}\mathbb{P}_x\left\{\tau_b^+ < \tau_a^- \wedge T\right\} \\
&= e^{-\Phi(\gamma)(b-x)}\mathbb{E}_{x-a}\left[e^{-\gamma\tau_{b-a}^+}; \tau_{b-a}^+ < \tau_0^-\right] \\
&= e^{-\Phi(\gamma)(b-x)}\frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)}
\end{aligned} \tag{5.14}$$

by the two-sided exit formula [28].

iii. Under the condition $I_T = a, S_T = b, H_I < H_S$, for $x < b, a < y < b$, the transition semigroups $P_t(x,\ \mathrm{d}y)$ of the post-H_S process can be found in terms of the transition probability density of the original process with an h-transform because

starting at x, the process is conditioned to stay above a and below b, before an exponential time T. As a result, the transition density of the post- H_S process under the given conditions is the h-transform of the transition density of the process until $T \wedge \tau_b^+ \wedge \tau_a^-$ with

$$\begin{aligned}
h(x) &= \mathbb{P}_x\left\{T < \tau_b^+ \wedge \tau_a^-\right\} \\
&= \mathbb{P}_x\left\{T < \tau_a^-, \tau_a^- < \tau_b^+\right\} + \mathbb{P}_x\left\{T < \tau_b^+, \tau_b^+ < \tau_a^-\right\} \\
&= \mathbb{P}_x\left\{\tau_a^- < \tau_b^+\right\} - \mathbb{E}_x\left[e^{-\gamma\tau_a^-}; \tau_a^- < \tau_b^+\right] + \mathbb{P}_x\left\{\tau_b^+ < \tau_a^-\right\} - \mathbb{E}_x\left[e^{-\gamma\tau_b^+}; \tau_b^+ < \tau_a^-\right] \\
&= 1 - \mathbb{E}_{x-a}\left[e^{-\gamma\tau_0^-}; \tau_0^- < \tau_{b-a}^+\right] - \mathbb{E}_{x-a}\left[e^{-\gamma\tau_{b-a}^+}; \tau_{b-a}^+ < \tau_0^-\right] \\
&= 1 - Z^{(\gamma)}(x-a) + \left(Z^{(\gamma)}(b-a) - 1\right)\frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)},
\end{aligned}$$

where the two-sided exit formulas given in [28] are used. □

Remark 5.3 (Comparison of the pre-H_I process (under $H_I < H_S$) with Salminen & Vallois ([52], Thm 3.5)). In Proposition 5.3, conditioned on $H_I < H_S, I_T = a, S_T = b$, the pre-$H_I$ process is an h-transform of the killed spectrally negative Lévy process with

$$h(x) = e^{-\Phi(\gamma)(b-x)}\frac{W^{(\gamma)\prime}(x-a)W^{(\gamma)}(b-a) - W^{(\gamma)\prime}(b-a)W^{(\gamma)}(x-a)}{(W^{(\gamma)}(b-a))^2}.$$

For Brownian motion, $W^{(\gamma)}(x) = \frac{2}{\sqrt{2\gamma}}\sinh(\sqrt{2\gamma}x)$ and $W^{(\gamma)\prime}(x) = 2\cosh(\sqrt{2\gamma}x)$. Substituting these gives

$$\begin{aligned}
h(x) = e^{-\sqrt{2\gamma}(b-x)}\big(&\cosh(\sqrt{2\gamma}(x-a))\sinh(\sqrt{2\gamma}(b-a)) \\
&- \cosh(\sqrt{2\gamma}(b-a))\sinh(\sqrt{2\gamma}(x-a))\big) = e^{-\sqrt{2\gamma}(b-x)}\sinh(\sqrt{2\gamma}(b-x)).
\end{aligned}$$

For x close to b, the approximation $\sinh(\sqrt{2\gamma}(b-x)) = \sqrt{2\gamma}(b-x) + o(b-x)$ holds. Hence

$$h(x) \sim e^{-\sqrt{2\gamma}(b-x)}(b-x),$$

which is precisely the h-function of a three-dimensional Bessel process (with parameter $\sqrt{2\gamma}$) started from b. This matches the description of the pre-H_I process given in Salminen & Vallois ([52], Theorem 3.5).

Moreover, the characterisations of the intermediate and post-supremum processes obtained for spectrally negative Lévy processes in [59] reduce in the Brownian case exactly to the h-transform representations established by Salminen & Vallois ([52],

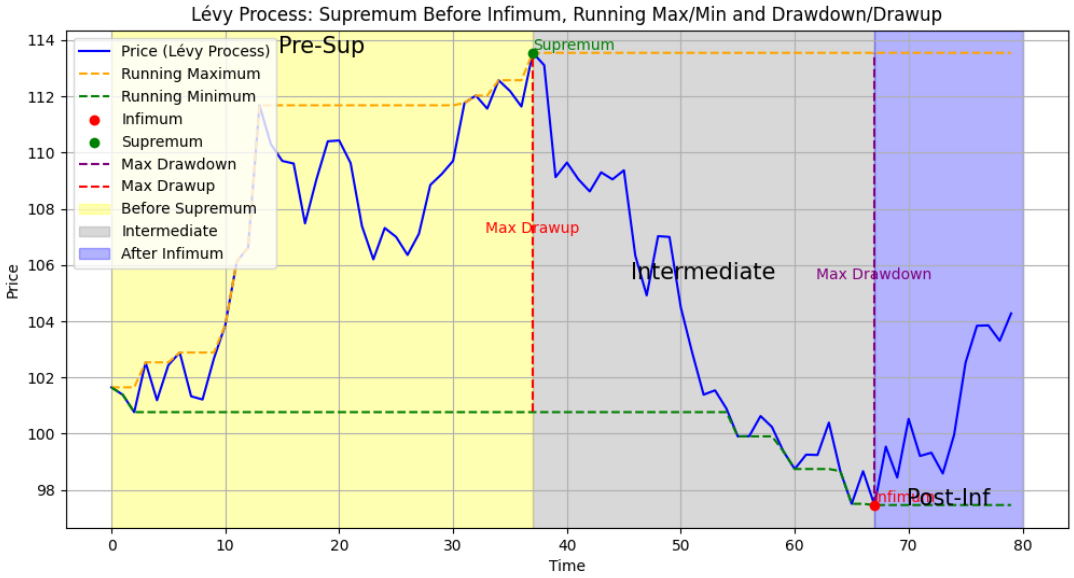


Figure 5.2: Path Decomposition at Extrema Given the Condition that the Supremum Occurs Before the Infimum

Theorem 3.5), confirming that the general Lévy framework is compatible with the classical Brownian path decomposition.

5.3 Path Decomposition at Extrema Given the Condition that the Supremum Occurs Before the Infimum

In this section, we characterize the pre-supremum, intermediate, and post-infimum parts under the condition that the supremum occurs before the infimum.

Proposition 5.4. *Conditionally on $H_S < H_I, I_T = a, S_T = b$, the pre-$H_S$ process, intermediate process and the post-H_I process are independent and have the following distributions: i. the law of the pre-supremum process is the h-transform of the law of the spectrally negative Lévy process with $h^*(.)$ killed at the first hitting time at b before the exponential time T where*

$$h^*(x) = -\frac{\gamma}{\Phi(\gamma)} \frac{W^{(\gamma)}(x-a)\left(W^{(\gamma)\prime\prime}(b-a)W^{(\gamma)}(b-a) - \left(W^{(\gamma)\prime}(b-a)\right)^2\right)}{\left(W^{(\gamma)}(b-a)\right)^2}.$$

ii. the transition semigroup of the intermediate process is given by

$$\begin{aligned} P_t(x, \mathrm{d}y) &= \mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a | H_S < H_I, S_T = b, I_T = a\right\} \\ &= \frac{h^{**}(y)}{h^{**}(x)} \mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a\right\} \end{aligned}$$

$h^{**}(.)$-transform of the post-H_S spectrally negative Lévy process killed at τ_a, where $a < x, y < b$, and the entrance law is obtained as $x \to b$ with;

$$\begin{aligned}
&\lim_{x\to b} h^{**}(x) \\
&= \lim_{x\to b} \Phi(\gamma) e^{-\Phi(\gamma)(b-x)} \frac{d}{da}\left[\frac{W^{(\gamma)}(x+a)}{W^{(\gamma)}(a+b)} + \frac{\gamma}{\Phi(\gamma)} W^{(\gamma)}(x) - Z^{(\gamma)}(a)\frac{W^{(\gamma)}(x)}{W^{(\gamma)}(a)}\right] \\
&= \Phi(\gamma) W^{(\gamma)}(b)\left[\frac{Z^{(\gamma)'}(a)W^{(\gamma)}(a) - Z^{(\gamma)}(a)W^{(\gamma)'}(a)}{(W^{(\gamma)}(a))^2}\right]
\end{aligned}$$

iii. the law of the post-H_I process is the h-transform of the law of the spectrally negative Lévy process killed at $T \wedge \tau_b^+ \wedge \tau_a^- \wedge \tau_a$ with

$$h^{***}(x) = \left(1 - e^{-\Phi(\gamma)(b-x)}\right)\left[1 - Z^{(\gamma)}(x-a) + \left(Z^{(\gamma)}(b-a) - 1\right)\frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)}\right].$$

Proof. i. By Proposition 5.2, the pre-H_I process is given. Let $\mathbb{P}^{h^1}$ denote the probability corresponding to this change of measure. Now, first note that the intermediate process being before the infimum is clearly governed by $\mathbb{P}^{h^1}$. Next, the supremum of the pre-infimum process is also equal to b under the given conditions. Therefore, the intermediate process starts at b and evolves independently from the past before the infimum as a process conditioned to stay below b and killed at the first passage time above a before the exponential time T.

$$\begin{aligned}
P_t(x,dy) &= P_x\{X_t \in dy, t < H_S \,|\, H_S < H_I, I_T = a, S_T = b\} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}b}P_x\{X_t \in dy, t < H_S, H_S < H_I, S_T \le b \,|I_T = a\}}{\frac{\mathrm{d}}{\mathrm{d}b}P_x\{H_S < H_I, S_T \le b \,|\, I_T = a\}} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}b}P_x^{h^1}\{X_t \in dy, t < \tau_b^+, S_{\tau_a} \le b\}}{\frac{\mathrm{d}}{\mathrm{d}b}P_x^{h^1}\{S_{\tau_a} \le b\}} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}b}P_x^{h^1}\{S_{\tau_a} \le b \,|X_t \in dy, t < \tau_b^+\}P_x^{h^1}\{X_t \in dy, t < \tau_b^+\}}{\frac{\mathrm{d}}{\mathrm{d}b}P_x^{h^1}\{S_{\tau_a} \le b\}} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}b}P_y^{h^1}\{S_{\tau_a} \le b\}}{\frac{\mathrm{d}}{\mathrm{d}b}P_x^{h^1}\{S_{\tau_a} \le b\}} P_x^{h^1}\{X_t \in dy, t < \tau_b^+\} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}b}P_y\{S_{H_I} \le b | I_T = a\}}{\frac{\mathrm{d}}{\mathrm{d}b}P_x\{S_{H_I} \le b | I_T = a\}} P_x^{h^1}\{X_t \in dy, t < \tau_b^+\} \\
&= \frac{h^2(y)}{h^2(x)}\frac{h^1(y)}{h^1(x)} P_x\{X_t \in dy, t < \tau_b^+\}
\end{aligned}$$

where $h^1(x) = \frac{\gamma}{\Phi(\gamma)}W^{(\gamma)\prime}(x-a) - \gamma W^{(\gamma)}(x-a)$ is the h-transform of pre-infimum

process and

$$h^2(x) = \frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_x\left\{S_{H_I} \le b \mid I_T = a\right\} =$$
$$\frac{d}{db}\left[1 - \frac{W^{(\gamma)\prime}(b-a) - \Phi(\gamma)W^{(\gamma)}(b-a)}{W^{(\gamma)\prime}(x-a) - \Phi(\gamma)W^{(\gamma)}(x-a)} \cdot \frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)}\right] =$$
$$-\frac{W^{(\gamma)}(x-a)}{\left(W^{(\gamma)}(b-a)\right)^2} \cdot \frac{W^{(\gamma)\prime\prime}(b-a)\,W^{(\gamma)}(b-a) - \left(W^{(\gamma)\prime}(b-a)\right)^2}{W^{(\gamma)\prime}(x-a) - \Phi(\gamma)W^{(\gamma)}(x-a)}$$

Therefore,

$$h^*(x) = h^1(x) \cdot h^2(x) =$$
$$-\frac{\gamma}{\Phi(\gamma)} \frac{W^{(\gamma)}(x-a)\left(W^{(\gamma)\prime\prime}(b-a)W^{(\gamma)}(b-a) - \left(W^{(\gamma)\prime}(b-a)\right)^2\right)}{\left(W^{(\gamma)}(b-a)\right)^2}$$

ii. When $H_S < H_I, S_T = b, I_T = a$, the transition semigroup of the intermediate process is given by

$$P_t(x,\ \mathrm{d}y) = \mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a | H_S < H_I, S_T = b, I_T = a\right\}$$

for $a < x, y < b$, and the entrance law is obtained as $x \to b$.

$$\lim_{x\to b} P_t(x,\ \mathrm{d}y) = \mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a \mid H_S < H_I, S_T = b, I_T = a\right\}$$
$$= \lim_{x\to b} \frac{\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a, H_S < H_I, S_T \le b, I_T \le a\right\}}{\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_x\left\{H_S < H_I, S_T \le b, I_T \le a\right\}}$$
$$= \lim_{x\to b} \frac{\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_x\left\{H_S < H_I, I_T \le a, S_T \le b \mid X_t = y, t < \tau_a\right\}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a\right\}}{\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_x\left\{H_S < H_I, I_T \le a, S_T \le b\right\}}$$
$$= \lim_{x\to b} \frac{\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_y\left\{H_S < H_I, I_T \le a, S_T \le b\right\}}{\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_x\left\{H_S < H_I, I_T \le a, S_T \le b\right\}}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a\right\}$$
$$= \lim_{x\to b} \frac{\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_y\left\{S_T \le b\right\}\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_y\left\{I_{\tau_{b^+},T} \le a | S_T = b\right\}}{\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_x\left\{S_T \le b\right\}\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_x\left\{I_{\tau_{b^+},T} \le a | S_T = b\right\}}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a\right\}$$
$$= \lim_{x\to b} \frac{h^1(y)h^2(y)}{h^1(x)h^2(x)}\mathbb{P}_x\left\{X_t \in \mathrm{d}y, t < \tau_a\right\}$$

where

$$h^{(**)}(x) = h^1(x)h^2(x)$$
$$= \Phi(\gamma)e^{-\Phi(\gamma)(b-x)}\frac{d}{da}\left[\frac{W^{(\gamma)}(x+a)}{W^{(\gamma)}(a+b)} + \frac{\gamma}{\Phi(\gamma)}W^{(\gamma)}(x) - Z^{(\gamma)}(a)\frac{W^{(\gamma)}(x)}{W^{(\gamma)}(a)}\right]$$

by Equations 5.7 and 5.8. And,

$$\lim_{x\to b} h^1(x) = \frac{\mathrm{d}}{\mathrm{d}b}\lim_{x\to b}\mathbb{P}_x\{S_T \le b\} = \lim_{x\to b}\frac{d}{db}\mathbb{P}_x\left\{T < \tau_b^+\right\}$$
$$= \lim_{x\to b}\Phi(\gamma)e^{-\Phi(\gamma)(b-x)} = \Phi(\gamma)$$

and

$$\lim_{x\to b} h^2(x) = \lim_{x\to b} \frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_x\left\{I_{H_S,T} \le a \mid S_T = b\right\} =$$
$$\frac{d}{da}\left[1 - \lim_{x\to b}\mathbb{P}_x\left(I_{\tau_b^+,T} > a \mid S_T = b\right)\right] =$$
$$\frac{d}{da}\left[\frac{\gamma}{\Phi(\gamma)}W^{(\gamma)}(b) + Z^{(\gamma)}(a)\frac{W^{(\gamma)}(b)}{W^{(\gamma)}(a)}\right] =$$
$$W^{(\gamma)}(b)\left[\frac{Z^{(\gamma)'}(a)W^{(\gamma)}(a) - Z^{(\gamma)}(a)W^{(\gamma)'}(a)}{(W^{(\gamma)}(a))^2}\right].$$

iii. The transition density of the post- H_I process under the given conditions is the h-transform of the transition density of the process until $T \wedge \tau_b^+ \wedge \tau_a^- \wedge \tau_a$ with where $h^1(x) = 1 - e^{-\Phi(\gamma)(b-x)}, x \le b$ is the h-transform of post-supremum process and

$$h^2(x) = P_x\left(T < \min\{\tau_a, \tau_a^-, \tau_b^+\}\right)$$
$$= P_x\left(I_T > a, S_T \le b\right) = 1 - Z^{(\gamma)}(x-a) + \left(Z^{(\gamma)}(b-a) - 1\right)\frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)}$$

Therefore,

$$h^{***}(x) = h^1(x)h^2(x)$$
$$= \left(1 - e^{-\Phi(\gamma)(b-x)}\right)\left[1 - Z^{(\gamma)}(x-a) + \left(Z^{(\gamma)}(b-a) - 1\right)\tfrac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)}\right].$$

□

Remark 5.4. In Proposition 5.4, we characterised the pre-supremum, intermediate and post-H_I processes for spectrally negative Lévy processes under the condition $H_S < H_I$. For Brownian motion, [52] derived the path decomposition explicitly only for the case $H_I < H_S$ and noted that the opposite case $H_S < H_I$ can be obtained by symmetry. We verify that the results obtained via symmetry in the Brownian motion case by Salminen & Vallois [52] coincide with the h-functions derived from our general spectrally negative Lévy framework.

Their decomposition consists of three independent components, each given as a Doob-h transform:

- **Pre-infimum process:** Distributed as $(b - X_t, 0 \le t < H_{b-a})$ under $Q^b_{\sqrt{2\lambda}}$, which is a three-dimensional Bessel process ($BES(3)$). The corresponding

Doob-h function is

$$h_1(x) = e^{-\sqrt{2\lambda}(b-x)} \sinh(\sqrt{2\lambda}(b-x)).$$

This form is well known to characterise a $BES(3)$ process.

- **Intermediate process:** The segment between infimum and supremum is distributed as a process killed upon hitting $b-a$, with Doob-h function

$$h_2(x) = e^{-\sqrt{2\lambda}(b-x)} \frac{\sinh(\sqrt{2\lambda}x)}{\sinh(\sqrt{2\lambda}b)}, \qquad 0 < x < b.$$

- **Post-supremum process:** The post-supremum process is killed at $T \wedge \tau_b^+ \wedge \tau_a^-$ with Doob-h function for $a < x < b$,

$$h_3(x) = 1 - \cosh(\sqrt{2\lambda}(x-a)) + (\cosh(\sqrt{2\lambda}(b-a)) - 1)\frac{\sinh(\sqrt{2\lambda}(x-a))}{\sinh(\sqrt{2\lambda}(b-a))}.$$

For Brownian motion, reflecting the path by setting $Y_t := -X_t$ does not change its distribution but reverses the order of the extrema: if $H_I < H_S$ holds for Y, then $H_S < H_I$ holds for X, and

$$I_T(Y) = -S_T(X), \qquad S_T(Y) = -I_T(X).$$

Consequently:

- the pre-H_S process of X corresponds to the pre-H_I process of Y;
- the intermediate process keeps the same functional form but with endpoints swapped;
- the post-H_I process of X corresponds to the post-H_S process of Y.

After reflection, distances measured from the original infimum a become distances measured from the original supremum b but in the opposite direction. A point x located $(x-a)$ above a in the original orientation corresponds, after symmetry, to a point $b-(x-a)$ measured downward from b. This shift of reference explains why, in the transformed h-functions, the variable x is replaced with $b-(x-a)$.

Applying this symmetry to h_1, h_2, h_3 gives:

$$h^*(x) = e^{-\sqrt{2\lambda}(b-x)} \sinh(\sqrt{2\lambda}(x-a)), \quad h^{**}(x) = e^{-\sqrt{2\lambda}(b-x)} \frac{\sinh(\sqrt{2\lambda}(x-a))}{\sinh(\sqrt{2\lambda}(b-a))},$$

$$h^{***}(x) = \big[1 - \cosh(\sqrt{2\lambda}(x-a)) + (\cosh(\sqrt{2\lambda}(b-a)) - 1)\tfrac{\sinh(\sqrt{2\lambda}(x-a))}{\sinh(\sqrt{2\lambda}(b-a))}\big] \times (1 - e^{-\sqrt{2\lambda}(b-x)}),$$

which coincide exactly with the h-functions derived in Proposition 5.4 for spectrally negative Lévy processes.

Substituting the Brownian scale functions $W^{(\gamma)}(x) = \frac{2}{\sqrt{2\gamma}} \sinh(\sqrt{2\gamma}x)$ and $Z^{(\gamma)}(x) = \cosh(\sqrt{2\gamma}x)$ into our general formulas and applying the symmetry transformation reproduces precisely the Salminen & Vallois [52] $H_S < H_I$ decomposition. This confirms that the characterisations obtained for spectrally negative Lévy processes are consistent with the classical Brownian decomposition, even when the supremum is attained before the infimum.

CHAPTER 6

MAXIMUM DRAWDOWNS AND MAXIMUM DRAWUPS

In this chapter, we focus on the calculation of the maximum drawdown and drawup functionals. We consider two distinct cases: when the infimum occurs before the supremum, and when the supremum occurs before the infimum. In the first case, the maximum drawdown is calculated separately for each of the decomposed and independent parts of the path. In the second case, we perform an analogous calculation for the maximum drawup.

6.1 Maximum Drawdowns

In this section, we compute the maximum drawdowns for the case where the infimum precedes the supremum. The calculation is carried out separately for each part of the path characterized in Chapter 5. Notably, the maximum drawdown associated with the pre-infimum part is obtained for the first time.

Theorem 6.1. *Conditionally on $H_I < H_S, I_T = a, S_T = b$, it follows that*

1. *the distribution of the maximum loss of the pre-infimum process, M^-_{0,H_I}, when $H_I < H_S, I_T = a, S_T = b$ is given by*

$$P\left\{M^-_{0,H_I} < d \mid H_I < H_S, I_T = a, S_T = b\right\} = f_1(d; a, b) \tag{6.1}$$
$$= \frac{e^{a\Phi(\gamma)} W'^{\phi(\gamma)}(0) W^{\phi(\gamma)}(b-a) W'^{\phi(\gamma)}(-a)}{W^{\phi(\gamma)}(d)\left[-W'^{\phi(\gamma)}(b-a) W^{\phi(\gamma)}(-a) + W'^{\phi(\gamma)}(-a) W^{\phi(\gamma)}(b-a)\right]}$$

for $-a < d < b - a$.

2. *the distribution of the maximum loss of the intermediate process,* $M^-_{H_I,H_S}$*, when* $H_I < H_S, I_T = a, S_T = b$ *is given by*

$$P\left\{M^-_{H_I,H_S} < d \mid H_I < H_S, I_T = a, S_T = b\right\} = f_2(d; a, b) \tag{6.2}$$
$$= \frac{W^{(\gamma)}(b-a)}{W^{(\gamma)}(d)} e^{-(b-a-d)\left(\frac{W^{(\gamma)\prime}(d).}{W^{(\gamma)}(d)} - \Phi(\gamma)W^{(\gamma)}(d) + \Phi(\gamma)\right)}$$

for $0 < d < b - a$.

3. *the distribution of the maximum loss of the post-supremum process* $M^-_{H_S,T}$ *when* $H_I < H_S, I_T = a, S_T = b$ *is given by*

$$P\left(M^-_{H_S,T} < d \mid H_I < H_S, I_T = a, S_T = b\right) = f_3(d; a, b) \tag{6.3}$$

$$= \frac{\left(Z^{(\gamma)}(d) - 1\right) \frac{W^{(\gamma)\prime}(d)}{W^{(\gamma)}(d)} - \gamma W^{(\gamma)}(d)}{\left(Z^{(\gamma)}(b-a) - 1\right) \frac{W^{(\gamma)\prime}(b-a)}{W^{(\gamma)}(b-a)} - \gamma W^{(\gamma)}(b-a)}$$

for $0 < d < b - a$.

Proof. 1. In view of the conditioning revealed by h in Proposition 5.3.i, we have

$$P\left\{M^-_{0,H_I} < d \mid H_I < H_S, I_T = a, S_T = b\right\}$$
$$= P_0^{h,\phi(\gamma)}(\tau_a < \tau^+_{a+d}) = \frac{h(a)}{h(0)} \frac{dP_0^{\phi(\gamma)}(I_{\tau^+_{a+d}} \le a)}{da}$$
$$= \frac{e^{a\Phi(\gamma)} W'^{\phi(\gamma)}(0) W^{\phi(\gamma)}(b-a) W'^{\phi(\gamma)}(-a)}{W^{\phi(\gamma)}(d)\left[-W'^{\phi(\gamma)}(b-a)W^{\phi(\gamma)}(-a) + W'^{\phi(\gamma)}(-a)W^{\phi(\gamma)}(b-a)\right]}$$

where

$$\begin{aligned} h(a) &= e^{-\Phi(\gamma)(b-a)} \frac{dP_a^{\phi(\gamma)}(I_{\tau_b^+} \le a)}{da} \\ &= e^{-\Phi(\gamma)(b-a)} \frac{d\left(1 - \frac{W^{\phi(\gamma)}(0)}{W^{\phi(\gamma)}(b-a)}\right)}{da} = \frac{-e^{-\Phi(\gamma)(b-a)} W'^{(\gamma)}(0)}{W^{\phi(\gamma)}(b-a)}, \end{aligned}$$

$$\begin{aligned} h(0) &= e^{-b\Phi(\gamma)} \frac{dP_0^{\phi(\gamma)}(I_{\tau_b^+} \le a)}{da} \\ &= e^{-b\Phi(\gamma)} \frac{-W'^{\phi(\gamma)}(b-a)W^{\phi(\gamma)}(-a) + W'^{\phi(\gamma)}(-a)W^{\phi(\gamma)}(b-a)}{\left[W^{\phi(\gamma)}(b-a)\right]^2} \end{aligned}$$

and

$$\frac{dP_0^{\phi(\gamma)}(I_{\tau^+_{a+d}} \le a)}{da} = \frac{d\left(1 - \frac{W^{\phi(\gamma)}(-a)}{W^{\phi(\gamma)}(d)}\right)}{da} = \frac{W'^{\phi(\gamma)}(-a)}{W^{\phi(\gamma)}(d)}.$$

2. For a generic spectrally negative Lévy process, we have

$$P_x\left\{\tau_b^+ < \alpha_d, \tau_b^+ < \tau_a^-, \tau_b^+ < T\right\} = E_x\left[e^{-q\tau_b^+}; \tau_b^+ \leq \alpha_d \wedge \tau_a^-\right] \quad (6.4)$$
$$= \frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(d)} e^{-(b-d-a)\frac{W^{(\gamma)\prime}(d)}{W^{(\gamma)}(d)}}$$

by identity ([4], Eq.20), top row and middle column, given in terms of the drawdown time α_d. In view of Eq. 5.12 in the proof of Proposition 5.3, and 6.4 above, we get

$$P\left\{M^-_{H_I,H_S} < d \mid H_I < H_S, I_T = a, S_T = b\right\} \quad (6.5)$$
$$= \lim_{x\to a} \frac{P_x^{\Phi(\gamma)}\left\{\tau_b^+ < \alpha_d, \tau_b^+ < \tau_a^-, \tau_b^+ < T\right\}}{P_x^{\Phi(\gamma)}\left\{\tau_b^+ < \tau_a^- \wedge T\right\}}$$
$$= \lim_{x\to a} \frac{\widetilde{W}^{(\gamma)}(b-a)}{\widetilde{W}^{(\gamma)}(x-a)} \frac{\widetilde{W}^{(\gamma)}(x-a)}{\widetilde{W}^{(\gamma)}(d)} e^{-(b-d-a)\frac{\tilde{W}^{(\gamma)\prime}(d)}{\widetilde{W}^{(\gamma)}(d)}}$$
$$= \frac{\widetilde{W}^{(\gamma)}(b-a)}{\widetilde{W}^{(\gamma)}(d)} e^{-(b-d-a)\frac{\tilde{W}^{(\gamma)\prime}(d)}{\widetilde{W}^{(\gamma)}(d)}}$$
$$= \frac{W^{(\gamma)}(b-a)}{W^{(\gamma)}(d)} e^{-(b-d-a)\frac{W^{(\gamma)\prime}(d)}{W^{(\gamma)}(d)-\Phi(\gamma)W^{(\gamma)}(d)+\Phi(\gamma)}}$$

for $0 < d < b-a$, where we used the identity $\widetilde{W}^{(0)}_{\Phi(\gamma)}(x) = e^{-\Phi(\gamma)x}W^{(\gamma)}(x)$.

3. In view of the conditioning revealed by h in Proposition 5.3.ii, we have

$$P\left\{M^-_{H_S,T} < d \mid H_I < H_S, I_T = a, S_T = b\right\} \quad (6.6)$$
$$= \lim_{x\to b-} \frac{P_x\left\{T < \tau^-_{b-d}, T < \tau_b^+ \wedge \tau_a^-\right\}}{P_x\left\{T < \tau_b^+ \wedge \tau_a^-\right\}}$$
$$= \lim_{x\to b-} \frac{P_x\left\{T < \tau^-_{b-d} \wedge \tau_b^+ \wedge \tau_a^-\right\}}{h(x)}$$
$$= \lim_{x\to b-} \frac{P_x\left\{T < \tau^-_{b-d} \wedge \tau_b^+\right\}}{h(x)}$$
$$= \lim_{x\to b-} \frac{1 - Z^{(\gamma)}(x-b+d) + \left(Z^{(\gamma)}(d)-1\right)\frac{W^{(\gamma)}(x-b+d)}{W^{(\gamma)}(d)}}{1 - Z^{(\gamma)}(x-a) + \left(Z^{(\gamma)}(b-a)-1\right)\frac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)}}$$
$$= \frac{\left(Z^{(\gamma)}(d)-1\right)\frac{W^{(\gamma)\prime}(d)}{W^{(\gamma)}(d)} - Z^{(\gamma)\prime}(d)}{\left(Z^{(\gamma)}(b-a)-1\right)\frac{W^{(\gamma)\prime}(b-a)}{W^{(\gamma)}(b-a)} - Z^{(\gamma)\prime}(b-a)}$$

where we have used $\tau^-_{b-d} < \tau_a^-$since $a < b-d$, and L'Hospital's rule to take the limit. The result follows by $Z^{(\gamma)\prime} = \Phi(\gamma)W^{(\gamma)}$.

□

Remark 6.1. Theorem 6.1 provides closed-form distributions for the maximum loss of the pre-infimum, intermediate and post-supremum processes, denoted f_1, f_2, f_3,

for general spectrally negative Lévy processes. For the Brownian motion case, substituting the scale functions
$W^{(\gamma)}(x) = \frac{2}{\sqrt{2\gamma}} \sinh(\sqrt{2\gamma}x)$ and $Z^{(\gamma)}(x) = \cosh(\sqrt{2\gamma}x)$ gives:

$$f_1(d; a, b) = e^{a\sqrt{2\gamma}} \sqrt{2\gamma} \, \frac{\sinh(\sqrt{2\gamma}(b-a)) \cosh(\sqrt{2\gamma}a)}{\sinh(\sqrt{2\gamma}d) \sinh(\sqrt{2\gamma}b)}.$$

For the intermediate and post-supremum maximum losses, these cases were already given in [59] and shown to coincide with the Brownian formulas of [52].

6.2 Maximum Drawups

In this section, we analyze the case where the supremum occurs before the infimum. Based on the characterizations derived in Chapter 5, we compute, for the first time, the maximum drawups corresponding to each part of the path.

Theorem 6.2. *Conditionally on $H_S < H_I, I_T = a, S_T = b$, it follows that*
i. the distribution of the maximum gain of the pre-supremum process, M^+_{0,H_S}, when $H_S < H_I, I_T = a, S_T = b$ is given by

$$P\left\{M^+_{0,H_S} \leq u \mid H_S < H_I, I_T = a, S_T = b\right\} = g_1(d; a, b)$$
$$= \frac{W^{(\gamma)}(b-a)}{W^{(\gamma)}(-a)} \frac{W^{(\gamma)}(u-b)}{W^{(\gamma)}(u)}$$

for $b < u < b-a$. ii.the distribution of the maximum gain of the intermediate process $M^+_{H_S,H_I}$ when $H_S < H_I, I_T = a, S_T = b$ is given by

$$P\left\{M^+_{H_S,H_I} \leq u \mid H_S < H_I, I_T = a, S_T = b\right\} = g_2(d; a, b) \tag{6.7}$$

$$= \frac{h^{**}(a)}{h^{**}(b)} \frac{W^{(\gamma)}(v+b-a)}{W^{(\gamma)}(v)} \tag{6.8}$$

where

$$h^{**}(x) = \Phi(\gamma) e^{-\Phi(\gamma)(b-x)} \frac{d}{da} \left[\frac{W^{(\gamma)}(x+a)}{W^{(\gamma)}(a+b)} + \frac{\gamma}{\Phi(\gamma)} W^{(\gamma)}(x) - Z^{(\gamma)}(a) \frac{W^{(\gamma)}(x)}{W^{(\gamma)}(a)} \right]$$

iii. the distribution of the maximum gain of the post-infimum process $M^+_{H_I,T}$ when $H_S < H_I, I_T = a, S_T = b$ is given by

$$P\left(M^+_{H_I,T} < u \mid H_S < H_I, I_T = a, S_T = b\right) = g_3(d; a, b) = \frac{Z^{(\gamma)}(u) - 1}{Z^{(\gamma)}(b-a) - 1}$$

for $b < u < b-a$.

Proof. i.

$$\begin{aligned} &P\left\{M^{+}_{0,H_S} \leq u \mid H_S < H_I, I_T = a, S_T = b\right\} \\ &= \frac{h^*(b)}{h^*(0)} P_0(I_{\tau_b^+} \geq b - u) = \frac{h^*(b)}{h^*(0)} \cdot \tfrac{W^{(\gamma)}(u-b)}{W^{(\gamma)}(u)} \end{aligned}$$

where

$$h^*(b) = -\frac{\gamma}{\Phi(\gamma)} \frac{\left(W^{(\gamma)\prime\prime}(b-a)W^{(\gamma)}(b-a) - \left(W^{(\gamma)\prime}(b-a)\right)^2\right)}{W^{(\gamma)}(b-a)},$$

$$h^*(0) = -\frac{\gamma}{\Phi(\gamma)} \frac{W^{(\gamma)}(-a)\left(W^{(\gamma)\prime\prime}(b-a)W^{(\gamma)}(b-a) - \left(W^{(\gamma)\prime}(b-a)\right)^2\right)}{\left(W^{(\gamma)}(b-a)\right)^2}.$$

Therefore, we have

$$\frac{h^*(b)}{h^*(0)} = \frac{W^{(\gamma)}(b-a)}{W^{(\gamma)}(-a)}.$$

ii) The intermediate process is sure to decrease from level b to level a. Note that, under the given conditions, the maximum drawdown for this process equals $b - a$. Therefore, by taking $u = b - a$ for $a - b \leq 0 \leq v \leq b - a$ we have

$$\begin{aligned} &P_b\left\{M^{+}_{H_S,H_I} \leq v \mid H_S < H_I, I_T = a, S_T = b\right\} \\ &= \frac{h^{**}(a)}{h^{**}(b)} P_b(M^{+}_{\tau_a} \leq v) = \frac{h^{**}(a)}{h^{**}(b)} P_0(M^{+}_{\tau_{a-b}} \leq v) \\ &= \frac{h^{**}(a)}{h^{**}(b)} \frac{W^{(\gamma)}(v+b-a)}{W^{(\gamma)}(v)} \end{aligned}$$

where

$$h^{**}(x) = \Phi(\gamma)e^{-\Phi(\gamma)(b-x)} \frac{d}{da}\left[\frac{W^{(\gamma)}(x+a)}{W^{(\gamma)}(a+b)} + \frac{\gamma}{\Phi(\gamma)} W^{(\gamma)}(x) - Z^{(\gamma)}(a)\frac{W^{(\gamma)}(x)}{W^{(\gamma)}(a)}\right]$$

iii) for $u < b - a$

$$
\begin{aligned}
&P\left(M^+_{H_I,T} < u \mid H_s < H_I, I_T = a, S_T = b\right) \\
&= \lim_{x\to a} \frac{P^h_x(T < \tau^+_{u+a}, T < \tau_a, T < \tau^+_b, T < \tau^-_a)}{P^h_x(T < \tau_a, T < \tau^+_b, T < \tau^-_a)} \\
&= \lim_{x\to a} \frac{P^h_x(T < \tau^+_{u+a}, T < \tau_a, T < \tau^-_a)}{P^h_x(I_T > a, S_T < b)} \\
&= \lim_{x\to a} \frac{P^h_x(I_T > a, S_T < u + a)}{P^h_x(I_T > a, S_T < b)} \\
&= \lim_{x\to a} \frac{1 - Z^{(\gamma)}(x-a) + [Z^{(\gamma)}(u+a-a) - 1]\dfrac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)}}{1 - Z^{(\gamma)}(x-a) + [Z^{(\gamma)}(b-a) - 1]\dfrac{W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)}} \\
&= \lim_{x\to a} \frac{Z'^{(\gamma)}(x-a) + [Z^{\gamma}(u) - 1]\dfrac{W'^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)}}{Z'^{(\gamma)}(x-a) + [Z^{(\gamma)}(b-a) - 1]\dfrac{W'^{(\gamma)}(x-a)}{W^{((\gamma))}(b-a)}} \\
&= \frac{Z'^{(\gamma)}(0) + [Z^{\gamma}(u) - 1]\dfrac{W'^{(\gamma)}(0)}{W^{(\gamma)}(b-a)}}{Z'^{(\gamma)}(0) + [Z^{(\gamma)}(b-a) - 1]\dfrac{W'^{(\gamma)}(0)}{W^{((\gamma))}(b-a)}} \\
&= \frac{Z^{(\gamma)}(u) - 1}{Z^{(\gamma)}(b-a) - 1}
\end{aligned}
$$

□

Remark 6.2. Theorem 6.2 gives the distributions g_1, g_2, g_3 of the maximum gain for the pre-supremum, intermediate and post-infimum processes, respectively, in the general spectrally negative Lévy setting. For the Brownian motion case, we now verify that these formulas reduce to the symmetric counterparts of the maximum decrease results of Salminen & Vallois ([52], Theorem 4.1).

(i) Pre-supremum Maximum Drawup. From Theorem 6.2:

$$g_1(u; a, b) = \frac{W^{(\gamma)}(b-a)}{W^{(\gamma)}(-a)} \frac{W^{(\gamma)}(u-b)}{W^{(\gamma)}(u)}, \qquad b < u < b - a.$$

For Brownian motion, $W^{(\gamma)}(x) = \frac{2}{\sqrt{2\gamma}} \sinh(\sqrt{2\gamma}x)$, so

$$W^{(\gamma)}(-a) = -\tfrac{2}{\sqrt{2\gamma}} \sinh(\sqrt{2\gamma}a), \quad W^{(\gamma)}(b-a) = \tfrac{2}{\sqrt{2\gamma}} \sinh(\sqrt{2\gamma}(b-a)),$$

$$W^{(\gamma)}(u-b) = \tfrac{2}{\sqrt{2\gamma}} \sinh(\sqrt{2\gamma}(u-b)), \quad W^{(\gamma)}(u) = \tfrac{2}{\sqrt{2\gamma}} \sinh(\sqrt{2\gamma}u).$$

Hence

$$g_1(u;a,b) = \frac{\sinh(\sqrt{2\gamma}(b-a))}{-\sinh(\sqrt{2\gamma}a)} \frac{\sinh(\sqrt{2\gamma}(u-b))}{\sinh(\sqrt{2\gamma}u)},$$

which is exactly the sign-changed (symmetric) version of the pre-infimum maximum loss formula in [52].

(ii) Intermediate Maximum Drawup. From Theorem 6.2:

$$g_2(u;a,b) = \frac{h^{**}(a)}{h^{**}(b)} \frac{W^{(\gamma)}(u+b-a)}{W^{(\gamma)}(u)},$$

where

$$h^{**}(x) = \Phi(\gamma)e^{-\Phi(\gamma)(b-x)} \frac{d}{da}\left[\frac{W^{(\gamma)}(x+a)}{W^{(\gamma)}(a+b)} + \frac{\gamma}{\Phi(\gamma)}W^{(\gamma)}(x) - Z^{(\gamma)}(a)\frac{W^{(\gamma)}(x)}{W^{(\gamma)}(a)}\right].$$

For Brownian motion, substituting $W^{(\gamma)}(x) = \frac{2}{\sqrt{2\gamma}}\sinh(\sqrt{2\gamma}x)$ and $Z^{(\gamma)}(x) = \cosh(\sqrt{2\gamma}x)$ yields

$$\frac{W^{(\gamma)}(u+b-a)}{W^{(\gamma)}(u)} = \frac{\sinh(\sqrt{2\gamma}(u+b-a))}{\sinh(\sqrt{2\gamma}u)},$$

which matches the symmetric counterpart of the intermediate-process maximum decrease distribution in [52].

(iii) Post-infimum Maximum Drawup. From Theorem 6.2:

$$g_3(u;a,b) = \frac{Z^{(\gamma)}(u)-1}{Z^{(\gamma)}(b-a)-1}.$$

For Brownian motion, $Z^{(\gamma)}(x) = \cosh(\sqrt{2\gamma}x)$, so

$$g_3(u;a,b) = \frac{\cosh(\sqrt{2\gamma}u)-1}{\cosh(\sqrt{2\gamma}(b-a))-1},$$

which is exactly the symmetric form of the post-supremum maximum loss distribution of [52].

Substituting the Brownian scale functions shows that g_1, g_2, g_3 are precisely the symmetric counterparts of the maximum decrease distributions in Salminen & Vallois ([52], Theorem 4.1), reflecting the pathwise symmetry of Brownian motion.

CHAPTER 7

JOINT DISTRIBUTIONS OF MAXIMUM DRAWDOWNS AND MAXIMUM DRAWUPS

In this chapter, we utilize the results obtained in Chapter 6 to derive the joint distribution of the maximum drawdown and drawup. This is achieved by calculating and summing the probabilities of disjoint events that jointly determine the values taken by the maximum drawdown and drawup.

Based on the conditional characterizations, the probability

$$\mathbb{P}\{H_I < H_S,\, I_T \leq a,\, S_T \leq b\},$$

which corresponds to the joint distribution of the infimum and supremum under the condition that the infimum occurs before the supremum, will be derived in Corollary 7.1. Similarly, the probability

$$\mathbb{P}\{H_S < H_I,\, I_T \leq a,\, S_T \leq b\},$$

which corresponds to the joint distribution of the infimum and supremum under the condition that the supremum occurs before the infimum, will be derived in Corollary 7.2.

These computations correspond to the conditional structure of the problem and will later be used to eliminate the conditioning in the final derivation of the unconditional joint distribution.

Corollary 7.1. *For $a < 0 < b$*

$$\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(H_I < H_S,\, I_T \le a,\, S_T \le b) = \phi(\gamma)\left(1 - \frac{W^{(\gamma)\prime}(b-a)\displaystyle\int_0^{b-a} W(z)\,dz}{(W^{(\gamma)}(b-a))^2}\right)$$
$$\times \frac{W^{(\gamma)\prime}(x-a)W^{(\gamma)}(b-a) - W^{(\gamma)\prime}(b-a)W^{(\gamma)}(x-a)}{W^{(\gamma)}(b-a)} \tag{7.1}$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(H_I < H_S,\, S_T \le b,\, I_T \le a)$$
$$= \frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(H_I < H_S,\, S_T \le b \mid I_T = a)\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_0(I_T \le a)$$

For the right-hand side of the equation, we know that

$$\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(H_I < H_S,\, S_T \le b \mid I_T = a) = \frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(H_I < H_S,\, S_{H_I,T} \le b \mid I_T = a)$$
$$= \frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(S_{H_I,T} \le b \mid I_T = a)\,\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(S_{H_I} < b \mid I_T = a),$$

where

$$S_{H_I,T} := \sup\{X_t : H_I \le t \le T\}.$$

By 5.2, we have

$$\mathbb{P}_0\left(S_{H_I} > b \mid I_T = a\right) = \frac{\left(W^{(\gamma)\prime}(b-a) - \Phi(\gamma)W^{(\gamma)}(b-a)\right)W^{(\gamma)}(x-a)}{\left(W^{(\gamma)\prime}(x-a) - \Phi(\gamma)W^{(\gamma)}(x-a)\right)W^{(\gamma)}(b-a)}.$$

So,

$$\mathbb{P}_0\left(S_{H_I} \le b \mid I_T = a\right) = 1 - \frac{\left(W^{(\gamma)\prime}(b-a) - \Phi(\gamma)W^{(\gamma)}(b-a)\right)W^{(\gamma)}(x-a)}{\left(W^{(\gamma)\prime}(x-a) - \Phi(\gamma)W^{(\gamma)}(x-a)\right)W^{(\gamma)}(b-a)} \tag{7.2}$$

for $b > a$. By 5.2 we also have

$$\mathbb{P}\left(S_{H_I,T} > b \mid I_T = a\right) = 1 - \frac{\Phi(\gamma)}{W^{(\gamma)}(b-a)}\int_0^{b-a} W^{(\gamma)}(z)\mathrm{d}z \tag{7.3}$$

So,

$$\mathbb{P}\left(S_{H_I,T} \le b \mid I_T = a\right) = \frac{\Phi(\gamma)}{W^{(\gamma)}(b-a)}\int_0^{b-a} W^{(\gamma)}(z)\mathrm{d}z = \frac{\Phi(\gamma)\left(Z^{(\gamma)}(b-a) - 1\right)}{\gamma W^{(\gamma)}(b-a)}$$

for $b > a$.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}(S_{H_I,T} \leq b \mid I_T = a) \\
&= \frac{\Phi(\gamma)\left[\left(Z^{(\gamma)}(b-a)\right)' \gamma W^{(\gamma)}(b-a) - \left(\gamma W^{(\gamma)}(b-a)\right)' \left(Z^{(\gamma)}(b-a)-1\right)\right]}{\left(\gamma W^{(\gamma)}(b-a)\right)^2} \\
&= \phi(\gamma)\left(1 - \frac{W^{(\gamma)}(b-a)' \int_0^{b-a} W(z)dz}{\left(W^{(\gamma)}(b-a)\right)^2}\right) \qquad (7.4)
\end{aligned}$$

by 5.2

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_0(I_T \leq a) = W^{(\gamma)\prime}(x-a) - \Phi(\gamma)W^{(\gamma)}(x-a) \qquad (7.5)$$

The formula 7.1 is now obtained from 7.2 and 7.4 when multiplying with the density of I_T given in 7.5. □

Corollary 7.2. *For $a < 0 < b$*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(H_S < H_I,\, I_T \leq a,\, S_T \leq b) \qquad (7.6) \\
&= \frac{W^{\Phi(\gamma)}(-a)}{W^{\Phi(\gamma)}(b-a)}\left(\Phi(\gamma)e^{-\Phi(\gamma)b}\right)\left(W^{(\gamma)}(b) - Z^{(\gamma)}(a)\cdot\frac{W^{(\gamma)}(b)W^{(\gamma)\prime}(a)}{\left(W^{(\gamma)}(a)\right)^2}\right)
\end{aligned}$$

Proof.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(H_S < H_I,\, S_T \leq b,\, I_T \leq a) = \\
&\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_0(H_S < H_I,\, I_T \leq a \mid S_T = b)\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(S_T \leq b)
\end{aligned}$$

For the right-hand side of the equation, we know that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_0(H_S < H_I,\, I_T \leq a \mid S_T = b) = \frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_0(H_S < H_I,\, I_{H_S,T} \leq a \mid S_T = b) \\
&= \frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_0(I_{H_S,T} \leq a \mid S_T = b)\,\mathbb{P}_0(I_{H_S} > a \mid S_T = b), \qquad (7.7)
\end{aligned}$$

where

$$I_{H_S,T} := \inf\{X_t : H_S \leq t \leq T\}.$$

By 5.1, we have

$$\mathbb{P}\left(I_{0,H_S} \leq a \mid S_T = b\right) = 1 - \frac{W^{\Phi(\gamma)}(-a)}{W^{\Phi(\gamma}(b-a)} \qquad (7.8)$$

for $b > a$.

$$\mathbb{P}\left(I_{H_S} > a \mid S_T = b\right) = \frac{W^{\Phi(\gamma)}(-a)}{W^{\Phi(\gamma}(b-a)} \qquad (7.9)$$

for $b > a$. By 5.1 we have

$$\mathbb{P}\left(I_{H_S,T} \leq a \mid S_T = b\right) = -\frac{\gamma}{\Phi(\gamma)} W^{(\gamma)}(b) + Z^{(\gamma)}(a) \frac{W^{(\gamma)}(b)}{W^{(\gamma)}(a)} \tag{7.10}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}a}\mathbb{P}_0(I_{H_S,T} \leq a \mid S_T = b) &= \frac{d}{da}\left(-\frac{\gamma}{\Phi(\gamma)} W^{(\gamma)}(b) + Z^{(\gamma)}(a) \cdot \frac{W^{(\gamma)}(b)}{W^{(\gamma)}(a)}\right) \\ &= Z^{(\gamma)\prime}(a) \cdot \frac{W^{(\gamma)}(b)}{W^{(\gamma)}(a)} - Z^{(\gamma)}(a) \cdot \frac{W^{(\gamma)}(b) W^{(\gamma)\prime}(a)}{\left(W^{(\gamma)}(a)\right)^2} \\ &= W^{(\gamma)}(b) - Z^{(\gamma)}(a) \cdot \frac{W^{(\gamma)}(b) W^{(\gamma)\prime}(a)}{\left(W^{(\gamma)}(a)\right)^2} \end{aligned} \tag{7.11}$$

for $b > a$.

$$\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{P}_0(S_T \leq b) = \Phi(\gamma) e^{-\Phi(\gamma) b} \tag{7.12}$$

The formula 7.6 is now obtained from 7.9 and 7.11 when multiplying with the density of S_T given in 7.12. □

Theorem 7.3. *For $\beta > 0$ and a bounded measurable function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, we have*

$$\mathbb{E}\big(\phi(M_T^+);\, M_T^- < \beta,\, H_I < H_S\big) = \Delta = \Delta_1 + \Delta_2,$$

where

$$\Delta_1 = \int_{-\infty}^{0} da \int_{\beta}^{\infty} du\; F(a, u; \beta), \qquad \Delta_2 = \int_{-\infty}^{0} da \int_{0}^{\beta} du\; F(a, u; \beta),$$

and the integrand is given by

$$\begin{aligned} F(a, u; \beta) = \phi(\gamma) &\left(1 - \frac{W^{(\gamma)\prime}(u-a)}{(W^{(\gamma)}(u-a))^2} \int_0^{u-a} W(z)\, dz\right) \\ &\times \left(W^{(\gamma)\prime}(x-a) - \frac{W^{(\gamma)\prime}(u-a)}{W^{(\gamma)}(u-a)}\, W^{(\gamma)}(x-a)\right) G(a, u; \beta), \end{aligned}$$

with

$$G(a, u; \beta) = \begin{cases} f_1(\beta; a, u)\, f_2(\beta; u)\, f_3(\beta; u), & u > \beta, \\ 1, & u < \beta. \end{cases}$$

Proof. Notice first that $H_I < H_S$ then

$$M_T^+ = S_T - I_T \quad \text{and} \quad M_T^- = M_{H_I}^- \vee M_{H_I,H_S}^- \vee M_{H_S,T}^-$$

and, therefore,

$$\begin{aligned}\Delta :&= \mathbb{E}\left(\phi(M_T^+; M_T^- < \beta, H_I < H_S)\right)\\ &= \mathbb{E}\left(\phi(S_T - I_T; M_{H_I}^- < \beta, M_{H_I,H_S}^- < \beta, M_{H_S,T}^- < \beta, H_I < H_S)\right)\end{aligned}$$

$$\Delta = \int_{-\infty}^{0} da \int_{0}^{+\infty} db\, \varphi(b-a)\, f(a,b)\mathbf{1}_{\{\beta>-a\}} \left(f_1(\beta;a,b)f_2(\beta;a,b)f_3(\beta;a,b)\right)$$
$$\mathbf{1}_{\{\beta<b-a\}} + \mathbf{1}_{\{\beta>b-a\}}$$

with f_i, $i = 1, 2, 3$, as in Theorem 6.1 and

$$f(a,b) = \frac{\mathrm{d}}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}b}\mathbf{P}(I_T \le a, S_T \le b, H_I < H_S)$$

by Corollary 7.1. We perform the change of variable

$$u = b - a \quad \text{so that} \quad b = a + u.$$

$$f_1(\beta;a,u) = \frac{e^{a\Phi(\gamma)}W'(0)W(u)W'(-a)}{W(\beta)\left[-W'(u)W(-a) + W'(-a)W(u)\right]}$$

$$f_2(\beta;u) = \frac{W(u)}{W(\beta)} \cdot \exp\left(-(u-\beta)\left(\frac{W'(\beta)}{W(\beta)} - \Phi(\gamma)W(\beta) + \Phi(\gamma)\right)\right)$$

$$f_3(\beta;u) = \frac{(Z^{(\gamma)}(\beta)-1)\cdot\dfrac{W'(\beta)}{W(\beta)} - \gamma W(\beta)}{(Z^{(\gamma)}(u)-1)\cdot\dfrac{W'(u)}{W(u)} - \gamma W(u)}$$

Assume $\varphi(u) = 1$. Then

$$\Delta := \mathbb{E}[\phi(M_T^+)\cdot\mathbf{1}_{\{M_T^-<\beta,\, H_I<H_S\}}] = \Delta_1 + \Delta_2$$

with

$$\Delta_1 = \int_{-\infty}^{0} da \int_{\beta}^{\infty} du\; F(a,u;\beta), \qquad \Delta_2 = \int_{-\infty}^{0} da \int_{0}^{\beta} du\; F(a,u;\beta),$$

where the integrand $F(a,u;\beta)$ is defined as

$$\begin{aligned}F(a,u;\beta) = \phi(\gamma)&\left(1 - \frac{W^{(\gamma)\prime}(u-a)}{(W^{(\gamma)}(u-a))^2}\int_0^{u-a} W(z)\,dz\right)\\ &\times\left(W^{(\gamma)\prime}(x-a) - \frac{W^{(\gamma)\prime}(u-a)}{W^{(\gamma)}(u-a)}\,W^{(\gamma)}(x-a)\right)G(a,u;\beta),\end{aligned}$$

and

$$G(a,u;\beta) := \begin{cases} f_1(\beta;a,u)\cdot f_2(\beta;u)\cdot f_3(\beta;u), & \text{if } u>\beta, \\ 1, & \text{if } u<\beta, \end{cases}$$

with

$$f_1(\beta;a,u) = \frac{e^{a\Phi(\gamma)}W'(0)W(u)W'(-a)}{W(\beta)\left[-W'(u)W(-a)+W'(-a)W(u)\right]},$$

$$f_2(\beta;u) = \frac{W(u)}{W(\beta)}\cdot\exp\left(-(u-\beta)\left(\frac{W'(\beta)}{W(\beta)}-\Phi(\gamma)W(\beta)+\Phi(\gamma)\right)\right),$$

$$f_3(\beta;u) = \frac{(Z^{(\gamma)}(\beta)-1)\cdot\dfrac{W'(\beta)}{W(\beta)}-\gamma W(\beta)}{(Z^{(\gamma)}(u)-1)\cdot\dfrac{W'(u)}{W(u)}-\gamma W(u)}.$$

□

Theorem 7.4. *Let $\beta>0$ and $\phi:\mathbb{R}_+\to\mathbb{R}_+$ be bounded and measurable. Then the expectation*

$$\Theta := \mathbb{E}\left[\phi(S_T-I_T)\cdot\mathbf{1}_{\{M_T^+<\beta,\ H_S<H_I\}}\right]$$

can be decomposed as

$$\Theta = \Theta_1+\Theta_2$$

with

$$\Theta_1 = \int_{-\infty}^{0} da\int_{\beta}^{\infty} du\ H(a,u;\beta), \qquad \Theta_2 = \int_{-\infty}^{0} da\int_{0}^{\beta} du\ H(a,u;\beta),$$

where the integrand $H(a,u;\beta)$ is defined as

$$H(a,u;\beta) := g(a,a+u)\cdot\phi(u)\cdot\Gamma(a,u;\beta),$$

and

$$\Gamma(a,u;\beta) := \begin{cases} g_1(\beta;a,a+u)\cdot g_2(\beta;a,a+u)\cdot g_3(\beta;a,a+u), & \textit{if } u>\beta, \\ 1, & \textit{if } u<\beta. \end{cases}$$

Theorem 7.5 (Joint Distribution of Maximum Drawdown and Drawup).
Let $M_T^- := \sup_{0\le t\le s\le T}(X_t-X_s)$ and $M_T^+ := \sup_{0\le t\le s\le T}(X_s-X_t)$ denote the maximum drawdown and maximum drawup of a spectrally negative Lévy process X up to time T. Then, for any $\alpha,\beta>0$, the joint distribution is given by

$$\mathbb{P}(M_T^-<\alpha,\ M_T^+<\beta) = \Delta+\Theta.$$

Proof.

$$\begin{aligned}&\mathbb{P}(M_T^- < \alpha,\, M_T^+ < \beta)\\&= \mathbb{P}(M_T^- < \alpha,\, M_T^+ < \beta,\, H_I < H_S) + \mathbb{P}(M_T^- < \alpha,\, M_T^+ < \beta,\, H_S < H_I),\end{aligned}$$

where:

1. Under $H_I < H_S$, we have $M_T^+ = S_T - I_T$, and the joint condition reduces to:

$$\begin{aligned}&\mathbb{P}(M_T^- < \alpha,\, M_T^+ < \beta,\, H_I < H_S)\\&= \mathbb{E}\left[\phi(S_T - I_T) \cdot \mathbf{1}_{\{M^-_{0,H_I} < \alpha,\, M^-_{H_I,H_S} < \alpha,\, M^-_{H_S,T} < \alpha,\, S_T - I_T < \beta,\, H_I < H_S\}}\right],\end{aligned}$$

   where ϕ is any bounded measurable function used for representation, and this expression is computed via the decomposition

$$\Delta = \int_{-\infty}^{0} da \int_{0}^{\infty} db\; f(a,b) \cdot \mathbf{1}_{\{\beta > b-a\}} \cdot f_1(\alpha; a, b) f_2(\alpha; a, b) f_3(\alpha; a, b),$$

   with functions f, f_1, f_2, f_3 defined as in Theorem 6.1 and Corollary 7.1.

2. Under $H_S < H_I$, we again have $M_T^+ = S_T - I_T$, and the joint condition reduces to:

$$\begin{aligned}&\mathbb{P}(M_T^- < \alpha,\, M_T^+ < \beta,\, H_S < H_I)\\&= \mathbb{E}\left[\psi(S_T - I_T) \cdot \mathbf{1}_{\{M^+_{0,H_S} < \beta,\, M^+_{H_S,H_I} < \beta,\, M^+_{H_I,T} < \beta,\, S_T - I_T < \alpha,\, H_S < H_I\}}\right],\end{aligned}$$

   where ψ is again a bounded measurable function, and the expectation is computed via the decomposition

$$\Theta = \int_{-\infty}^{0} da \int_{0}^{\infty} db\; g(a,b) \cdot \mathbf{1}_{\{\alpha > b-a\}} \cdot g_1(\beta; a, b) g_2(\beta; a, b) g_3(\beta; a, b),$$

   where g, g_1, g_2, g_3 are the corresponding joint and conditional densities from Theorem 6.2 and Corollary 7.2.

Consequently, we conclude:

$$\mathbb{P}(M_T^- < \alpha,\, M_T^+ < \beta) = \Delta + \Theta.$$

□

CHAPTER 8

CONCLUSION

In this thesis, we investigated the joint behavior of the maximum drawdown and maximum drawup for spectrally negative Lévy processes up to an independent exponential time. Since these processes do not have the symmetry that Brownian motion does, we needed a different approach and built our method around decomposing the path at its extrema. We considered two distinct scenarios: one in which the infimum occurs before the supremum, and another in which the supremum occurs before the infimum. For each case, we decomposed the sample path into meaningful segments—pre-infimum, pre-supremum, intermediate, post-supremum, and post-infimum—and characterized their distributions, with several of these being identified for the first time. These decompositions allowed us to compute the maximum drawdown and drawup separately for each part. By combining the results with the probabilities of disjoint conditioning events, we ultimately derived the joint distribution of the maximum drawdown and drawup, which constitutes the main contribution of this study.

Our findings make a significant contribution to the fluctuation theory of Lévy processes by extending the path decomposition framework in a direction that had not been addressed before. Unlike the Brownian case—where symmetry simplifies the analysis—our method provides a general approach applicable to *asymmetric and discontinuous paths*. These results can serve as a theoretical foundation for risk assessment in various domains, such as *financial modeling*, *insurance mathematics*, and *extreme event analysis*, where understanding the co-movement of losses and gains is crucial.

As with any theoretical study, our work is subject to certain limitations. We restricted our analysis to *spectrally negative Lévy processes* and assumed an *independent exponential stopping time*. These assumptions simplify the mathematical framework but may limit applicability in settings where stopping is endogenous or the underlying process has both upward and downward jumps.

As a future extension of this study, a natural direction is to investigate the joint distribution of the maximum drawdown, maximum gain, and their durations. Since the decomposed parts of the path are conditionally independent, the joint distribution of the maximum drawdown and its duration can be derived using similar techniques. This would provide deeper insight into not only the magnitude but also the timing of extreme events, which is especially relevant in financial risk management.

Another important direction is the application of the theoretical results to real financial data. This would involve selecting appropriate data sets that exhibit Lévy-like characteristics, verifying their compatibility with the model assumptions, and estimating the relevant parameters. Once validated, the model could be used to compute joint risk measures and assess their predictive power in practical settings.

Furthermore, the theoretical results may be used to design insurance contracts aimed at protecting investors against large losses. The fair pricing of such contracts and their optimal exercise times could be determined using the proposed methodology.

This thesis presents a comprehensive framework for analyzing the the joint distribution of the maximum drawdown and the maximum drawup in the setting of spectrally negative Lévy processes. By addressing key gaps in the literature and introducing new techniques for path decomposition, it establishes a foundation for deeper understanding of extremal behavior in stochastic models. We believe that the results obtained here not only contribute to the theoretical development of Lévy process analysis, but also offer valuable tools for practical applications in finance, insurance, and risk management. It is our hope that this work will inspire future studies that continue to connect advanced probability theory with real-world decision-making under uncertainty.

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