SOME ARITHMETIC RAMSEY THEORETIC RESULTS VIA ULTRAFILTERS AND THE STONE-ČECH COMPACTIFICATION

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ABSTRACT

SOME ARITHMETIC RAMSEY THEORETIC RESULTS VIA ULTRAFILTERS AND THE STONE-ČECH COMPACTIFICATION

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In this thesis, with the goal of illustrating how ultrafilter techniques are used to prove results in arithmetic Ramsey theory, we survey two recent results. After providing the necessary background regarding ultrafilters and the algebra of the Stone-Čech compactification of a discrete semigroup, we first present the proof of Hindman's conjecture for two colors given by Bowen and then present a result by by Di Nasso and Ragosta that can be considered as an infinitary exponential version of Schur's theorem.

Keywords: Ultrafilters, Stone–Čech Compactification, Ramsey Theory

ÖZ

ULTRAFİLTRELER VE STONE-ČECH KOMPAKTİFİKASYONU ARACILIĞIYLA BAZI ARİTMETİK RAMSEY KURAMSAL SONUÇLAR

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Bu tezde, ultrafiltre tekniklerinin aritmetik Ramsey kuramında sonuçların kanıtlanmasında nasıl kullanıldığını göstermek amacıyla, iki güncel sonucu inceleyeceğiz. Ultrafiltreler ve ayrık yarıgrupların Stone-Čech kompaktifikasyonuna ilişkin gerekli arka planı sağladıktan sonra, önce iki renk için Hindman'ın sanısının Bowen tarafından yapılan kanıtını sunacağız, daha sonra Di Nasso ve Ragosta'nın Schur'un teoreminin sonsuz üstel versiyonu sayılabilecek bir sonucunu sunacağız.

Anahtar Kelimeler: Ultrafiltreler, Stone–Čech Kompaktifikasyonu, Ramsey Kuramı

To infinity and beyond!

—Buzz Lightyear

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CHAPTER 1

INTRODUCTION

In this thesis, we have two main goals. First, we wish to study problems like the following one: Given any finite partition of $\mathbb{N}$, is there always a cell containing a set of the form $\{x, y, x + y, x \cdot y\}$? This open problem is known as Hindman's conjecture. Second, we wish to illustrate how ultrafilters and the algebra of the Stone–Čech compactification are connected to such problems, providing a gentle introduction to arithmetic Ramsey theory. We will show that some Ramsey theoretic results are equivalent to the existence of specific ultrafilters on $\mathbb{N}$. One way to think about ultrafilters is to think of them as finitely additive measures; so, some Ramsey theory results imply the existence of finitely additive measures on $\mathbb{N}$ of a special type.

We will present the proofs of the two following recent results using the algebra of the Stone–Čech compactification of $\mathbb{N}$:

Theorem 1. [Bow25] Let $\mathbb{N} = C_1 \sqcup C_2$ be a partition with 2 cells. Then there is a cell C_i and $x, y \in \mathbb{N}$ such that $\{x, y, x + y, x \cdot y\} \subseteq C_i$.

Theorem 2. [DNR24] Given any finite cell partition of $\mathbb{N}$, there is a cell C and an infinite sequence $\{b_n\} \subseteq \mathbb{N}$ such that $b_n, (b_{n+1})^{b_n} \in C$.

The structure of the thesis is as follows. Chapter 2 introduces the basic theory of ultrafilters. We develop the general preliminaries regarding ultrafilters and define ultrafilter quantifiers, products, and push-forwards.

Chapter 3 starts with an overview of Ramsey theory and partition regularity. All the combinatorial results that we survey in this thesis are about the partition regularity of specific patterns. We will define partition regularity and give equivalent definitions in terms of ultrafilters. We will briefly go over all the combinatorial results we will sur-

vey in this thesis before moving on to the fundamental results of arithmetic Ramsey theory. Section 2 of this chapter focuses on Schur's theorem and its generalizations. *Schur's theorem states that any finite partition of* $\mathbb{N}$ *has a cell containing* $\{x, y, x+y\}$ *for some* $x, y \in \mathbb{N}$. A particular strengthening of Schur's theorem is Hindman's theorem, that is, *for any finite partition of* $\mathbb{N}$*, there is a cell containing an infinite set and all the non-repeating finite sums of this infinite set.* We will present the Galvin-Glazer proof of Hindman's theorem. This proof technique is a frequently used method of ultrafilter proofs of Ramsey theory results, and we will use Galvin-Glazer type proofs several times in the rest of this thesis. After briefly surveying the strengthenings of Hindman's theorem, we will move on to the last section of this chapter, where we will give a recent proof of van der Waerden's theorem from [DNR24] using ultrafilters.

The proofs of Hindman's theorem and van der Waerden's theorem both utilize a sum operation of ultrafilters. In particular, the proof of Hindman's theorem hints that the idempotent elements of this operation play a central role in the combinatorial results. To study these idempotents, first we need to endow a proper algebraic structure on the set of all ultrafilters on $\mathbb{N}$. Indeed, there is a natural topological semigroup structure on the set of all ultrafilters; this semigroup structure is constructed by the lifting property of a natural topology on the set of all ultrafilters on $\mathbb{N}$. This natural topology is actually a specific compactification of $\mathbb{N}$, namely, the Stone–Čech compactification of $\mathbb{N}$. We will construct these topological and algebraic structures for arbitrary semigroups in the following two chapters.

In Chapter 4, we focus on the topological structure. We will put a topology on the set of all ultrafilters on a set S. We will show that this space βS of all ultrafilters on a discrete space S is in fact the Stone–Čech compactification of S. To do so, we will first define the Stone–Čech compactification for arbitrary spaces and define the limit along an ultrafilter, which is a superior notion for convergence compared to the usual (sequential) limits. The Stone–Čech compactification of a space X is defined by a universal property, that is, every map from X to a compact Hausdorff space C can be lifted to a map from the Stone–Čech compactification of X to C. So Stone–Čech compactification is the universal Hausdorff compactification of X. Nearly all results in this chapter are from [HS98, Chapter 3].

With the topological structure at hand, we will move on to constructing a semigroup structure on βS and studying the idempotent elements of this semigroup. Chapter 5 starts with a brief survey of the theory of semigroups and idempotent elements, leading up to Ellis' lemma, which states that idempotent elements of compact topological semigroups exist. Then we define the semigroup structure of βS and show it is indeed a compact semigroup, so we can apply Ellis' lemma. Section 4 of Chapter 5 focuses on the relation between idempotent ultrafilters and Hindman's theorem. We will see that "being close to" an idempotent ultrafilter in the topological semigroup βS is equivalent to witnessing Hindman's theorem. The final section of Chapter 5 focuses on idempotent ultrafilters that are contained in minimal ideals; these idempotents witness rich combinatorial patterns, and they have relations with notions of combinatorial largeness. We will define some of these notions, such as thick and syndetic, and show the relations with minimal idempotent ultrafilters. After this, we have explored the algebra of βS enough to cover the preliminaries for the two recent results we want to survey. Most of the results in Chapter 5 are from [HS98].

In Chapter 6, we present the recent proof of Theorem 1 by Matt Bowen from [Bow25], and in the final chapter, Chapter 7, we present a Galvin-Glazer style proof of Theorem 2 due to Mauro Di Nasso and Mariaclara Ragosta from [DNR24].

We should note that, if the set $\mathbb{N}$ is defined to include 0, many additive problems like Schur's theorem and Theorem 1 become trivial, because we can take $x = y = 0$ so that $\{x, y, x + y\} = \{0\}$ would be trivially contained in a cell. Indeed, we are tempted to start $\mathbb{N}$ from 2, since, in a similar fashion, $1 \in \mathbb{N}$ makes some exponential and multiplicative theorems trivial. Observe that 0 and 1 are idempotent elements in the respective semigroups $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$. To avoid getting these results trivially, we actually want $\mathbb{N}$ to have no idempotents, but we will take $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the sake of not being controversial. Usually, our proofs will work for partitions of any $\{n, n + 1, n + 2, \ldots\}$ for any natural number n. Therefore, if the reader wants to, they can start the set $\mathbb{N}$ from any large enough natural number. Also, we generally want the elements of patterns to be distinct; for example, $\{x, y, x + y, x \cdot y\}$ should generally be a set with four distinct elements in Theorem 1.

CHAPTER 2

ULTRAFILTERS

2.1 Filters and Ultrafilters

In this chapter, we will give an overview of ultrafilters and well-known facts about them. For a more in-depth look to the material in this chapter, the reader can refer to [HS98, Gol22].

Throughout this chapter, S will denote an infinite set.

Definition 1. A **filter** $\mathcal{F}$ on S is a non-empty collection of subsets of S satisfying the following three conditions:

1. $\varnothing \notin \mathcal{F}$ and $S \in \mathcal{F}$.
2. $\mathcal{F}$ is closed under finite intersections, that is, if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
3. $\mathcal{F}$ is closed upwards, that is, if $A \in \mathcal{F}$ and $A \subseteq B \subseteq S$, then $B \in \mathcal{F}$.

Example 1. The collection

$$\mathcal{F}_{cof} = \{A \subseteq S \mid S \setminus A \text{ is finite}\}$$

is a filter on S. This is known as the **Fréchet filter** or the **cofinite filter** on S.

Example 2. Let $B \subseteq S$ be any non-empty subset. The collection

$$\mathcal{F}_B = \{A \subseteq S \mid B \subseteq A\}$$

is a filter on S. We call this filter, **filter generated by set** B.

We can try to further this example by taking a collection $\mathcal{A} \subseteq \mathcal{P}(S)$ and defining the filter $\mathcal{F}$ generated by the collection A as

$$\mathcal{F}_{\mathcal{A}} = \{B \subseteq S \mid A \subseteq B \text{ for some } A \in \mathcal{A}\},$$

However, $\mathcal{F}_{\mathcal{A}}$ might not be a filter, as some finite intersection of its elements may be an empty set. Thus, we define the following property to characterise when a collection of sets generates a filter.

Definition 2. A family $\mathcal{A}$ of subsets of a set S has the **finite intersection property (FIP)** if every finite subcollection of $\mathcal{A}$ has a non-empty intersection. That is, for any integer $n \geq 1$ and sets $A_1, A_2, \ldots, A_n \in \mathcal{A}$,

$$\bigcap_{i=1}^{n} A_i \neq \emptyset.$$

Any family of sets with the finite intersection property generates a filter and every filter has FIP.

Our main objects throughout this text will be maximal filters, namely ultrafilters.

Definition 3. $\mathcal{U}$ is an **ultrafilter** on S if $\mathcal{F} = \mathcal{U}$ for any filter $\mathcal{F}$ on S with $\mathcal{U} \subseteq \mathcal{F}$, that is, $\mathcal{U}$ is a maximal filter with respect to subset inclusion.

We will give the following well-known equivalent definitions of an ultrafilter.

Proposition 1. Let $\mathcal{U} \subseteq \mathcal{P}(S)$ be a collection of subsets. The following are equivalent.

(1) U is an ultrafilter.

(2) U has FIP and for every $B \notin \mathcal{U}$ there is $A \in \mathcal{U}$ with $B \cap A = \varnothing$.

(3) U is a maximal collection with FIP.

(4) U is a filter such that for every $A \subseteq S$, either $A \in \mathcal{U}$ or $S \setminus A \in U$.

(5) For every finite partition $S = A_1 \sqcup \ldots \sqcup A_n$, there is unique $i \leq n$ such that $A_n \in U$.

The following lemma is simple but crucial for our arguments.

Lemma 1. Let $\mathcal{U}$ be an ultrafilter, $A_1, ..., A_n \subseteq S$ with $A_1 \sqcup ... \sqcup A_n \in \mathcal{U}$.

Then exactly one of A_i is in $\mathcal{U}$.

Proof. For every $i \leq n$, either $A_i \in \mathcal{U}$ or $S \setminus A_i \in \mathcal{U}$. If $A_i \in \mathcal{U}$, any other $A_j \subseteq S \setminus A_i$ cannot be in U. Assume, towards a contradiction, that is, for each i, assume $S \setminus A_i \in U$. Then $(A_1 \sqcup ... \sqcup A_n) \cap (\bigcap_{i \leq n} S \setminus A_i) = \varnothing \in U$. Contradiction. So, there is exactly one $i \leq n$ with $A_i \in \mathcal{U}$. □

The converse of the last lemma is true as well. So, we have this important characterization of ultrafilters through finite partitions. This characterization is due to Galvin [GH70]. It highlights how ultrafilters are about choosing and building things consistently.

Proposition 2. [GH70] Let $\mathcal{U} \subseteq \mathcal{P}(S)$ be a collection of subsets of S. Then the following are equivalent.

(1) For every finite partition $S = A_1 \sqcup \ldots \sqcup A_n$, there is unique i with $A_i \in \mathcal{U}$.

(2) For every 3-cell partition $S = A_1 \sqcup A_2 \sqcup A_3$, there is unique i with $A_i \in \mathcal{U}$.

(3) $\mathcal{U}$ is an ultrafilter.

Proof. $(3) \implies (1)$: Follows directly from the previous lemma.
$(1) \implies (2)$: Clear.
$(2) \implies (3)$: Let $\mathcal{U} \subseteq \mathcal{P}(S)$ be a collection such that for every partition $S = A_1 \sqcup A_2 \sqcup A_3$, $A_i \in \mathcal{U}$ for a unique i. We will show $\mathcal{U}$ is an ultrafilter.

- As $S = S \sqcup \varnothing \sqcup \varnothing$ is a 3-cell partition. $\varnothing$ cannot be in $\mathcal{U}$, since there should be a unique cell in $\mathcal{U}$. Thus, $\varnothing \notin \mathcal{U}$ and $S \in \mathcal{U}$
- Let $A \in \mathcal{U}$ and $A \subseteq B$. Consider the partitions $S = A \sqcup (B \setminus A) \sqcup (S \setminus B)$ and $S = B \sqcup S \setminus B \sqcup \varnothing$. If $\mathcal{U}$ contains $S \setminus B$ in the second partition, it cannot contain A from the first partition. $\mathcal{U}$ cannot contain $\varnothing$ as well, and so, $B \in \mathcal{U}$.

- Let $A, B \in \mathcal{U}$.

  Consider partitions $S = A \sqcup (S \setminus A) \sqcup \varnothing$ and $S = (A \cap B) \sqcup A \setminus B \sqcup (S \setminus A)$. $S \setminus A \notin \mathcal{U}$ from the first partition. So either $A \cap B \in U$ or $A \setminus B \in U$. But in the partition $S = B \sqcup (A \setminus B) \sqcup (S \setminus (A \cup B))$, $\mathcal{U}$ cannot contain both $A \setminus B$ and B. Thus $A \cap B \in \mathcal{U}$.

  Therefore, we have shown $\mathcal{U}$ is a filter.

- Let $A \subseteq S$. Then $S = A \subseteq S \setminus A$ is a partition. So either $A \in \mathcal{U}$ or $S \setminus A \in \mathcal{U}$.

Thus $\mathcal{U}$ is an ultrafilter. □

Another important way to define ultrafilters is through finitely-additive 0-1 measures.

Definition 4. A function $\mu : \mathcal{P}(S) \to \{0, 1\}$ is a **finitely additive 0-1 measure** on S if it satisfies:

1. $\mu(S) = 1$.
2. μ is finitely addtivie, that is if $A, B \subseteq S$ are disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Finitely aditive 0-1 measures are not measures in the usual sense since they are not σ-additive. While this is an important weakening from the usual measures, their strength is that they can be defined on countable spaces in non-trivial ways, and they are defined on the whole power set instead of some proper σ-algebra. While a usual measure on $\mathbb{N}$ will only depend on the measures on finitely many singletons, our measure can have very wild examples.

What makes these measures especially important for us is that they are actually just ultrafilters in a nice "measure" trenchcoat. A finitely additive 0-1 measure on set S carries exactly the same information as an ultrafilter. Thus, we will treat them the same after showing this fact.

Theorem 3. There is a canonical bijection between the set of all ultrafilters on S and the set of all finitely additive 0-1 measures on S.

Proof. Let $\mathcal{U}$ be an ultrafilter on S. Define a function $\mu_{\mathcal{U}} : \mathcal{P}(S) \to \{0, 1\}$ by:

$$\mu_{\mathcal{U}}(A) = \begin{cases} 1 & \text{if } A \in \mathcal{U} \\ 0 & \text{if } A \notin \mathcal{U} \end{cases}$$

We leave the verification of $\mu_{\mathcal{U}}$ being a finitely additive measure to the reader.

Let $\mu : \mathcal{P}(S) \to \{0, 1\}$ be a finitely additive measure. Define the collection of sets:

$$\mathcal{U}_\mu = \{A \subseteq S \mid \mu(A) = 1\}$$

Again, we leave the verification of $\mathcal{U}_\mu$ being an ultrafilter to the reader. Observe that given an ultrafilter $\mathcal{U}$, $\mathcal{U}_{\mu_{\mathcal{U}}} = \mathcal{U}$ and given a measure μ, $\mu_{\mathcal{U}_\mu} = \mu$. So these assignments are indeed bijections. □

We have shown that each ultrafilter induces a measure. This gives the intuition that sets in an ultrafilter are "large" while sets that are not in an ultrafilter are "small". In fact, from now on, we will usually say A is large, instead of $\mu_{\mathcal{U}}(A) = 1$. We might also say A is $\mathcal{U}-$large, instead of $A \in \mathcal{U}$ if there is more than one ultrafilter in play.

Using the measure approach, if a property is true in a large set, $A \in \mathcal{U}$, then this property holds everywhere almost surely. We will formalize and use this intuition in the next section.

Hopefully, it is evident by now that ultrafilters pick a cell from each partition of the base set, and not just the base set, but any large set. This is going to be the key idea on the rest of this text. We will use this idea in more advanced ways and try to find nice structures in a given partition. The whole study of finding structures in arbitrary partitions is the main goal of Ramsey theory. We will delve deeper into partition Ramsey theory in the next chapter; for now, let us focus on the existence of ultrafilters.

Definition 5. An ultrafilter $\mathcal{U}$ on S is **principal** if it contains a singleton. For each $x \in S$, the ultrafilter $\mathcal{U}_x = \{A \mid x \in A\}$ is called the principal ultrafilter of x.

An ultrafilter $\mathcal{U}$ is **non-principal** if it is not principal. Equivalently, $\mathcal{U}$ is non-principal if it does not contain any finite set.

While the existence of principal ultrafilters, which correspond the Dirac measures concentrated on singletons, is quite straightforward, the existence of a non-principal ultrafilter requires a bit of an effort.

Proposition 3 (Ultrafilter lemma)**.** Let $\mathcal{F}$ be a filter on S. There is an ultrafilter $\mathcal{U}$ extending $\mathcal{F}$, that is $\mathcal{F} \subseteq \mathcal{U}$.

Proof. Let $\mathcal{F}$ be a filter on S. We will use Zorn's lemma. Consider the partial order $P = \{\mathcal{G} : \mathcal{F} \subseteq \mathcal{G}\}$ ordered by $\subseteq$. Let $C = \{\mathcal{F}_\alpha\}_{\alpha \in I} \subseteq P$ be a chain of filters that contains $\mathcal{F}$. We first claim that $\bigcup C$ is a filter.

- S is in $\bigcup C$ as $S \in \mathcal{F}$. If $\varnothing \in \bigcup C$, then there is some $\mathcal{F}_\alpha$ such that $\varnothing \in \mathcal{F}_\alpha$, but this contradicts $\mathcal{F}_\alpha$ being a filter. Therefore $\varnothing \in \mathcal{F}$.
- Let $A \in \bigcup C$ and $B \subseteq A$. $A \in \bigcup C$ means there is some $\mathcal{F}_\alpha$ such that $A \in \mathcal{F}_\alpha$. Thus $B \in \mathcal{F}_\alpha$, since $\mathcal{F}_\alpha$ is a filter. Hence $B \in \bigcup C$.
- Let $A, B \in \bigcup C$. Then there is some α and β with $A \in \mathcal{F}_\alpha$ and $B \in \mathcal{F}_\beta$. Since C is a chain, either $\alpha < \beta$ or $\beta < \alpha$. Without loss of generality, say $\alpha < \beta$. Hence $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$. therefore $A, B \in \mathcal{F}_\beta$. So $A \cap B \in \mathcal{F}_\beta \subseteq \bigcup C$. So $\bigcup C$ is an upper bound for the chain C. Therefore, every chain has an upper bound. By Zorn's lemma, there is a maximal element $\mathcal{U}$ in $(P, \subseteq)$. $\mathcal{U}$ is a maximal filter, hence an ultrafilter and every filter in P extends $\mathcal{F}$. Thus $\mathcal{U}$ is an ultrafilter extending $\mathcal{F}$.

□

Corollary 4. There exists a non-principal ultrafilter.

Proof. Let $\mathcal{F}_{cof}$ be the cofinite filter. Using the ultrafilter lemma, we extend $\mathcal{F}_{cof}$ into an ultrafilter $\mathcal{U} \supseteq \mathcal{F}_{cof}$. If there is a finite set $A \subseteq \mathcal{U}$, $S \setminus A \in \mathcal{F}_{cof} \subseteq U$; thus $\varnothing = A \cap S \setminus A \in \mathcal{U}$, contradiction. Therefore, $\mathcal{U}$ does not contain any finite set. $\mathcal{U}$ is non-principal. □

In this proof, we used the axiom of choice; however, the existence of non-principal ultrafilters is weaker than the axiom of choice. Indeed, it is well-known that the

ultrafilter lemma is equivalent to the Boolean prime ideal theorem [Jec08, Theorem 2.2], which is known to be strictly weaker than the axiom of choice [HL71].

2.2 Ultrafilter Quantifiers and Tensor Product

Let S be an infinite set and $\mathcal{U}$ be an ultrafilter on S.

Definition 6. Let $\varphi(x)$ be a formula with one free variable. We define the **ultrafilter quantifier** as follows.

$$\mathcal{U}x\ \varphi(x) \iff \{a \in S \mid \varphi(a)\} \in \mathcal{U}$$

In other words, the statement $\mathcal{U}x\,\varphi(x)$ says that $\varphi(x)$ holds on a large set, that is, $\varphi(x)$ holds almost surely with respect to $\mu_{\mathcal{U}}$. Note that the quantifier $\mathcal{U}x$ is stronger than $\exists x$, but weaker than $\forall x$.

First, we observe some properties of this quantifier.

Proposition 4. Let φ, ψ be formulas with one free variable. Let $\mathcal{U}$ be an ultrafilter.

(1) $\mathcal{U}x\ \varphi(x) \wedge \psi(x) \iff (\mathcal{U}x\ \varphi(x)) \wedge (\mathcal{U}x\ \psi(x))$

(2) $\mathcal{U}x\ \varphi(x) \vee \psi(x) \iff (\mathcal{U}x\ \varphi(x)) \vee (\mathcal{U}x\ \psi(x))$

(3) $\mathcal{U}x\ (\neg\varphi(x)) \iff \neg(\mathcal{U}x\ \varphi(x))$

Proof. We will prove (1), and the rest is left as an exercise.

$$\begin{aligned}
\mathcal{U}x\ (\varphi(x) \wedge \psi(x)) &\iff \{x \mid \varphi(x) \wedge \psi(x)\} \in \mathcal{U} \\
&\iff \{x \mid \varphi(x)\} \cap \{x \mid \psi(x)\} \in \mathcal{U} \\
&\iff \{x \mid \varphi(x)\} \in \mathcal{U} \ \wedge\ \{x \mid \psi(z)\} \in \mathcal{U} \\
&\iff (\mathcal{U}x\ \varphi(x)) \ \wedge\ (\mathcal{U}x\ \psi(x))
\end{aligned}$$

□

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on S. Let $\varphi(x, y)$ be a formula with two variables. Then $\mathcal{U}x\,\varphi(x, y)$ is a formula with one free variable y. So we can quantify the free variable

with a quantifier. Hence, using the definition, we have

$$\begin{aligned}\mathcal{V}y\mathcal{U}x\,\varphi(x,y) &\iff \{y \mid \mathcal{U}x\,\varphi(x,y)\} \in \mathcal{V} \\ &\iff \{y \mid \{x \mid \varphi(x,y)\} \in \mathcal{U}\} \in V\end{aligned}$$

In particular, if we have a set $A \subseteq S \times S$, we have

$$\mathcal{V}y\mathcal{U}x\,(x,y) \in A \iff \{y \mid \{x \mid (x,y) \in A\} \in \mathcal{U}\} \in \mathcal{V}$$

One important observation is that ultrafilter quantifiers usually do not commute, that is,

$$\mathcal{V}y\mathcal{U}x\,\varphi(x,y) \iff \mathcal{U}x\mathcal{V}y\,\varphi(x,y)$$

in general, as shown by the following example.

Example 3. Let $S = \mathbb{N}$. Let $\mathcal{U}$ and $\mathcal{V}$ be non-principal ultrafilters on $\mathbb{N}$. Define $\psi(x,y)$ as $x < y$.

- The statement $\mathcal{V}y\mathcal{U}x\; x < y$ is false because for any $y \in \mathbb{N}$, $\{x \mid x < y\}$ is a finite set and not in $\mathcal{U}$. Therefore $\{y \mid \{x \mid x < y\} \in \mathcal{U}\} = \varnothing \notin \mathcal{V}$.

  But if we change the order of quantifiers;

- The statement $\mathcal{U}x\,\mathcal{V}y\; x < y$ is true because for every $x \in \mathbb{N}$ $\{y \mid x < y\}$ is a cofinite set and hence is in $\mathcal{V}$. Therefore $\{x \mid \{y \mid x < y\} \in \mathcal{V}\} \in U$.

Example 4. We will further the last example. Let $\mathcal{U}$ and $\mathcal{V}$ be distinct non-principal ultrafilters on the infinite set S. We will find $X \in S \times S$ such that $\mathcal{V}y\mathcal{U}x\,(x,y) \in X$ is not equivalent to $\mathcal{U}x\,\mathcal{V}y\,(x,y) \in X$.

Let $A \in \mathcal{U} \setminus \mathcal{V}$. Consider the set $X = A \times S$.

- $\mathcal{V}x\mathcal{U}y\,(x,y) \in X$ is false. Given $x \in S$, $\{y \mid (x,y) \in X\}$ is $\varnothing$ if $x \notin A$, and, S if $x \in A$. In particular, $\{y \mid (x,y) \in X\} \in \mathcal{V}$ iff $x \in A$.

  Therefore, $\{x \mid \{y \mid (x,y) \in X\} \in \mathcal{V}\} = A \notin \mathcal{U}$.

- $\mathcal{V}y\mathcal{U}x\,(x,y) \in X$ is true. Given $x \in S$, $\{x \mid (x,y) \in X\} = A \in \mathcal{U}$. Therefore

$$\{y \mid \{x \mid (x,y) \in X\} \in \mathcal{U}\} = S \in \mathcal{V}$$

We have one more observation on the commutativity of the quantifiers.

Proposition 5. Let $s \in S$. Let $\mathcal{U}_s$ be a principal ultrafilter generated by s and $\mathcal{U}$ be any ultrafilter on S. Let $\psi(x, y)$ be a formula with two free variables. Then

$$\mathcal{U}y\,\mathcal{U}_s x\;\psi(x,y) \iff \mathcal{U}_s x\,\mathcal{U}y\;\psi(x,y)$$

Ultrafilter quantifiers always commute with principal ultrafilter quantifiers. Throughout this text, the commutativity properties of ultrafilter quantifiers are important to keep track of.

An important property of ultrafilter quantifiers is that they help us obtain ultrafilters on the cartesian product S^n whenever we have ultrafilters on S.

Definition 7. Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on S.

We define the **tensor product** $\mathcal{U} \otimes \mathcal{V}$ as the ultrafilter

$$\mathcal{U} \otimes \mathcal{V} = \{A \subseteq S^2 \mid \mathcal{U}x\mathcal{V}y\;(x,y) \in A\}$$

on set $S \times S$. In other words, for every $A \subseteq S \times S$

$$A \in \mathcal{U} \otimes \mathcal{V} \iff \mathcal{U}x\,\mathcal{V}y\;(x,y) \in A$$

In particular,

$$A \in \mathcal{U} \otimes \mathcal{V} \iff \{x \mid \{y \mid (x,y) \in A\} \in \mathcal{V}\} \in \mathcal{U}$$

We leave it to the reader to verify that $\mathcal{U} \otimes \mathcal{V}$ is indeed an ultrafilter.

The (tensor) product $\mathcal{U} \otimes \mathcal{V}$ is sometimes called **Fubini product** of U and V. The terminology is inspired by Fubini's theorem because we have

$$\int_{S\times S} f\; d\mu_{\mathcal{U}\otimes\mathcal{V}} = \int_S \int_S f\; d\mu_{\mathcal{V}}\; d\mu_{\mathcal{U}}$$

We would like to note that, unlike Fubini's theorem for σ-additive measures, the order of integration here cannot be changed in general because $\mathcal{U} \otimes \mathcal{V}$ is rarely equal to $\mathcal{V} \otimes \mathcal{U}$, as we have demonstrated in the previous examples.

2.3 Push-forward Ultrafilter

We will define push-forwards of ultrafilters, which will be quite useful in the next chapters.

Definition 8. Let $f : S \to T$ be a function and $\mathcal{U}$ be an ultrafilter on S. The **push-forward ultrafilter** $f_*(\mathcal{U})$ is defined as

$$A \in f_*(\mathcal{U}) \iff f^{-1}(A) \in \mathcal{U}$$

It is straightforward to show $f_*(\mathcal{U})$ is an ultrafilter on T. Moreover, $A \in \mathcal{U}$ implies $f(A) \in f_*(\mathcal{U})$ for every $A \in \mathcal{U}$, because $f^{-1}(f(A)) \supseteq A \in \mathcal{U}$.

Proposition 6. Let $f : S \to T$ be a function and $\mathcal{U}$ be an ultrafilter on S. Then

$$B \in f_*(\mathcal{U}) \iff \mathcal{U}x\ (f(x) \in B) \iff f_*(\mathcal{U})y\ (y \in B)$$

Proof. We only show the second if and only if statement and leave the first one to the reader to verify.

$$\begin{aligned} f_*(\mathcal{U})y\ (y \in B) &\iff f^{-1}[\{y \in T \mid y \in B\}] \in \mathcal{U} \\ &\iff \{x \in S \mid f(x) \in B\} \in U \\ &\iff \mathcal{U}x\ (f(x) \in B) \end{aligned}$$

□

Example 5. (Push-forward via a constant map) We have $c_*(\mathcal{U}) = \mathcal{U}_t$ for every $\mathcal{U}$ whenever $c : S \to T$ is a constant function with value $t \in T$.

In particular, if $\mathcal{U}_s$ is a principal ultrafilter generated by $s \in S$, we have $f_*(\mathcal{U}_s) = \mathcal{U}_{f(s)}$ for every function f. We have non-trivial examples as well.

Example 6. We want to show that addition on $\mathbb{N}$ gives us an addition on the ultrafilters on $\mathbb{N}$. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function such that $f(n, m) = n + m$. Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on $\mathbb{N}$. Denote pushforward $f_*(\mathcal{U} \otimes \mathcal{V})$ by $\mathcal{U} \oplus \mathcal{V}$.

Let $A \subseteq \mathbb{N}$. Then $f^{-1}(A) = \{(n,m) \mid n+m \in A\}$, and so,

$$\begin{aligned} A \in \mathcal{U} \oplus \mathcal{V} = f_*(\mathcal{U} \otimes \mathcal{V}) &\iff f^{-1}(A) \in \mathcal{U} \otimes \mathcal{V} \\ &\iff \{n \mid \{m \mid (n,m) \in f^{-1}(A)\} \in \mathcal{V}\} \in \mathcal{U} \\ &\iff \{n \mid \{m \mid n+m \in A\} \in \mathcal{V}\} \in \mathcal{U} \\ &\iff \mathcal{U}n\, \mathcal{V}m\, (n+m \in A) \end{aligned}$$

In particular, we have

$$A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \mid A - n \in \mathcal{V}\} \in \mathcal{U}$$

where $A - n = \{m \mid n+m \in A\}$.

Observe that an ultrafilter $\mathcal{U}$ with $\mathcal{U} = \mathcal{U} \oplus \mathcal{U}$ satisfies

$$A \in \mathcal{U} \iff \{n \mid A - n \in \mathcal{U}\} \in \mathcal{U}$$

that is, if we think $\mathcal{U}$ as a measure, then $\mathcal{U}$ has some sort of shift invariance property.

Example 7. We will take a function mapping the additive structure $(\mathbb{N}, +)$ to the multiplicative structure $(\mathbb{N}, \cdot)$. We want to illustrate that an ultrafilter witnessing an additive combinatorial property, containing all the finite non-repeating sums, will be mapped to an ultrafilter witnessing a multiplicative combinatorial property, containing all the finite non-repeating products of an infinite set.

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ such that for every $A \in \mathcal{U}$, there is an infinite set $X \subseteq \mathbb{N}$ with all of its non-repeating finite sums inside A, that is,

$$FS(X) = \left\{ \sum_{x \in F} x \mid F \subseteq X \text{ is finite} \right\} \subseteq A$$

Let $f : \mathbb{N} \to \mathbb{N}$ with $f(n) = 2^n$. We will show that every element $B \in f_*(\mathcal{U})$ has an infinite set Y with all of its finite non-repeating multiplications inside A, that is $FP(Y) = \{\prod_{y \in F} y \mid F \subseteq Y \text{ is finite}\} \subseteq B$.

Observe that $f(FS(X)) = FP(\{2^x \mid x \in X\})$. Let $B \in f_*(\mathcal{U})$, then $f^{-1}(B) \in \mathcal{U}$. So, there is an infinite $X \subseteq \mathbb{N}$ with $FS(X) \subseteq f^{-1}(B)$, hence, the infinite set $Y = \{2^x \mid x \in X\}$ satisfies $FP(Y) \in B$. Therefore, $f_*(\mathcal{U})$ witnesses what we want.

A follow-up question can be asked: Given an ultrafilter $\mathcal{U}$, when do two functions give the same push-forward? It turns out that two functions give the same push-forward whenever they agree on a $\mathcal{U}$-large set.

Proposition 7. [Gol22] Let $\mathcal{U}$ be an ultrafilter on S and $f, g : S \to T$ be functions.

1. If $\mathcal{U}x\ (f(x) = g(x))$, then $f_*(\mathcal{U}) = g_*(\mathcal{U})$.

2. If $f_*(\mathcal{U}) = g_*(\mathcal{U})$, then we have $\mathcal{U}s\ f(s) \in g(A)$ for all $A \in \mathcal{U}$.

Proof. Let $f, g : S \to T$ be a function and $\mathcal{U}$ be an ultrafilter.

(1) Let $A \in f_*(\mathcal{U})$. then $f^{-1}(A) \in \mathcal{U}$. Since $X = \{x \mid f(x) = g(x)\} \in U$, we have $X \cap f^{-1}(A) \in \mathcal{U}$. But $X \cap f^{-1}(A) = X \cap g^{-1}(A) \subseteq g^{-1}(A)$. Hence $g^{-1}(A) \in \mathcal{U}$. Thus, $A \in g_*(\mathcal{U})$. $f_*(\mathcal{U}) \subseteq g_*(\mathcal{U})$. Since $f_*(\mathcal{U})$ is maximal, $f_*(\mathcal{U}) = g_*(\mathcal{U})$

(2) Let $A \in \mathcal{U}$. Then $g(A) \in g_*(\mathcal{U})$. Our assumption is $f_*(\mathcal{U}) = g_*(\mathcal{U})$. Thus, $g(A) \in f_*(\mathcal{U})$ and so, $f^{-1}(g(A)) = \{s \mid f(s) \in g(A)\} \in \mathcal{U}$.

□

A folklore result is that the tensor product of push-forwards is a push-forward of the product of functions that appears in [DN26].

Proposition 8. Let $S_1, ..., S_n$ and $T_1, ..., T_n$ be infinite sets. Let $\mathcal{U}_i$ be an ultrafilter on S_i and $f_i : S_i \to T_i$ be a function for every $i \leq n$. Then

$$(f_1, \ldots, f_n)_*(\mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_n) = f_{1*}(\mathcal{U}_1) \otimes \ldots \otimes f_{n*}(\mathcal{U}_n)$$

Proof. We will show this only for $n = 2$. Let $B \subseteq T_1 \times T_2$.

$$\begin{aligned}
B \in (f_1, f_2)_*(\mathcal{U}_1 \otimes \mathcal{U}_2) &\iff (f_1, f_2)^{-1}(B) \in \mathcal{U}_1 \otimes \mathcal{U}_2 \\
&\iff \mathcal{U}_1 x_1\, \mathcal{U}_2 x_2\ ((f(x_1), f(x_2)) \in B) \\
&\iff \mathcal{U}_1 x_1\ (\{s_2 \mid (f_1(x_1), f_2(s_2)) \in B\} \in \mathcal{U}_2) \\
&\iff f_{1*}(\mathcal{U}_1) y_1\ (\{s_2 \mid (y_1, f(s_2)) \in B\} \in \mathcal{U}_2) \\
&\iff f_{1*}(\mathcal{U}_1) y_1\, \mathcal{U}_2 x_2\ \ (y_1, f(x_2) \in B) \\
&\iff f_{1*}(\mathcal{U}_1) y_1\ f_{2*}(U_2) y_2\ \ ((y_1, y_2) \in B) \\
&\iff f_{1*}(\mathcal{U}_1) \otimes f_{2*}(\mathcal{U}_2)
\end{aligned}$$

The proof for the case $n > 2$ can be done in a similar manner. □

CHAPTER 3

RAMSEY THEORY VIA ULTRAFILTERS

3.1 Introduction to Ramsey Theory

Ramsey theory is the study of patterns that necessarily appear in partitions of a given large structure. Questions of Ramsey theory are generally of the following form: *Given a structure S and a partition $S = \bigsqcup_{i \in I} C_i$, what patterns must always appear in one of the cells?*

Ramsey theory is named after Frank Plumpton Ramsey, who proved two theorems, finite and infinite Ramsey's theorems, in 1930 that serve as prototypical examples of results in this area. The classical formulation of Ramsey's theorem is in terms of graphs, but we will reformulate it in terms of partitions of sets. First, we state the classical formulation of infinite Ramsey's theorem with graphs here.

Theorem 5 (Infinite Ramsey's Theorem, 1930)**.** Let G be a countably infinite complete graph, and let the edges of G be partitioned into finite cells. Then there is some cell containing the edges of a countably infinite complete induced subgraph of G.

The patterns that always arise in partitions are called partition regular. Being partition regular is equivalent to the existence of specific ultrafilters; we will give full ultrafilter characterizations of partition regularity in this section later.

In this thesis, we will focus on arithmetic patterns. The study of arithmetic partition regular patterns is called arithmetic Ramsey theory. Arithmetic Ramsey theory historically starts with two important theorems, Schur's theorem and van der Waerden's theorem.

Theorem 6 (Schur's Theorem, 1916)**.** Let $\mathbb{N}$ be partitioned into finitely many cells,

then there is a cell containing $\{x, y, x+y\}$ for some $x, y \in \mathbb{N}$.

Theorem 7 (Van der Waerden's Theorem, 1927)**.** Let $\mathbb{N}$ be partitioned into finitely many cells, then there is a cell containing arbitrarily long arithmetic progressions.

We will prove both of these theorems with ultrafilter methods and survey various strengthenings of Schur's theorem, such as Hindman's theorem.

Before we proceed further, we want to note that we are strictly interested in infinite structures and finite partitions in this thesis. The customary terminology of Ramsey theory is in terms of colorings instead of partitions. So, given a partition

$$S = C_1 \sqcup \cdots \sqcup C_r$$

we call the partitions with finitely many cells a finite coloring and call the cells themselves colors.

Definition 9. Let S be an infinite set and let $n \geq 1$.

(1) A **coloring** on S with colors $(C_i)_{i\in I}$ is a function $r : S \to \{C_i \mid i \in I\}$. For simplicity, we will abuse the notation and denote it as a partition

$$S = \bigsqcup_{i\in I} C_i$$

instead of $S = \bigsqcup_{i\in I} r^{-1}(C_i)$.

(2) Define $[S]^n = \{A \subseteq S \mid |A| = n\}$ as the collection of subsets of S with n many elements. In the case $(S, \leq)$ is a well-ordered set, we will treat $[S]^n$ as, the set of increasing n-tuples,

$$[S]^n = \{(s_1, \cdots, s_n) \mid s_i < s_{i+1} \text{ for all } i = 1, \ldots, n-1\}$$

(3) Let $[S]^n = \bigsqcup_{i\in I} C_i$ be a coloring. A subset $A \subseteq S$ is said to be C_i**-monochromatic** if $[A]^n \subseteq C_i$. We will also say A **is monochromatic** if A is C_i-monochromatic for some color C_i.

As baby examples of how ultrafilters can be used to derive Ramsey theoretic statements, we will now derive some basic results such as infinite pigeonhole principle and infinite Ramsey's theorem using ultrafilters.

Proposition 9 (Infinite Pigeonhole Principle)**.** Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be a finite coloring, then there is an infinite monochromatic set $X \subseteq \mathbb{N}$.

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be a finite coloring. Then there is i with $C_i \in \mathcal{U}$. Since $\mathcal{U}$ is non-principal, C_i is infinite. □

Now, we return to infinite Ramsey's theorem. Observe that, given a set S, the set $[S]^2$ can be seen as the edge set of the complete graph on the vertex set S. Using this intuition, we will reformulate and prove the infinite Ramsey's theorem. But before that, we need a lemma.

Lemma 2. Let $\mathcal{U}$ and $\mathcal{V}$ be non-principal ultrafilters on infinite set $\mathbb{N}$. Then

$$[\mathbb{N}]^2 \in \mathcal{U} \otimes \mathcal{V}$$

and consequently, the tensor product of ultrafilters on $\mathbb{N}$ gives us an ultrafilter on the edge set $[\mathbb{N}]^2$.

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on the infinite set $\mathbb{N}$. Recall the definition of product of ultrafilters;

$$A \in \mathcal{U} \otimes \mathcal{V} \iff \mathcal{U}n\, \mathcal{V}m \Big((n,m) \in A \Big)$$

We have

$$\begin{aligned} [\mathbb{N}]^2 = \{(n,m) \mid n < m\} \in \mathcal{U} \otimes \mathcal{V} &\iff \mathcal{U}n\, \mathcal{V}m \left((n,m) \in [\mathbb{N}]^2 \right) \\ &\iff \mathcal{U}n\, \mathcal{V}m \left(n < m \right) \\ &\iff \{n \mid \{m \mid n < m\} \in \mathcal{V}\} \in \mathcal{U} \end{aligned}$$

Since for every $n \in \mathbb{N}$, $\{m \mid n < m\}$ is cofinite, $[\mathbb{N}]^2 \in \mathcal{U} \otimes \mathcal{V}$. □

In fact, a more general fact, which will help us prove infinite Ramsey's theorem for higher dimensions, is the following.

Proposition 10. Let $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \ldots \mathcal{U}^{(m)}$ be ultrafilters on $\mathbb{N}$. Then

$$[\mathbb{N}]^m \in \mathcal{U}^{(1)} \otimes \ldots \otimes \mathcal{U}^{(n)}$$

Theorem 8. (Infinite Ramsey's Theorem) Let $[\mathbb{N}]^2 = C_1 \sqcup \cdots \sqcup C_r$, be a finite coloring. Then there is an infinite monochromatic subset $A \subseteq \mathbb{N}$.

We will give two proofs. The first one is using an ultrafilter, and the second one uses ultrafilter quantifiers.

Proof I. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $n_0 \in \mathbb{N}$. Define

$$C_i(n_0) = \{m \mid (n_0, m) \in C_i\}$$

as set of natural numbers m such that the edge (n_0, m) has color C_i. We can partition

$$\mathbb{N} = C_1(n_0) \sqcup \cdots \sqcup C_r(n_0) \sqcup \{n_0\}$$

Then there exists i_0 such that $C^0 = C_{i_0}(n_0) \in \mathcal{U}$. Pick $n_1 \in C^0$. As before, $C_i(n_1)$'s partition $\mathbb{N}$. So there exists i_1 such that $C_{i_1}(n_1) \in \mathcal{U}$. Set $C^1 = C^0 \cap C_{i_1}(n_1) \in \mathcal{U}$.

Pick $n_2 \in C^1$ and continue the process inductively to get a sequence $(i_k)_{k\in\mathbb{N}}$ with $i_k \in \{1, \ldots, r\}$, strictly decreasing sequence of sets C^k and a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that $n^{k+1} \in C^k$ with $\{n_k, n_i\}$ is of color C_k for all $i < k$. By the pigeonhole principle, (i_k) has a constant subsequence $(i_{k_t})_{t\in\mathbb{N}}$ with $i_{k_1} = i_{k_t}$ for all t.

Then the set $A = \{n_{k_t} \mid t \in \mathbb{N}\}$ is an infinite monochromatic set with $\{n, m\} \in C_{i_1}$ for every $n, m \in A$. □

This proof can be rewritten using the pigeonhole principle repeatedly instead of an ultrafilter, which is actually the standard, well-known proof. The ultrafilter version emphasizes "just go towards the largely colored way", since $C_i(x) \in \mathcal{U}$ means that a large set of things are colored C_i with x. The second proof illustrates "just go towards the largely colored way" even better, without alternating between colors in the construction like Proof I.

Proof II. Let $\mathcal{U}$ be a non-principal ultrafilter. Let $[\mathbb{N}]^2 = C_1 \sqcup \cdots \sqcup C_r$ be a coloring with r colors.

By the earlier proposition $[\mathbb{N}]^2 \in \mathcal{U} \otimes \mathcal{U}$. Then exactly one of the colors must be in $\mathcal{U} \otimes \mathcal{U}$, that is $C = C_i \in \mathcal{U} \otimes \mathcal{U}$ for some i.

$$C \in \mathcal{U} \otimes \mathcal{U} \iff \mathcal{U}n\, \mathcal{U}m\ (n, m) \in C$$

Let $C^* = \{n \mid \mathcal{U}m\ (n, m) \in C\} \in \mathcal{U}$ and $C(n) = \{m \mid \{n, m\} \in C\}$. Pick $n_0 \in C^*$. Then $C(n_0) \in \mathcal{U}$ and we can pick $n_1 \in C^* \cap C(n_0) \in \mathcal{U}$. As before, $C(n_1) \in \mathcal{U}$ as $n_1 \in C^*$. Set $C^1 = C^* \cap C(n_0) \cap C(n_1) \in \mathcal{U}$. Pick $n_2 \in C^1$. Similarly, set $C^2 = C^1 \cap C(n_2) \in \mathcal{U}$. Continue picking $n_{k+1} \in C^k$ and setting $C^{k+1} = C^k \cap C(n_{k+1})$ inductively.

Thus $A = \{n_k \mid k \in \mathbb{N}\} \subseteq \mathbb{N}$ is an infinite monochromatic subset of $\mathbb{N}$ such that $\{n_k, n_l\} \in C_i$ for every $l, k \in \mathbb{N}$. □

Proof II is especially important as it is a baby version of the general proof template we are frequently going to use. Also, we can generalize Proof II to higher dimensions; in fact, the ultrafilter product directly gives the infinite Ramsey's theorem for higher dimensions.

Theorem 9 (Infinite Ramsey's Theorem for Higher Dimensions)**.** Let m, r be natural numbers and let $[\mathbb{N}]^m = C_1 \sqcup \cdots \sqcup C_r$ be a coloring. There is an infinite monochromatic subset A.

The proof with ultrafilter quantifiers can be found in [Tod10].

As we mentioned before, we are trying to understand partition regular patterns, which are patterns that always occur in every partition. Patterns are formulated as follows. Given a set S and a property φ, we consider the collection

$$\mathcal{G}_\varphi = \{A \subseteq S \mid A \text{ satisfies } \varphi\}$$

and we usually identify the property φ and the collection $\mathcal{G}_\varphi$. Now, we define patterns $\mathcal{G}_\varphi$ that always occur in every finite partition.

Definition 10. Let S be a set and $\mathcal{G} \subseteq \mathcal{P}(S)$ be a collection of subsets of S.

(1) $\mathcal{G}$ is **weakly partition regular** if every finite coloring $S = C_1 \sqcup \cdots \sqcup C_r$ has some monochromatic $A \in \mathcal{G}$.

A property φ is weakly partition regular if $\mathcal{G}_\varphi$ is weakly partition regular.

(2) $\mathcal{G}$ is **partition regular** if for every $A \in \mathcal{G}$ and for every finite coloring $A = C_1 \sqcup \cdots \sqcup C_r$ of A, there is some monochromatic $X \in \mathcal{G}$.

A property φ is partition regular if $\mathcal{G}_\varphi$ is partition regular.

Every result we have shown in this chapter until now is weakly partition regularity of specific collections. For example, infinite Ramsey's theorem says that the collection $\mathcal{G} = \left\{[A]^k \mid A \subseteq S\right\}$ with $S = [\mathbb{N}]^k$ is weakly partition regular.

If $S \in \mathcal{G}$, then partition regularity implies weakly partition regularity. Usually, we will pick $\mathcal{G}$ as upwards closed collections, so partition regularity will imply weakly partition regularity.

Ultrafilters are partition regular, and they characterize being both weakly partition regular and partition regular. So we have the following theorems.

Proposition 11. [HS98, Theorem 5.7] Let $\mathcal{G} \subseteq \mathcal{P}(S)$ be a collection of subsets of S. Then the following are equivalent.

(1) $\mathcal{G}$ is weakly partition regular.

(2) There is an ultrafilter $\mathcal{U}$ on S such that for every $A \in \mathcal{U}$, there is some $G \in \mathcal{G}$ with $G \subseteq A$.

With this characterisation, we can now show that a collection is weakly partition regular by finding an ultrafilter witnessing that property. We will use this characterisation frequently. In fact, most of the time, the ultrafilter that witnesses the property we are looking for will also witness other properties as well. This gives us a hierarchy of properties. If a property is witnessed by an ultrafilter, we can think of this property as some notion of largeness. We will investigate some useful notions of largeness in later chapters and their deep relationship with algebraic properties of βS, the space of all ultrafilters on S.

We have a similar characterization for partition regularity as well.

Proposition 12. [HS98, Theorem 3.11] Let $\mathcal{G} \subseteq \mathcal{P}(S)$ be a collection. Define

$$\mathcal{G}^\uparrow = \{B \subseteq S \mid A \subseteq B \text{ for some } A \in \mathcal{G}\}$$

Then the following are equivalent.

(1) G is partition regular.

(2) Given $A \in \mathcal{G}$, there is an ultrafilter $\mathcal{U}$ on S such that $A \in \mathcal{U} \subseteq \mathcal{G}^{\uparrow}$.

We should note that the non-trivial side to prove is $(1) \implies (2)$ for both propositions.

A particular category of properties we are interested in is the solutions of systems of equations. Given a system of equations, we are interested in when the collection of its solutions is (weakly) partition regular.

For example, consider the homogeneous linear system $A \cdot \vec{v} = \vec{0}$ where A is an integer $m \times n$ matrix and $\vec{v} \in \mathbb{N}^n$. Let $\mathcal{G} = \{\{v_1, \ldots, v_n\} \subseteq \mathbb{N} \mid A\vec{v} = \vec{0}\}$. We say $A \cdot \vec{v} = \vec{0}$ is (weakly) partition regular if $\mathcal{G}$ is partition regular. Rado's theorem gives an exact criterion for the partition regularity of a linear homogeneous system.

Theorem 10 (Rado's Theorem)**.** [Bol98, Theorem VI.27] Let A be an $m \times n$ integer matrix and $A\vec{v} = \vec{0}$ a linear system of equations. Let c_i denote the i^{th} column of A. Then the following are equivalent.

(1) $A\vec{v} = 0$ is a partition regular system of equations.

(2) A satisfies the **columns property**, that is there is a partition $C_1 \sqcup \cdots \sqcup C_r = \{1, \ldots, n\}$, where n is the number of columns of A such that

- $\sum_{i \in C_1} c_i = 0$, that is the sum of all the columns in C_1 is zero.
- $\sum_{i \in C_j} c_i$ can be written as a $\mathbb{Q}$-linear combination of columns $c \in C_2 \cup \cdots \cup C_j$.

A thorough look at the partition regularity of linear systems, including Rado's theorem, can be found in [HS98, Chapter 15].

The question of which classes of non-linear systems of equations are partition regular is an active open problem with various recent results, but they are outside the scope of this thesis.

Example 8. We will give some explicit examples of systems of equations and remark on their partition regularity properties.

1. We look at Schur's theorem, which states that given any finite partition of $\mathbb{N}$, there is a monochromatic $\{x, y, x+y\}$. Observe that $\{x, y, x+y\}$ is the solution of the system

$$x_1 + x_2 - x_3 = 0$$

So, let $A_1 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$, and so, Schur's theorem states that $A_1\vec{v} = \vec{0}$ is weakly partition regular.

2. We consider the length $t+1$ arithmetic progressions

$$AP_t(a, b) = \{a, a + b \ldots, a + tb\}$$

for some $a, b \in \mathbb{N}$. Then $AP_t(a, b)$ is a solution of the system

$$x_3 - x_2 = x_2 - x_1$$

$$\ldots$$

$$x_t - x_{t-1} = x_2 - x_1$$

For example for $t = 5$, we have the matrix $A_2 = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$.
Then van der Waerden's theorem implies that $A_2\vec{v} = \vec{0}$ is weakly partition regular for every $t \in \mathbb{N}$.

3. We will further Schur's theorem and give an example of an infinite system of linear equations. Consider the following infinite system

$$x_1 = y_1$$

$$x_1 + x_2 = y_2$$

$$x_1 + x_2 + x_3 = y_3$$

$$\ldots$$

$$x_1 + \ldots + x_n = y_n$$

$$\ldots$$

The set $\{x_n\}_{n\in\mathbb{N}} \cup \{\sum_{i=1}^{n} x_i\}_{n\in\mathbb{N}}$ is a solution set for the given system. This system is partition regular; by Hindman's theorem, we will actually have a stronger result.

4. We can also try to further van der Waerden's theorem by considering the following infinite system

$$x_3 - x_2 = x_2 - x_1$$
$$\dots$$
$$x_t - x_{t-1} = x_2 - x_1$$
$$\dots$$

Any solution of this system is an infinite length arithmetic progression. Infinite length arithmetic progressions are not weakly partition regular.

5. Pythagorean triples are the solutions of the non-linear system

$$a^2 + b^2 = c^2$$

This system is weakly partition regular. A proof can be found in [FKM25]

6. An example of a non-linear Diophantine equation that is not weakly partition regular is the following:

$$x + y = z^2$$

This system is not weakly partition regular; a coloring without any monochromatic solution of this system can be found in [CGS12].

7. Consider the following non-linear system of equations:

$$x_1 + x_2 = x_3$$
$$x_1 \cdot x_2 = x_4$$

A solution of this system is of the form $\{x, y, x+y, x\cdot y\}$. The weakly partition regularity of this system is an open problem known as Hindman's conjecture. In Chapter 6, we will prove that there is always a monochromatic solution of this system for 2-colorings of $\mathbb{N}$.

8. The exponential version of Schur's theorem states that there is a monochromatic $\{x, y, y^x\}$ for any finite coloring of $\mathbb{N}$, which can be stated as the following equation is weakly partition regular:

$$(x_2)^{x_1} = x_3$$

9. We will give an infinite strengthening of the last example, that is, consider the following infinite non-linear system:

$$\begin{aligned}(x_2)^{x_1} &= x_3\\ (x_5)^{x_4} &= x_6\\ &\dots\\ (x_{3t-1})^{x_{3t-2}} &= x_{3t}\\ &\dots\end{aligned}$$

This system is weakly partition regular, that is, given a finite coloring of $\mathbb{N}$, there is a color C and a sequence $\{b_n\}_{n\in\mathbb{N}}$ such that $\{b_n, b_{n+1}, b_{n+1}^{b_n}\} \subseteq C$ for every n. We will present a proof of this result and the exponential version of Schur's theorem in Chapter 7.

We will focus on the first three examples in the rest of this chapter. Schur's theorem is a direct corollary of Rado's theorem, but we will give the classical proof of it. Also, we are interested in the infinite version of Schur's theorem, which is Hindman's theorem. We will prove this result in the next section, and we will prove van der Waerden's theorem in the final section of this chapter. Both proofs will rely heavily on ultrafilters and the machinery we defined in the last chapter.

3.2 Schur's Theorem and IP sets

In this section, we will prove the result that kick-started arithmetic Ramsey theory, Schur's theorem. To prove Hindman's theorem, an infinite version of Schur's theorem, we will introduce IP sets. We will prove that IP sets are weakly partition regular. Finally, we will briefly survey the partition regularity results related to IP sets.

Theorem 11 (Schur's Theorem)**.** Let $\mathbb{N} = C_1 \sqcup \dots \sqcup C_r$ be a finite coloring. Then, there exists a color C_i and $x, y \in \mathbb{N}$ such that $\{x, y, x+y\}$.

The classical proof idea is to use Ramsey's theorem after inducing a coloring on $[\mathbb{N}]^2$ from a given coloring of $\mathbb{N}$.

Proof. Let $\mathcal{U}$ be a ultrafilter on $\mathbb{N}$ and $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any finite coloring of $\mathbb{N}$. We will define a coloring on $[\mathbb{N}]^2 = \bar{C}_1 \sqcup \cdots \sqcup \bar{C}_r$ by

$$(a, b) \in \bar{C}_i \iff |a - b| \in C_i$$

Then $\mathcal{U} \otimes \mathcal{U}$ is an ultrafilter on $[\mathbb{N}]^2$, so there is a unique color $\bar{C}_i$ such that $\bar{C}_i \in \mathcal{U} \otimes \mathcal{U}$. Thus,

$$A = \left\{x \in \mathbb{N} \mid \left\{y \in \mathbb{N} \mid (x, y) \in \bar{C}_i\right\} \in \mathcal{U}\right\} \in \mathcal{U}$$

Pick $a \in A$, then $\bar{C}_i(a) = \left\{y \in \mathbb{N} \mid (a, y) \in \bar{C}_i\right\} \in \mathcal{U}$. So, we can pick $b \in \bar{C}_i(a) \cap A \in \mathcal{U}$ with $b > a$. Now, $b \in A$ implies $\bar{C}_i(b) \in \mathcal{U}$. So, we can pick $c \in \bar{C}_i(a) \cap \bar{C}_i(b) \cap A \in \mathcal{U}$ such that $c > b > a$.

Therefore, $(a, b), (a, c), (b, c) \in \bar{C}_i$, that is $|a - b|, |a - c|, |b - c| \in C_i$.

Set $x = b - a$ and $y = c - b$ to get

$$\begin{aligned} x = b - a &\in C_i \\ y = c - b &\in C_i \\ x + y = c - a &\in C_i \end{aligned}$$

□

We want to prove a stronger version of Schur's theorem. Instead of having a monochromatic $\{x_1, x_2, x_1 + x_2\}$, we want $\{x_n\}_{n \in \mathbb{N}}$ with the set of all of its non-repeating sums monochromatic. To prove this, we are going to use the existence of an ultrafilter $\mathcal{U}$ satisfying $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$. Recall that in Example 6 of the last chapter, we observed that if an ultrafilter satisfies $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$, it can be seen as a shift-invariant measure. In this section, we will assume shift-invariant ultrafilters exist without proof. Their existence will be proven in Chapter 5.

First, we will remind the terminology of finite sums and recall the sum $\mathcal{U} \oplus \mathcal{V}$ of ultrafilters from Examples 6 and 7.

Definition 11. Let $X \subseteq \mathbb{N}$ be a subset. Define

$$FS(X) = \left\{\sum_{x \in F} x \mid F \subseteq X \text{ is finite}\right\}$$

$$FP(X) = \left\{ \prod_{x \in F} x \mid F \subseteq X \text{ is finite} \right\}$$

$FS(X)$ denotes the non-repeating finite sums of elements in X, while $FP(X)$ denotes the same thing for multiplication.

Let $A \subseteq \mathbb{N}$. The set A is an **IP set** if there is an infinite set X with $FS(X) \subseteq A$. The set A is a **multiplicative IP set** if there is an infinite set Y with $FP(Y) \subseteq A$.

We want to note that the naming of IP set stands for infinite-dimensional parallelopiped, as one can visualize the set $FS(x_1, x_2) = \{x_1, x_2, x_1 + x_2\}$ as a two-dimensional parallelopiped, when x_1 and x_2 are driven as vectors.

Schur's theorem can be reformulated as, for every finite coloring of $\mathbb{N}$, there are $x_1, x_2 \in \mathbb{N}$ with monochromatic $FS(\{x_1, x_2\})$. The infinite version of Schur's theorem can be reformulated as the collection of IP sets is weakly partition regular.

Definition 12. Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on $\mathbb{N}$. We define the sum $\mathcal{U} \oplus \mathcal{V}$ of ultrafilters by

$$A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \in \mathbb{N} \mid A - n \in \mathcal{V}\} \in \mathcal{U}$$

where $A - n = \{m \in \mathbb{N} \mid n + m \in A\}$

We give the set on the right side a special name, as it behaves like a 'subtraction'. Let $A \subseteq \mathbb{N}$ be any subset, then

$$A - \mathcal{V} = \{n \mid A - n \in \mathcal{V}\}$$

By definition, for every ultrafilters $\mathcal{U}, \mathcal{V}$ and subset $A \subseteq \mathbb{N}$, we have

$$A - \mathcal{V} \iff A \in \mathcal{U} \oplus \mathcal{V}$$

Thus, if we have an ultrafilter $\mathcal{U}$ with $\mathcal{U} = \mathcal{U} \oplus \mathcal{U}$, then for every $A \subseteq \mathbb{N}$,

$$A - \mathcal{U} \iff A \in \mathcal{U} \oplus \mathcal{U}$$

Theorem 12. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ satisfying $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$. If $C \in \mathcal{U}$, then C is an IP set.

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ such that $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$. Let $C \in \mathcal{U}$. Then we have $C - \mathcal{U} \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter with $C, C - \mathcal{U} \in \mathcal{U}$, we have

$$C^* = \{x \in C \mid C - x \in \mathcal{U}\} = C \cap (C - \mathcal{U}) \in \mathcal{U}$$

We will recursively construct the set $X = \{x_i\}_{i \in \mathbb{N}}$. Let $x_1 \in C^*$, then $C - x_1 \in \mathcal{U}$. Let

$$A_1 = C \cap (C - x_1) \in \mathcal{U}$$
$$B_1 = A_1 - \mathcal{U}$$

Since $A_1 \in \mathcal{U} \oplus \mathcal{U}$, we have $B_1 = A_1 - \mathcal{U} \in \mathcal{U}$. So we can pick $x_2 \in A_1 \cap B_1$, so $x_1, x_2 \in C$ with $x_1 + x_2 \in C$.

We have $FS(x_1, x_2) \subseteq A$. We have shown the base step, and we can move on to the induction step, however we will show one more step to illustrate the process clearly.

Since $x_2 \in B_1 = A_1 - \mathcal{U}$, $A_1 - x_2 \in \mathcal{U}$. We can set

$$A_2 = A_1 \cap (A_1 - x_2) \in \mathcal{U}$$
$$B_2 = A_2 - \mathcal{U}$$

Since $A_2 \in \mathcal{U} \oplus \mathcal{U}$, we have $B_2 = A_2 - \mathcal{U} \in \mathcal{U}$. We can pick $x_3 \in A_2 \cap B_2$. Since $x_3 \in A_1 \subseteq A_2$, we have $x_3 \in C - x_1$ and $x_3 + x_1 \in C$. Observe that $x_3 \in A_2 \implies x_3 \in A_1 - x_2$, that is $x_2 + x_3 \in A_1 = C \cap (C - x_1)$. So $x_2 + x_3 \in C$ and $x_2 + x_3 \in C - x_1$. Therefore, $x_1 + x_2 + x_3 \in C$. So $FS(x_1, x_2, x_3) \in C$.

Now we will do the recursion step.

Assume for every $n \leq k$, $A_n, B_n \in \mathcal{U}$, $(x_i)_{i<k}$ are defined such that $x_n \in A_n \cap B_n$ and $B_n = A_n - \mathcal{U}$. We define

$$x_{k+1} \in A_k \cap B_k$$
$$A_{k+1} = A_k \cap (A_k - x_k) \in \mathcal{U}$$
$$B_{k+1} = A_{k+1} - \mathcal{U}$$

$A_{k+1} \in \mathcal{U}$, since $x_k \in B_k$ and $B_{k+1} \in \mathcal{U}$, since $A_{k+1} \in \mathcal{U} \oplus \mathcal{U}$.

A simple way to check $FS(\{x_k\}_{k \in \mathbb{N}}) \subseteq C$ is, take an element $x = x_{i_1} + \cdots + x_{i_k} \in FS(\{x_n\})$ where the indices are increasing. Then $x_{n_k} \in A_{n_k}$ and $A_{n_k} \subseteq$

$A_{n_{k-1}} - x_{n_{k-1}}$. Then $x_{n_{k-1}} + x_{n_k} \in A_{n_{k-1}}$ and $A_{n_{k-1}} \subseteq A_{n_{k-2}} - x_{n_{k-2}}$. Then $x_{n_{k-2}} + x_{n_{k-1}} + x_{n_k} \in A_{k_{n-2}}$. We continue doing this until $x_{n_1} + \ldots + x_{n_k} \in C$.

□

Corollary 13. IP sets are weakly partition regular.

Proof. Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any finite coloring and let $\mathcal{U}$ be an ultrafilter satisfying $\mathcal{U} = \mathcal{U} \oplus \mathcal{U}$. Then there is some i such that $C_i \in \mathcal{U}$. By the previous theorem, there is an infinite set $X \subseteq \mathbb{N}$ with $FS(X) \subseteq C_i$. □

The proof of the theorem is due to Galvin and Glazer; the exact proof and a quick survey of its history can be found in [HS98, Chapter 5]. We will present a more recent and refined version of this proof in Chapter 5. The general proof template of utilising an ultrafilter that satisfies a specific property to recursively build up a nice set is used frequently. Here, the property we used is rather algebraic; we used $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$, but usually this property is either algebraic, combinatorial, or some sort of density statement that the ultrafilter is witnessing. The reason we call the property $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$ algebraic is that it is a property of an important topological semigroup, namely the Stone–Čech compactification of $\mathbb{N}$.

We have proven that IP sets are weakly partition regular; in Chapter 5, we will prove that they are partition regular. This result is known as Hindman's theorem.

Theorem 14. (Hindman's Theorem) IP sets are partition regular.

We will prove this theorem in Chapter 5. We have shown that for ultrafilters $\mathcal{U}$ with $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$, every element of $\mathcal{U}$ is an IP set. The converse holds as well, that is, given an IP set A, there is an ultrafilter $\mathcal{U}$ with $A \in \mathcal{U}$ and $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$. We will define IP sets for arbitrary semigroups and prove these results for arbitrary semigroups, as well, though in this chapter, we are only considering results for $\mathbb{N}$. In particular, the multiplicative version of the results we have proved also holds, but we will give an easier proof. We will show that the weakly partition regularity of multiplicative IP sets is an easy corollary of the weakly partition regularity of IP sets.

Theorem 15 (Multiplicative Hindman's Theorem)**.** Multiplicative IP sets are weakly partition regular.

Proof. Let $\mathbb{N} = C_1 \sqcup \cdots C_r$ be any finite coloring. Consider the function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = 2^n$. We will pull back the coloring on the range of f to the domain of f, that is, consider the coloring

$$\mathbb{N} = f^{-1}(C_1) \sqcup \cdots \sqcup f^{-1}(C_r)$$

Then by, Hindman's theorem, there is $X \subseteq \mathbb{N}$ such that $FS(X) \subseteq f^{-1}(C_i)$ for some i. Thus $Y = f(X)$ is a subset with $f(FS(X)) = FP(Y) \subseteq C_i$. □

So for any finite coloring of $\mathbb{N}$, there is a color C_i that is an IP set and a color C_j that is a multiplicatively IP set. A natural follow-up question is, can it be that $i = j$? That is, can we have infinite sets X and Y with monochromatic $FS(X) \cup FP(Y)$, and the answer is positive; given any finite coloring, there is always a color C_i that is both additively and multiplicatively IP set. Again, we are going to prove this in Chapter 5. An even stronger question is, can we have $X = Y$?

Given any finite coloring of $\mathbb{N}$, is there an infinite set X such that $FS(X) \cup FP(X)$ is monochromatic? The answer is negative as we shall learn later. This leads us to the following simple open question.

Question 1 (Hindman's Conjecture)**.** Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any finite coloring. Is there $x, y \in \mathbb{N}$ such that $\{x, y, x + y, xy\}$ is monochromatic?

This question is still open for finite colorings, but for colorings with only two colors, the answer is positive. This was recently proven in [Bow25]. We will go over this paper and present the proof in Chapter 6. Along the way, we will build all the necessary background.

At first sight, it will seem as if our general machinery works for only associative operations. Though this turns out to be not the case, as we shall prove that there are monochromatic triples for exponentials in the later chapters using a Galvin-Glazer type induction. This recent proof is from [DNR24] and is due to Mauro Di Nasso and Mariaclara Ragosta.

For now, we turn to the arithmetic progressions, specifically to van der Waarden's theorem. At the end of the last section, we stated that AP_t is weakly partition regular

for every t. This is a natural historical continuation from Schur's theorem and again a very crucial result of arithmetic Ramsey theory. Though the result is historically central, the proof we are going to present is quite recent.

3.3 Van der Waerden's Theorem

In this section, we will present a proof of van der Waerden's theorem from [DN26]. This proof is due to Mauro Di Nasso, and it is elementary in the sense that it only uses the tools developed in the last chapter, such as push-forwards and the product of ultrafilters.

Theorem 16 (Van der Waerden's Theorem)**.** Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be a finite coloring. Then, there is a color C_i such that for every t, there is a monochromatic arithmetic progression of length t contained in C_i, that is there is $a, b \in \mathbb{N}$ with

$$\{a, a+b, \ldots, a+tb\} \subseteq C_i$$

Van der Waerden's theorem was first published in [VdW27] by B. L. van der Waerden. It is an important milestone in arithmetic Ramsey theory. Over the last century, this theorem has had many variants and proofs. Though we are only going to present the standard version with an ultrafilter proof, there are many variants of van der Waarden with different proofs. A recent survey of the history of van der Waerden and its variants is [BR26]. In this survey, there are three proofs of the standard van der Waerden: a combinatorial one, a topological dynamic one, and an algebraic proof, which is similar to one we are presenting. There are also a variety of variants of van der Waerden; in fact, we will use a dynamical variant of van der Waerden, namely the IP* variant, in Chapter 7.

Now, we go back to the prerequisites for the proof we are presenting.

Definition 13. (Ultrafilter Sum and Product)

Again, recall Example 7 in the last chapter. Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on $\mathbb{N}$ and $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $f(n, m) = n + m$, then $f_*(\mathcal{U} \otimes V) = \mathcal{U} \oplus V$.

That is $A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \in \mathbb{N} \mid A - n \in \mathcal{V}\} \in \mathcal{U}$.

We denote the repeated sums and products as

$$\mathcal{U}^{\otimes n+1} = \mathcal{U} \otimes \mathcal{U}^{\otimes n}$$

If $f : \mathbb{N}^k \to \mathbb{N}$ with $f((n_i)_{i\leq k}) = \sum_{i\leq k} n_i$, then we denote $f_*(\mathcal{U}^{\otimes k}) = \mathcal{U}^{\oplus k}$. So repeated sums of $\mathcal{U}$ with itself are denoted with $\mathcal{U}^{\otimes n}$.

Recall that we defined

$$A - \mathcal{V} = \{n \in \mathbb{N} \mid A - n \in \mathcal{V}\}$$

$A - \mathcal{V}$ is the set of elements that shifts A into a $\mathcal{V}$-large set. Thus, we have

$$A \in \mathcal{U} \oplus \mathcal{V} \iff A - \mathcal{V} \in \mathcal{U}$$

Define $T_i : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $T_i(a, b) = a + ib$. Therefore, given an ultrafilter $\mathcal{W}$ on $\mathbb{N}^2$ and set $A \in \mathbb{N}$, we have

$$\begin{aligned} A \in T_{i*}(\mathcal{W}) &\iff T_i^{-1}(A) \in \mathcal{W} \\ &\iff \left\{(n, m) \in \mathbb{N}^2 \mid n + i \cdot m \in A\right\} \in \mathcal{W} \end{aligned}$$

Lemma 3. Let $t \in \mathbb{N}$. Then the following are equivalent.

(1) For any finite coloring of $\mathbb{N}$, there exists $a, b \in \mathbb{N}$ such that $\{a, a + b \ldots, a + tb\} \subseteq C$, for some color C.

(2) There exists an ultrafilter $\mathcal{W}$ on $\mathbb{N} \times \mathbb{N}$ such that $T_{0*}(\mathcal{W}) = T_{j*}(\mathcal{W})$ for all $j \leq t$.

Proof.

$(\iff)$: Let $\mathcal{W}$ be an ultrafilter on $\mathbb{N} \times \mathbb{N}$ such that $T_{0*}(\mathcal{W}) = T_{j*}(\mathcal{W})$ for all $j \leq t$.

Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be a finite coloring. Since $T_{0*}(\mathcal{W})$ is an ultrafilter on $\mathbb{N}$, there is a color $C \in T_{0*}(\mathcal{W})$. By the definition of push-forward, for every $i \leq t$,

$$C \in T_{i*}(\mathcal{W}) \iff T_i^{-1}(C) \in \mathcal{W}$$

So, $\{(n,m) \in \mathbb{N}^2 \mid n + im \in C\} = T_i^{-1}(C) \in \mathcal{W}$ for every $i \leq t$. Then pick

$$(a,b) \in \bigcap_{i \leq t} T_i^{-1}(C) \in \mathcal{W}$$

Since $(a,b) \in T_i^{-1}(C)$, $a + ib \in C$. Therefore $\{a, a+b, \ldots, a+tb\} \subseteq C$.

$(\implies)$: Let $A \subseteq \mathbb{N}$ be subset. Denote $AP_t(a,b) = \{a, a+b, \ldots, a+tb\}$.

Define

$$X(A) = \left\{(a,b) \in \mathbb{N}^2 \mid AP_t(a,b) \subseteq A \text{ or } AP_t(a,b) \subseteq \mathbb{N} \setminus A\right\}$$

$X(A)$ is the set of all pairs, whose t-term arithmetic progressions are not split up by A.

Define

$$\mathcal{X} = \{X(A) \mid A \subseteq \mathbb{N}\}$$

$\mathcal{X}$ is a collection with FIP:

Let $X(A_1), \ldots, X(A_n) \in \mathcal{X}$. We will define a coloring by coloring each natural number k by whether they are in A_i or not, for every i.

Let $(s_1, \ldots, s_n) \in \{0,1\}^n$ be a string of length n. Define $B_s = A_1^{s_1} \cap \ldots \cap A_n^{s_n}$ where $A_i^0 = A_i^c$, the complement of A_i and $A_i^1 = A_i$. Thus, $\mathbb{N} = \bigsqcup_s B_s$ is a finite coloring. By our assumption, there is a color B_s, and there are $a, b \in \mathbb{N}$ such that $AP_t(a,b) \in B_s$.

Thus $(a,b) \in X(A_i)$ for every $i \leq n$. Therefore, $\mathcal{X}$ has FIP.

By the ultrafilter lemma, there is an ultrafilter $\mathcal{W}$ extending $\mathcal{X}$.

$\mathcal{W}$ satisfies $T_0(\mathcal{W}) = T_i(\mathcal{W})$ for every $i \leq l$:

Say $A \in T_{0*}(\mathcal{W})$, that is $T_0^{-1}(A) \in \mathcal{W}$. Then $X(A) \cap T_0^{-1}(A) \in \mathcal{W}$.

Observe that

$$T_i^{-1}(A) \cap X(A) = T_0^{-1}(A) \cap X(A) \in \mathcal{W}$$

Thus $T_i^{-1}(A) \in \mathcal{W}$, hence $A \in T_{i*}(\mathcal{W})$. So $T_{0*}(\mathcal{W}) \subseteq T_{i*}(\mathcal{W})$ and $T_{0*}(\mathcal{W})$ is a maximal filter. Therefore, $T_{0*}(\mathcal{W}) = T_{i*}(\mathcal{W})$ for all $i \leq t$. $\square$

Now, we will present the main proof, but to illustrate it clearly, we will do the proof when there are only two colors. First, we state the following lemma.

Lemma 4. Let $f : S \to T$ and $G : T^n \to L$ be functions, $B \subseteq L$ and $\mathcal{U}$ be an ultrafilter on S. Define $F = (f, \ldots, f) : S^n \to T^n$. Then,

$$f_*(\mathcal{U})y_1 \cdots f_*(\mathcal{U})y_n \big(G(y_1, \ldots, y_n) \in B\big) \iff \mathcal{U}x_1 \ldots \mathcal{U}x_n \big((G \circ F)(x_1, \ldots, x_n)\big)$$

This is a corollary of Proposition 8.

Proof of van der Waerden's theorem. Let $\mathbb{N} = C_1 \sqcup C_2$ be a finite coloring.

We will do induction on t. For $t = 1$, every coloring of $\mathbb{N}$ has a non-empty color. So we are done.

Assume that there is a color C and $a, b \in \mathbb{N}$ with $\{a, a+b, \ldots, a+(t-1)b\} \in C$. So there is an ultrafilter $\mathcal{W}$ on $\mathbb{N}^2$ with $T_{i*}(\mathcal{W}) = T_{0*}(\mathcal{W})$ for all $i \leq t-1$. We will denote the ultrafilter $T_{i*}(\mathcal{W})$ with T^i. We have $T^i = T^0$ for all $i < t$.

Let

$$\begin{aligned}
\mathcal{Z}_1 &= T^t \oplus T^t \oplus T^t \oplus T^0 \\
\mathcal{Z}_2 &= T^t \oplus T^t \oplus T^0 \oplus T^0 \\
\mathcal{Z}_3 &= T^t \oplus T^0 \oplus T^0 \oplus T^0
\end{aligned}$$

Since there are only two colors and each of the three ultrafilters $\mathcal{Z}_s$ contain exactly one color, there is a color $C \in \mathcal{Z}_q, \mathcal{Z}_p$ for some $p < q \in \{1, 2, 3\}$.

Then

$$C \in \mathcal{Z}_p \cap \mathcal{Z}_q = \big((T^t)^{\oplus q} \oplus (T^0)^{\oplus 4-q}\big) \cap \big((T^t)^{\oplus p} \oplus (T^0)^{\oplus 4-p}\big)$$

Thus,

$$\Gamma = C - (T^0)^{\oplus 4-q} \in \big((T^t)^{\oplus q}\big) \cap \big((T^t)^{\oplus p} \oplus (T^0)^{\oplus q-p}\big)$$

Set

$$\Gamma_1 = \Gamma - (T^t)^{\oplus q-p} = \left\{k' \mid \Gamma - k' \in (T^t)^{\oplus q-p}\right\} \in (T^t)^{\oplus p}$$
$$\Gamma_2 = \Gamma - (T^0)^{\oplus q-p} = \left\{k' \mid \Gamma - k' \in (T^0)^{\oplus q-p}\right\} \in (T^t)^{\oplus p}$$

Now $\Gamma_1 \cap \Gamma_2 \in (T^t)^{\oplus q-p}$. Pick an element $k' \in \Gamma_1 \cap \Gamma_2$.

We have "subtracted" the equal left summands of $\mathcal{Z}_p$ and $\mathcal{Z}_p$ from the color set C to get Γ first, and then "substracted" the different summands from Γ this time to get Γ_1 and Γ_2, two sets, in the same ultrafilters $(T^t)^{\oplus}p$.

By definition, $\Gamma - k' \in (T^0)^{\oplus q-p} \cap (T^t)^{\oplus q-p}$ and recall that $T^0 = T^j = T_0 * (\mathcal{W})$ for all $j < t$. So by unraveling the definitions, from $\Gamma - k' \in (T^j)^{\oplus q-p}$ we get

$$\begin{aligned}
\Gamma - k' \in (Tj)^{\oplus q-p} &\iff T_{j*}(\mathcal{W})x_1 \cdots T_{j*}(\mathcal{W})x_{q-p}\big(\sum_{i=1}^{q-p} x_i \in \Gamma - k'\big) \\
&\iff \mathcal{W}(n_1, m_1) \cdots \mathcal{W}(n_{q-p}, m_{q-p})\big(\big(\sum \circ T_j\big)(n_i, m_i)_{i<q-p} \in \Gamma - k'\big) \\
&\iff \mathcal{W}(n_1, m_1) \cdots \mathcal{W}(n_{q-p}, m_{q-p})\big(k' + \sum_{i=1}^{q-p}(n_i + jm_i) \in \Gamma\big)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Gamma - k' \in (T^t)^{\oplus q-p} &\iff T_{t*}(\mathcal{W})x_1 \cdots T_{t*}(\mathcal{W})x_{q-p}\big(\sum_{i=1}^{q-p} x_i \in \Gamma - k'\big) \\
&\iff \mathcal{W}(n_1, m_1) \cdots \mathcal{W}(n_{q-p}, m_{q-p})\big(\big(\sum \circ T_t\big)(n_i, m_i)_{i<q-p} \in \Gamma - k'\big) \\
&\iff \mathcal{W}(n_1, m_1) \cdots \mathcal{W}(n_{q-p}, m_{q-p})\big(k' + \sum_{i=1}^{q-p}(n_i + tm_i) \in \Gamma\big)
\end{aligned}$$

So, we have

$$\mathcal{W}(n_1, m_1) \cdots \mathcal{W}(n_{q-p}, m_{q-p})\big(\big(k' + \sum_{i=1}^{q-p}(n_i + tm_i) \in \Gamma\big) \wedge \big(k' + \sum_{i=1}^{q-p}(n_i + jm_i) \in \Gamma\big)\big)$$

There are $(n_1, m_1), \ldots, (n_{q-p}, m_{q-p}) \in \mathbb{N}^2$ such that

$$k' + \sum_{i=1}^{q-p}(n_i + jm_i) \in \Gamma \ \forall j \leq t$$

Recall that $\Gamma = C - (T^0)^{\oplus 4-q} = \{k \in \mathbb{N} \mid C - k \in (T^0)^{\oplus 4-q}\}$. Thus, for $j \leq t$, we have

$$C - \big(k' + \sum_{i=1}^{q-p}(n_i + jm_i)\big) \in (T^0)^{\oplus 4-q}$$

In particular, $\bigcap_{j\leq t} C - \left(k' + \sum_{i=1}^{q-p}(n_i + jm_i)\right)$ is non-empty. Hence, there exists k such that

$$k + \left(k' + \sum_{i=1}^{q-p}(n_i + jm_i)\right) \in C$$

Set, $a = k + k' + \sum_{i=1}^{q-p} n_i$ and $b = \sum_{i=1}^{q-p} m_i$. Therefore, $\{a, a+b, \ldots, a+tb\} \in C$. □

The proof for more than two colors is not very different. Say $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ is a coloring, we define the initial ultrafilters as

$$\mathcal{Z}_s = (T^t)^{\oplus s} \oplus (T^0)^{\oplus 2-s+r}$$

for $s \in \{1, \ldots, r+1\}$. This way we can still use the pigeonhole argument and get a color C with $C \in \mathcal{Z}_q, \mathcal{Z}_p$ for some $p < q \leq r+1$. The rest of the proof goes nearly the same, except instead of 4 we have $r+2$ many ultrafilters. So replace the $4-q$'s with $r+2-q$.

Now that we have proven van der Waerden's theorem, we should note that there is an ultrafilter witnessing it.

Corollary 17. There is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that for every $A \in \mathcal{U}$ and for every $l \in \mathbb{N}$ there is $\{a, a+b, \ldots, a+lb\} \subseteq A$. In particular, there is an ultrafilter $\mathcal{W}$ on $\mathbb{N} \times \mathbb{N}$ with $T_{i*}(\mathcal{W}) = T_{j*}(\mathcal{W})$ for every $i \leq j$.

Proof. We will do the same proof with the lemma.

Given $A \subseteq \mathbb{N}$ and $l \in \mathbb{N}$, define

$$\mathcal{X}_l(A) = \{(a,b) \mid AP_l(a,b) \subseteq A \text{ or } AP_l(a,b) \subseteq \mathbb{N} \setminus A\}$$

We have to show collection $\mathcal{X} = \{\mathcal{X}_l(A) \mid A \subseteq \mathbb{N}, l \in \mathbb{N}\} \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ has FIP. Let $\mathcal{X}_{l_1}(A_1), \ldots \mathcal{X}_{l_n}(A_n) \in \Gamma$. Then let $(s_1, \ldots, s_n) \in \{0,1\}^n$ be a string of length n. Define $B_s = A_1^{s_1} \cap \ldots \cap A_n^{s_n}$ where $A_i^0 = A_i^c$, the complement of A_i and $A_i^1 = A_i$. So $\mathbb{N} = B_1 \sqcup \cdots \sqcup B_s$ is a finite coloring. Then by van der Waerden's for any l there is $a, b \in \mathbb{N}$ with monochromatic $\{a, a+b, \ldots, a+lb\}$. Set $l = max(l_1, \ldots, l_n)$. Thus, $(a,b) \in \bigcap_{i\leq n} \mathcal{X}_l(A_i)$ and hence $\mathcal{X}$ has FIP. So extend $\mathcal{X}$ into an ultrafilter $\mathcal{W}$ on $\mathbb{N} \times \mathbb{N}$.

Define $T_i : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $T_i(n, m) = n + im$ for $l \in \mathbb{N}$. We claim that $T_{i*}(\mathcal{W}) = T_{j*}(\mathcal{W})$ for every $i \leq j$.

Let $B \in T_i(\mathcal{W})$. Then $T_i^{-1}(B) \in \mathcal{W}$. That is $\{(n, m) \mid n + im \in B\} \in \mathcal{W}$. Since $\mathcal{X}_j(B) \in \mathcal{W}$, we have $\mathcal{X}_j(B) \cap T_i^{-1}(B) \in \mathcal{W}$.

Observe that if a pair (b, c) is in $\mathcal{X}_j(B) \cap T_i^{-1}(B)$, then $\{a, a + b, \ldots, a + ib, \ldots, a + jb\} \subseteq B$. Therefore $\mathcal{X}_j(B) \cap T_i^{-1}(B) = \mathcal{X}_j(B) \cap T_j^{-1}(B)$, Thus $B \in T_j(\mathcal{W})$ and hence $T_{i*}(\mathcal{W}) \subseteq T_{j*}(\mathcal{W})$. By maximality of ultrafilters $T_{i*}(\mathcal{W}) = T_{j*}(\mathcal{W})$

Set $\mathcal{U} = T_{i*}(\mathcal{W})$. We claim that for all $A \in \mathcal{U}$, there is $\{a, a + b, \ldots, a + lb\} \subseteq A$.

Let $A \in \mathcal{U}$ and $l \in \mathbb{N}$. Then for every $t \leq l$, $\mathcal{X}_t(A) \in \mathcal{W}$. In particular $A' = T_1^{-1}(A) \cap (\bigcap_{t \leq l} \mathcal{X}_t(A)) \in \mathcal{W}$. Pick an element $(a, b) \in A'$, so there is some c with $\{a, b, a + b, \ldots, a + lb\} \subseteq A$. □

Observe that in the proof of van der Waerden, like the last section, we defined and used an addition on ultrafilters, and by "subtracting" sets by ultrafilters, we shifted the sets until we get Γ_1 and Γ_2, two large sets with respect to the same ultrafilter. So the proof relied mostly on the subtraction this time instead of idempotentness ($\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$). If we were to understand the algebraic structure of this addition, we might be able to utilise it to get some other arithmetic Ramsey theory results. This is, in fact, the case. We will develop the algebraic structure of ultrafilters in the following two chapters. First, we will briefly investigate the topological space of all ultrafilters and then define the topological semigroup structure on it.

To be complete, we should note that there are also two other related methods to solve such problems, topological dynamics and nonstandard methods. Both methods are deeply related to ultrafilters; in fact, the proof we presented was first obtained from iterated non-standard extensions. Unfortunately, both non-standard analysis and topological dynamics are outside the scope of this thesis. For the use of nonstandard methods in Ramsey theory, one can check [DNGL19].

CHAPTER 4

SPACE OF ULTRAFILTERS AND STONE–ČECH COMPACTIFICATION

In the previous chapters, we have defined a sum on ultrafilters using the natural sum on $\mathbb{N}$. We want to do a similar thing with arbitrary semigroups, that is, given an infinite semigroup $(S, \cdot)$, we want to define an operation on the set

$$\beta S = \{\mathcal{U} : \mathcal{U} \text{ is an ultrafilter on } S\}$$

of all ultrafilters on S using the operation of S. To do this, we will put a topology on βS and show that βS with this topology is a special type compactification of the discrete semigroup S. In the next chapter, we will use the universal property of this compactification and extend the operation of S onto βS. In what follows, all topological spaces are assumed to be Hausdorff.

This special compactification is called the Stone–Čech compactification. It is the "most extensive" compactification of a topological space. This "most extensiveness" is captured with the following universal property: Let X be a topological space. Then every map from X to any compact space C can be lifted to the Stone–Čech compactification of X.

We will show βS, endowed with an appropriate topology, is the Stone–Čech compactification of the discrete space S. We will also define a generalization of sequential limits in a topological space, the limits along an ultrafilter. Limits along an ultrafilter are generally a better convergence notion than sequential limits. We will use them to characterize lifts in the universal property of the Stone–Čech compactification of S.

All the results we give in this chapter are well-known, and they can be found with more depth in [HS98, Chapter 3] and [Gol22]. Throughout the chapter, whenever we say a map or mapping, we mean a continuous functions.

4.1 Space of Ultrafilters

Let S be an infinite discrete topological space.

Definition 14. Consider the set of all ultrafilters on S and the topology generated by all sets of the form

$$\widehat{A} = \{\mathcal{U} \mid A \in \mathcal{U}\}$$

We denote this topological space with βS.

It is clear that if A is non-empty, then $\widehat{A}$ contains a principal ultrafilter, and hence is non-empty.

We will give some basic properties.

Proposition 13. [HS98, Theorem 3.17] Let S be an infinite discrete space and $A, B \subseteq S$. Then

(1) $\widehat{S \setminus A} = \beta S \setminus \widehat{A}$.

(2) $\widehat{A \cap B} = \widehat{A} \cap \widehat{B}$.

(3) $\widehat{A \cup B} = \widehat{A} \cup \widehat{B}$.

(4) $\widehat{A} \subseteq \widehat{B} \iff A \subseteq B$

(5) $\widehat{A} = \widehat{B} \iff A = B$.

(6) If $B \setminus A$ is infinite, then $\widehat{B} \setminus \widehat{A} \subsetneq \widehat{B \setminus A}$.

If B is infinite and $B \setminus A \neq \varnothing$ is finite, $\widehat{B \setminus A} \subsetneq \widehat{B} \setminus \widehat{A}$

(7) $\widehat{A}$ is closed and open, so it is clopen.

The proofs follow directly from the definition of ultrafilters and basic properties of set operators. It is easily seen that βS is a zero-dimensional space, that is, it has a basis consisting of clopen sets. Moreover, we have the following.

Proposition 14. [HS98, Theorem 3.18.a] βS is a compact Hausdorff space.

A very important property of βS is that S can be embedded into it. We will identify principal ultrafilters $\mathcal{U}_x$ with $x \in S$. This identification is more than an injection; it is an embedding.

Proposition 15. [HS98, Theorem 3.18.e] Let $\iota : S \to \beta S$ be the function such that $\iota(s) = U_s$ the principal ultrafilter containing $\{s\}$. Then ι is an embedding and $\iota(S)$ is dense in βS.

Proof. We leave the verification of ι being an embedding to the reader and prove that $\iota(A)$ is dense in βS.

Let $O \subseteq \beta S$ be an open set. Then there is some $A \subseteq S$ with $\widehat{A} \subseteq O$. So $\iota(A) \subseteq \widehat{A} \subseteq O$. Thus, $\iota(S)$ intersects every open set. Hence $cl(\iota(S)) = \beta S$. $\iota(S)$ is dense in βS. □

From now on, we identify $\iota(s)$ with s. In the following chapters, we usually abuse the notation and denote them the same.

Proposition 16. [HS98, Theorem 3.18.e] Let $\mathcal{U} \in \beta S$.

$$\mathcal{U} \text{ is an isolated point of } \beta S \iff \mathcal{U} \in \iota(S).$$

Proof. Let $s \in S$. Then $\widehat{\{s\}} = \{\mathcal{U}_s\}$, so $\iota(s) = \mathcal{U}_s$ is isolated. Let $\mathcal{U}$ be an isolated point of βS, then $\{\mathcal{U}\}$ is open. Since $\iota(S)$ is dense in βS, it intersects the open set $\{\mathcal{U}\}$. Hence $\mathcal{U} \in \iota(S)$. □

So the only isolated points of βS are the principal ultrafilters. Thus, the interesting part of the βS is the remainder $\beta S \setminus S$ of the space.

Until now, our basic open and closed sets were $\widehat{A}$, the collection of ultrafilters that extend $\{A\}$. We will extend this notion to have a better grasp of the closed sets of βS.

Proposition 17. [HS98, Theorem 3.20.a] Let $\mathcal{F}$ be a filter on S. The subset $\widehat{\mathcal{F}} = \{\mathcal{U} \mid \mathcal{F} \subseteq \mathcal{U}\} \subseteq \beta S$ is closed.

Proof. Let $\mathcal{F}$ be a filter on S. Consider $O = \bigcup\{\widehat{A} \mid S \setminus A \in \mathcal{F}\} \subseteq \beta S$. O is a union of open sets, so it is open. $O = \beta S \setminus \widehat{F}$. Therefore, $\widehat{\mathcal{F}}$ is closed. □

Observe that given a subset B of βS, the intersection $\bigcap B$ of all of the elements of B is a filter. So we can use the previous proposition to get $\widehat{\bigcap B}$, the collection of all ultrafilters that extend the filter $\bigcap B$, is closed.

Thus, we have the following characterization of closures.

Proposition 18. [HS98, Theorem 3.20.b] Let $\mathcal{B} \subseteq \beta S$ be a non-empty subset.

$$cl(\mathcal{B}) = \widehat{\bigcap \mathcal{B}} = \{\mathcal{U} \mid A \in \mathcal{U} \text{ for some } \mathcal{V} \in \mathcal{B} \text{ and } A \in \mathcal{V}\}$$

This characterization is our main method to generate non-trivial closed subsets in βS.

4.2 Convergence Along a Filter

We will generalise the notion of convergence of a sequence in a space X given a filter on an infinite set S.

Let X be a topological space.

Definition 15. Let $\mathcal{U}$ be an ultrafilter on set S and $(x_i)_{i \in S}$ be a sequence of X.

We say $\lim_{\mathcal{U}} x_n = x$ if for every neighborhood V of x,

$$\{i \mid x_i \in V\} \in \mathcal{U}$$

If $\lim_{\mathcal{U}} x_i = x$, we say x is **the limit of** (x_i) **along the filter** $\mathcal{U}$ and we sometimes denote it by $x_i \xrightarrow{\mathcal{U}} x$ or $\lim_{i \to \mathcal{U}} x_i$. Limit along an ultrafilter is also called **Ultralimit**.

When $S = \mathbb{N}$, $(x_n)_{n \in \mathbb{N}}$ is just a sequence, $\lim_{\mathcal{U}} x_n = x$ can be seen as $x_n \to x$ almost surely with the measure from $\mathcal{U}$. Assuming $\mathcal{U}$ is non-principal, this is a more general definition than the standard limit. In fact, we will see that it has much nicer properties and $\lim_{\mathcal{U}}$ respects $\lim$ whenever $\lim$ exists, and $\mathcal{U}$ is non-principal.

It is straightforward to see that if $\mathcal{U}$ is non-principal and $\lim_{n\to\infty} x_n$ exists, then

$$\lim_{n\to\infty} x_n = \lim_{\mathcal{U}} x_n$$

because, since $\mathcal{U}$ is non-principal, it contains all the cofinite subsets of $\mathbb{N}$, and by $\lim_{n\to\infty} x_n = x$, for every neighborhood V of x, we have the cofinite set $\{n \mid x_n \in V\} \in \mathcal{U}$.

The definition of ultralimit helps us generalise sequential convergence in two main ways. Firstly, we can have limits of sequences with bigger cardinalities. For example, if $S = \mathbb{R}$, the sequence $(x_i)_{i\in\mathbb{R}}$ is of length continuum, and the limit is defined similarly. For every neighborhood, a 'large' part of the sequence lies inside the neighborhood. Secondly, as we stated before, for $S = \mathbb{N}$, ultralimits generalise the usual limits. Ultralimits have really nice properties and determine the topology completely, unlike the usual sequential limits.

A nice property of $\lim_{\mathcal{U}}$ is that compactness is equivalent to sequential compactness without the need for first countability or a metric.

Proposition 19. [Gol22, Theorem 3.1.9] The following are equivalent.

1. X is compact
2. For every sequence $(x_i)_{i\in I}$ in X and for every ultrafilter $\mathcal{U}$ on I with $|I| \leq |X|$, the ultralimit $\lim_{\mathcal{U}} x_i$ exists.

Proof. Let X be a topological space and $\mathcal{U}$ an ultrafilter on X

($\implies$): Let (x_n) be a sequence in X. We will do a proof by contradiction, assume not. Then for every $x \in X$, there is a neighborhood V_x of x such that $I_x = \{n \mid x_n \in V_x\} \notin \mathcal{U}$.

The collection of all V_x forms an open cover of X, and by compactness, there is a finite subcover $V_{x_1}, \ldots, V_{x_n}$. Since $\bigcup V_{x_n} = X$, $\bigcup I_{x_n} = \mathbb{N} \in \mathcal{U}$. Therefore there is $I_{x_n} \in \mathcal{U}$. Which is a contradiction. Hence there is some $x \in X$ with every neighborhood V of x, $\{n \mid x_n \in V\} \in \mathcal{U}$. Thus $x = \lim_{\mathcal{U}} x_n$.

($\Longleftarrow$): Let $\mathcal{O}$ be an open cover of X with no finite subcover. Well-order the open cover, so $\mathcal{O} = \{V_i\}_{i \in I}$ for some infinite set I.

We will construct a sequence with index set $\mathcal{P}_{fin}(I) = \{F \subseteq I \mid F \text{ is finite}\}$. For every $i \in I$, let $F_i = \{F \in \mathcal{P}_{fin}(I) \mid i \in F\}$. The collection $\{F_i\}_{i \in I}$ has FIP. So by the Ultrafilter Lemma, there is an ultrafilter $\mathcal{U}$ extending it. Now we will construct a sequence $(x_F)_{F \in \mathcal{P}_{fin}(I)}$ that does not have an ultralimit along $\mathcal{U}$.

$\{V_i\}$ has no finite subcover, so $X \setminus \bigcup_{i \in F} V_i$ is non-empty for every finite $F \subseteq I$. Let $x_F \in X \setminus \bigcup_{i \in F} V_i$.

We will now show $\lim_{\mathcal{U}} x_F$ does not exist. Assume $x \in X$ is the ultralimit. Then $x \in V_i$ for some $i \in I$. V_i is a neighborhood of x, so $A = \{F \in \mathcal{P}_{fin}(I) \mid x_F \in V_i\} \in \mathcal{U}$. Thus $A \cap F_i \in \mathcal{U}$. But then for any $F \in A \cap F_i$, $x_F \in V_i$ and $x_F \in X \setminus \bigcup_{i \in F} V_i$ which is a contradiction.

□

Another nice property of ultralimit is the following.

Proposition 20. [Gol22, Theorem 3.10] Let X be a topological space. The following are equivalent.

1. X is Hausdorff
2. For any sequence $(x_i)_{i \in I}$ and any ultrafilter $\mathcal{U}$, ultralimit $\lim_{\mathcal{U}} x_i$ is unique whenever it exists.

Proof. Let X be a topological space.

($\Longrightarrow$): Let x, y be distinct $\mathcal{U}$-ultralimits of x_i. Let U and V be open sets seperating x and y. Then $\{i \mid x_i \in U\}, \{i \mid x_i \in V\} \in \mathcal{U}$ as x, y are limits. But $\mathcal{U}$ and V are disjoint so $\varnothing = \{i \mid x_i \in U\} \cap \{i \mid x_i \in V\} \in \mathcal{U}$ a contradiction.

($\Longleftarrow$): Let $x, y \in X$ be distinct. Denote the set of all neighborhoods of x as $\mathcal{A}$ and denote the set of all neighborhoods of y as $\mathcal{B}$. If $\mathcal{A} \cup \mathcal{B}$ does not have FIP,

then there are open sets separating x, y, so we are done. Assume $\mathcal{A} \cup \mathcal{B}$ has FIP. Then we can extend this collection to an ultrafilter $\mathcal{U}$ by the Ultrafilter Lemma. Then the constant sequence $(x)_{i \in I}$ has $\mathcal{U}$-ultralimits x and y, contradicting our assumption.

□

We will characterize the continuity of a function in terms of ultralimits.

Proposition 21. [HS98, Theorem 3.54] Let $f : X \to Y$ be a function and $x \in X$. Then f is **continuous at** x if and only if for any sequence $(x_i)_{i \in I}$ and for any ultrafilter $\mathcal{U}$ on I,

$$f(\lim_{\mathcal{U}} x_i) = \lim_{\mathcal{U}} f(x_i)$$

4.3 Stone–Čech Compactification of a Discrete Space

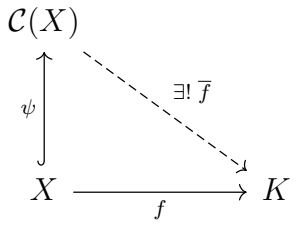
We have shown that S embeds densely into the compact Hausdorff space βS. That is, βS is a compactification of S. It is actually a special compactification called the Stone–Čech compactification.

Every space in this subsection is Hausdorff.

Definition 16. Let X be a topological space and $(\mathcal{C}(X), \psi)$ be a pair such that $\mathcal{C}(X)$ is a compact space, $\psi : X \to \mathcal{C}(X)$ is an embedding with $\psi(X) \subseteq \mathcal{C}(X)$ is dense. Then $(\mathcal{C}(X), \psi)$ is a **compactification of** X.

A compactification $(\mathcal{C}(X), \psi)$ of X is the **Stone–Čech compactification of** X if every continuous map $f : X \to K$ with compact space K can be *uniquely* lifted to a continuous map $\overline{f} : \mathcal{C}(X) \to K$ with $\overline{f} \circ \psi = f$.

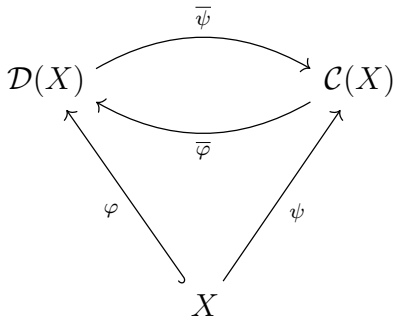
That is, the following diagram commutes.



This universal property completely determines the space up to a homeomorphism. We call this property **Universal Property of Stone–Čech compactification**.

Proposition 22. Let $(\mathcal{C}(X), \psi)$ and $(\mathcal{D}(X), \varphi)$ be Stone–Čech compactifications of space X. Then $\mathcal{C}(X)$ is homeomorphic to $\mathcal{D}(X)$.

Proof. Let $\mathcal{C}(X)$ and $\mathcal{D}(X)$ be such compact spaces. Then we have extensions $\overline{\varphi}$: $\mathcal{D}(X) \to \mathcal{C}(X)$ and $\overline{\psi} : \mathcal{C}(X) \to \mathcal{D}(X)$ satisfying the following diagram.



Since $\overline{\varphi} \circ \psi = \varphi$ and vice versa, we have $(\overline{\psi} \circ \overline{\varphi}) \circ \psi = \psi$. So $(\overline{\psi} \circ \overline{\varphi}) : \mathcal{C}(X) \to \mathcal{C}(X)$ is the *unique* extension of ψ, but the identity function $id_{\mathcal{C}(X)}$ satisfies $id_{\mathcal{C}(X)} \circ \psi = \psi$

Therefore $\overline{\psi} \circ \overline{\varphi} = id_{\mathcal{C}(X)}$. By the same argument, $\overline{\varphi} \circ \overline{\psi} = id_{\mathcal{D}(X)}$.

Thus $\overline{\varphi} : \mathcal{C}(X) \to \mathcal{D}(X)$ is an homeomorphism. □

Now, we can show βS is *the* Stone–Čech compactification when S is an infinite discrete space.

We will start with how we lift $f : S \to K$.

Let $f : S \to K$ be a map and K be compact. Observe that $(s)_{s \in S}$ is a sequence indexed by S. So, we can look at its ultralimit.

Define $\overline{f} : \beta S \to K$ by

$$\overline{f}(\mathcal{U}) = \lim_{\mathcal{U}} f(s)$$

Since K is compact and Hausdorff, $\lim_{\mathcal{U}} f(s)$ exists and is unique. Thus, $\overline{f}$ is well-defined.

$\overline{f}$ lifts f: Recall that $\iota(s) = \mathcal{U}_x$ is the principal ultrafilter. So, $\overline{f}(\mathcal{U}_x) = \lim_{\mathcal{U}_x} f(s) = f(x)$. Therefore, $\overline{f} \circ \iota = f$.

$\overline{f}$ is continuous: Let $\mathcal{U} \in \beta S$, we want to show $\overline{f}$ is continuous at $\mathcal{U}$. Let V be a neighborhood of $\overline{f}(\mathcal{U}) = \lim_{\mathcal{U}} f(s)$. S is discrete, so S is normal. Thus, we can find an open $O \subseteq V$ with $cl(O) \subseteq V$. Then, by definition of ultralimit, $A = \{s \mid f(s) \in O\} \in \mathcal{U}$.

For every $\mathcal{F} \in \widehat{A}$, $\lim_{\mathcal{F}} s \in cl(O)$. Thus, $\overline{f}(\widehat{A}) \subseteq cl(O) \subseteq V$. So $\overline{f}$ is continuous at the point $\mathcal{U}$ for every $\mathcal{U}$. So $\overline{f}$ is continuous.

Proposition 23. [HS98, Theorem 3.27] Let S be an infinite discrete space.

$$(\beta S, \iota) \text{ is the Stone–Čech compactification of } S.$$

Proof. Let K be a compact space with a mapping $f : S \to K$. Then, as we have shown, $\overline{f} : \beta S \to K$, $\overline{f}(\mathcal{U}) = \lim_{\mathcal{U}} f(s)$ lifts f.

Therefore, βS is the Stone–Čech compactification of S. □

Proposition 24. Let S and T be discrete spaces and $f : S \to T$ be a map. Then there is a map $\beta f : \beta S \to \beta T$ with the following commutative diagram.

$$\begin{array}{ccc} \beta S & \overset{\exists! \beta f}{\dashrightarrow} & \beta T \\ {\scriptstyle \iota_S}\uparrow & & \uparrow{\scriptstyle \iota_T} \\ S & \xrightarrow[f]{} & T \end{array}$$

Furthermore, $\beta f(\mathcal{U}) = f_*(\mathcal{U})$ is the push-forward ultrafilter.

Proof. $\iota_T \circ f : S \to \beta T$ is a map to a compact space. Since βS is the Stone-Čech Compactification, there is a unique lift $\beta f = \overline{\iota_T \circ f}$ defined by $\beta f(\mathcal{U}) = \lim_{\mathcal{U}} \beta f(s)$.

Observe that $\iota_T(t) = \mathcal{V}_t$, the principal ultrafilter of T. Thus,

$$\beta f(\mathcal{U}) = \lim_{\mathcal{U}} \beta f(s) = \lim_{\mathcal{U}} \mathcal{V}_{f(s)}$$

Let $B \in \beta f(\mathcal{U})$. So, $\widehat{B} \subseteq \beta T$ is a neighborhood of $\beta f(\mathcal{U})$. Then, by definition of ultralimit, $\{s \mid \mathcal{V}_{f(s)} \in \widehat{B}\} \in \mathcal{U}$. That is, $f^{-1}(B) = \{s \in S \mid f(s) \in B\} \in \mathcal{U}$. Therefore, $\mathcal{V} \in f^{-1}(B)$.

So $\beta f(\mathcal{U}) \subseteq f_*(\mathcal{U})$ and $\beta f(\mathcal{U})$ is an ultrafilter and hence maximal.

Therefore, $\beta f(\mathcal{U}) = f_*(\mathcal{U})$. □

After the last observation, given $f : S \to T$, we can carry over many results from push-forward ultrafilters to βf.

Proposition 25. Let $f, g : S \to T$ be mappings and S, T be discrete.

1. If $B \in \beta f(\mathcal{U})$, then $f^{-1}(B) \in \mathcal{U}$.
2. If $A \in \mathcal{U}$, then $f(A) \in \beta f(\mathcal{U})$.
3. If $S = T$, then
$$\mathcal{U} = \beta f(\mathcal{U}) \iff \{s \mid s = f(s)\} \in \mathcal{U}$$
4. If $\mathcal{U}x\ f(x) = g(x)$, then $f(\mathcal{U}) = g(\mathcal{U})$, that is
$$\{s \mid f(s) = g(s)\} \in \mathcal{U} \implies f(\mathcal{U}) = g(\mathcal{U})$$

Proposition 26. [HS98, Lemma 3.32] Let $f : S \to S$ be a mapping. Then the following are equivalent.

(1) $\beta f : \beta S \to \beta S$ has a fixed point.

(2) For every finite partition $S = A_1 \sqcup \ldots \sqcup A_n$, $A_i \cap f(A_i) \neq \varnothing$ for some $i \leq n$.

Proof. Recall that $\beta f(\mathcal{U}) = f_*(\mathcal{U})$.

($\implies$) : Assume not, that is assume $S = \bigsqcup_{i \leq n} A_i$ with $A_i \cap f(A_i) = \varnothing$ for all $i \leq n$. Let $\mathcal{U}$ be an ultrafilter on S. Then $A_i \in \mathcal{U}$ for some i. So, $f(A) \in \beta f(\mathcal{U})$, but $A \cap f(A) = \varnothing$. Thus, $\mathcal{U} \neq \beta f(\mathcal{U})$. Therefore, βf has no fixed point.

($\Longleftarrow$) : Assume not, that is assume $\mathcal{U} \neq \beta f(\mathcal{U})$ for all $\mathcal{U} \in \beta S$. We will construct a partition of S. For every $\mathcal{U} \in \beta S$, pick $A_{\mathcal{U}} \in \mathcal{U} \setminus \beta f(\mathcal{U})$. Then $S \setminus A \in \beta f(\mathcal{U})$. Set $B_{\mathcal{U}} = f^{-1}(S \setminus A) \in \mathcal{U}$. Finally, consider $C_{\mathcal{U}} = A_{\mathcal{U}} \cap B_{\mathcal{U}} \in \mathcal{U}$. The collection $\{\widehat{C_U} \mid \mathcal{U} \in \beta S\}$ is an open cover of βS.

Since βS is compact, there is a finite subcover, say, $\{\widehat{C_{\mathcal{U}^{(1)}}}, \ldots, \widehat{C_{\mathcal{U}^{(k)}}}\}$.

$$\beta S = \widehat{C_{\mathcal{U}^{(1)}}} \cup \ldots \cup \widehat{C_{\mathcal{U}^{(k)}}} = \widehat{C_{\mathcal{U}^{(1)}}, \ldots, C_{\mathcal{U}^{(k)}}} = \widehat{S}$$

.

□

We give the following three propositions as our final facts for this chapter.

Proposition 27. [HS98, Theorem 3.34] Let $f : S \to S$. If f has no fixed point, βf has no fixed points.

Proposition 28. [HS98, Theorem 3.37] Any countable union of nowhere dense subsets of $\beta S \setminus S$ is a nowhere dense subset of $\beta S \setminus S$.

Proposition 29. Let $A \subseteq S$. Then βA is homeomorphic to $cl(A) \subseteq \beta S$.

As a consequence, we identify βA with $cl(A)$ for every $A \subseteq S$.

We have developed the necessary theory of Stone–Čech compactification βS for a discrete space S, but there is always more that can be said. Yet, to not diverge from our main topic, arithmetic Ramsey theory and its relation with the algebra of the Stone–Čech compactification of $\mathbb{N}$, we end the chapter here.

Given a non-discrete space X, there are several methods to construct the Stone–Čech compactification of X. A construction with zero-set ultrafilters can be found in [Gol22].

CHAPTER 5

SEMIGROUP EXTENSIONS TO $\beta\mathbb{N}$

In this chapter, our main aim is to introduce the algebraic structure of Stone–Čech compactification βS of a semigroup $(S, \cdot)$ and extend some of the partition regularity results in Chapter 3 to arbitrary semigroups. The idempotent elements of βS will be especially important to show the partition regularity results. Thus, in the first section, we will briefly survey the theory of semigroups and idempotents of arbitrary semigroups. In the second section, we will survey compact semigroups to show that these idempotent elements exist by Ellis' lemma. Then, we will introduce the semigroup structure on βS with the help of the last chapter. The rest of the chapter will focus on how the partition regularity of arithmetic properties on S and the theory of idempotents interact.

We prove Hindman's theorem in full generality in Section 4, and we will survey minimal idempotent ultrafilters and their interactions with certain notions of largeness in the final section of this chapter. Almost all material in this chapter can be found scattered in the first five chapters of [HS98].

5.1 Semigroups

In this subsection, we briefly introduce semigroups. A non-empty set S paired with an associative binary operation $\cdot : S \times S \to S$ is called a **semigroup**.

Definition 17. Let $(S, \cdot)$ and $(T, *)$ be semigroups.

1. A function $f : S \to T$ is a **semigroup homorphism** if $f(x \cdot y) = f(x) * f(y)$.
2. A semigroup homomorphism $f : S \to T$ is an **isomorphism** if f is a bijection.

3. An element $e \in S$ is a **left (respectively, right) identity** if $e \cdot x = x$ (respectively, $x \cdot e = x$) for every $x \in S$.

4. An element e is an **identity** if e is a both left and right identity.

We will usually call a semigroup homomorphism a homomorphism or a morphism.

If a semigroup has an identity, it is unique. If a semigroup has only a left identity, it does not have to be unique. Consider the semigroup (S, π_2), where π_2 is the projection map given by $\pi_2(s, y) = y$ for every y. Note that every element is a left-sided identity.

Definition 18. Let $(S, \cdot)$ be a semigroup.

1. Let $s \in S$. Define $\lambda_s : S \to S$ as $\lambda_s(x) = s \cdot x$.

2. Let $s \in S$. Define $\rho_s : S \to S$ as $\rho_s(x) = x \cdot s$.

3. $L(S) = \{\lambda_s \mid s \in S\}$.

4. $R(S) = \{\rho_s \mid s \in S\}$.

Both $(L(S), \circ)$ and $(R(S), \circ)$ are semigroups. In fact $\lambda : S \to L(S)$ defined by $\lambda(s) = \lambda_s$ is a semigroup morphism. However, $\rho : S \to R(S)$ defined by $\rho(s) = \rho_s$ is not necessarily a morphism as $\rho(s \cdot t) = \rho_t \circ \rho_s$ might not equal to $\rho_s \circ \rho_t$. Indeed, ρ is an anti-morphism, that is $\rho(s \cdot t) = \rho(t) \cdot \rho(s)$.

Also, λ is not necessarily an isomorphism. Consider (S, π_1). Then $L(S)$ is a singleton while S can be infinite.

Definition 19. Let $(S, \cdot)$ be a semigroup.

1. A subset $T \subseteq S$ is a **subsemigroup** if T is closed under the opertion $\cdot$ of S. That is $t_1 \cdot t_2 \in T$ for all $t_1, t_2 \in T$.

   Equivalently, T is a subsemigroup if $T \cdot T \subseteq T$.

2. A subsemigroup $G \subseteq S$ is a **subgroup** if G is a group under the operation $\cdot$ of S. That is, there is a two-sided identity, and for every element there is an inverse.

3. An element $e \in S$ is an **idempotent** if $e \cdot e = e$.

4. Define $E(S) = \{x \in S \mid x \cdot x = x\}$ to be the set of all idempotents of S.

5. Let e be an idempotent element. Define

$$H(e) = \bigcup \{G \mid e \in G \text{ and } G \text{ is a subgroup}\}$$

It turns out that $H(e)$ is a maximal subgroup of S when e is an idempotent [HS98, Theorem 1.18].

We can order the set $E(S)$ of idempotents of S with

$$\begin{aligned} x \leq_L y &\iff x \cdot y = x \\ x \leq_R y &\iff y \cdot x = x \\ x \leq y &\iff x \cdot y = y \cdot x = x \end{aligned}$$

An idempotent element is **minimal idempotent** if it minimal with respect to $\leq$.

After defining the ideals, we will see that this order is deeply related to the ideals of the semigroup S.

Definition 20. Let $(S, \cdot)$ be a semigroup and $I \subseteq S$ be a non-empty subset.

1. I is a **left ideal** of S if, $x \cdot i \in I$ for every $x \in S$ and $i \in I$. Equivalently, $S \cdot I \subseteq I$.

2. Similarly, I is a **right ideal** if $I \cdot S \subseteq S$.

3. I is an **ideal** if it is both a left ideal and a right ideal.

4. A left (right) ideal I is a **minimal ideal** if, whenever $J \subseteq I$ is a left (right) ideal, $J = I$.

5. S is simple if S is a minimal ideal of S.

Whenever $x \in S$, x generates the left ideal $S \cdot x$, the right ideal $x \cdot S$ and the ideal $S \cdot x \cdot S$.

Proposition 30. [HS98, Theorem 1.38] Let $(S, \cdot)$ be a semigroup and $e \in E(S)$ be an idempotent. Then

(1) If e is inside a minimal ideal, then e is a minimal idempotent.

(2) If S is simple and e is minimal, then $S \cdot e$ is a minimal left ideal

(3) If every left ideal of S contains an idempotent and e is minimal, then $S \cdot e$ is a minimal left ideal.

(4) If every left ideal of S contains an idempotent (or if S is simple), then the following are equivalent:

- e is minimal.
- e is contained in some minimal left ideal.
- $S \cdot e$ is a minimal left ideal.

Definition 21. Let $(S, \cdot)$ be a semigroup. Define $K(S)$ to be the smallest ideal of S.

Observe that not every semigroup has a smallest ideal $K(S)$. Consider $(\mathbb{N}, +)$, an ideal of $\mathbb{N}$ is of the form $I_k = \{n \mid n \geq k\}$. $\mathbb{N}$ has no smallest ideal.

We give three facts from [HS98] to understand $K(S)$ better.

Proposition 31. [HS98, Theorem 1.51, 1.53,1.54] Let $(S, \cdot)$ be a semigroup.

(1) If S has a minimal left ideal, then the smallest ideal $K(S)$ is the union of all minimal left ideals.

(2) If L is a minimal left ideal and R is a minimal right ideal, then $K(S) = L \cdot R$.

(3) If $K(S)$ exists and $e \in E(S)$ is an idempotent, then $e \in K(S) \iff S \cdot e \cdot S = K(S)$.

Assuming $K(S)$ exists, every minimal left ideal is contained in $K(S)$, and so, every minimal idempotent is contained in $K(S)$.

5.2 Topological Semigroups and Ellis' Lemma

We will briefly introduce topological semigroups. Again, for a more thorough introduction, the reader can check [HS98] as everything in this section can be found there with greater depth.

Every topological space in this chapter is assumed to be Hausdorff.

Definition 22. Let $(S, \cdot, \tau)$ be a triple such that (S, τ) is a topological space and $(S, \cdot)$ is a semigroup.

1. $(S, \cdot, \tau)$ is a **right topological semigroup** if for every $s \in S$, the map $\rho_s : S \to S$ with $\rho_s(x) = x \cdot s$ is continuous.

2. $(S, \cdot, \tau)$ is a **left topological semigroup** if for every $s \in S$, the map $\lambda_s : S \to S$ with $\lambda_s(x) = s \cdot x$ is continuous.

3. $(S, \cdot, \tau)$ is a **semi-topological semigroup** if it is both a left topological and right topological semigroup.

4. $(S, \cdot, \tau)$ is a **topological semigroup** if $\cdot : S \times S \to S$ is continuous.

We will usually abuse the notation and say S is a right topological semigroup instead of $(S, \cdot, \tau)$

Now, we will state one of the most important theorems for our purposes.

Theorem 18 (Ellis' Lemma)**.** Let S be a compact space and a right topological semigroup. Then S contains an idempotent element. ($E(S) \neq \varnothing$.)

Proof. We will consider the collection of subsemigroups of S that are compact and order them by reverse inclusion to apply Zorn's lemma. Then we will show that there is an idempotent inside the subsemigroup given by Zorn's lemma.

Consider the collection

$$\{A \subseteq S \mid \text{A is compact and } A \cdot A \subseteq A\}$$

Order this collection by reverse inclusion, that is, $A_1 \leq A_2$ if $A_2 \subseteq A_1$. So a maximal element of this partial order will be a minimal element with the usual inclusion.

Let C be a chain. Observe that elements of C are compact subsets of S, and hence they are closed. Since C is a chain, any finite intersection of elements of C is non-empty. Therefore, C is a family of closed sets with the Finite Intersection Property. By compactness, the intersection $\bigcap C$ is non-empty. So chain C has an upper bound $\bigcap C$. Hence, we can apply Zorn's lemma.

Say A is the minimal element of the collection with the usual inclusion by Zorn's lemma. Let $x \in A$. We will show that x is an idempotent.

Since S is a right topological semigroup, ρ_x is a continuous function. Then $\rho_x(A) = A \cdot x$ is the image of a compact set under a continuous map, hence compact. Since ρ_x is a morphism, $\rho_x(A) \subseteq A$ is a subsemigroup of S. A is a minimal subsemigroup that is compact, so $\rho_x(A)$ cannot be smaller than A. Thus $A = \rho_x(A)$.

Let

$$C = \rho_x^{-1}(x) \cap A = \{y \in A \mid y \cdot x = x\}$$

Since $\rho_x^{-1}(x)$ and A are closed, so is C. So C is compact. C is also a semigroup as for every $c_1, c_2 \in C$, $c_1 \cdot c_2 \cdot x = c_1 \cdot x = x$, and hence, $c_1 \cdot c_2 \in C$. Therefore, $C \subseteq A$ is a semigroup that is compact, and by minimality of A, $C = A$. Thus $x \in A = \rho_x^{-1}(x) \cap A$, and $x \cdot x = \rho_x(x) = x$. □

Corollary 19. [HS98, Corollary 2.7] Let S be a compact right topological semigroup. Then every left ideal of S contains a minimal left ideal.

Moreover, minimal left ideals are closed, and they contain an idempotent.

Proof. Let L be a left ideal of S and let $x \in L$.

Observe that ρ_x is a continuous map and $\rho_x(S) = S \cdot x$ is a compact space. So every left ideal contains a closed ideal. In particular, minimal left ideals contain a closed ideal as well, but by their minimality, they must be a closed ideal.

Now, we will apply Zorn's lemma to the collection

$$\{T \subseteq S \mid T \text{ is a closed ideal and } T \subseteq L\}$$

ordered by reverse inclusion. Similarly to the last proof, any chain is a collection of closed sets with FIP and by compactness, has a non-empty intersection. Thus, by Zorn's lemma, we have a minimal element T of the collection. We have shown that every left ideal contains a closed ideal, so any left ideal lying inside T will contain another closed ideal, which contradicts the minimality of T. So T is a minimal ideal.

Since every minimal ideal is closed, it is compact. By Ellis' lemma, they contain an idempotent. □

While there are many things to note on the theory of semigroups and topological semigroups, our main interest is in a very specific right topological semigroup, namely the Stone–Čech compactification $(\beta S, \odot)$. So, we go back to βS.

5.3 βS as a Semigroup

It is customary to denote points of βS with letters like p, q, but we will usually denote them with calligraphic capital letters like $\mathcal{U}$ and $\mathcal{V}$. We will use lowercase p and q when we want emphasize that they are points in the space βS. We will use $\mathcal{U}$ and $\mathcal{V}$ when we want to emphasize that they are ultrafilters, but keep in mind that we will use both notations interchangeably.

Also, recall that given $s \in S$, we identify s with $\mathcal{U}_s$, the principal ultrafilter induced by $\{s\}$. So, whenever we say $s \in \beta S$, recall that s also denotes the principal ultrafilter $\mathcal{U}_s$.

Let $(S, \cdot)$ be a semigroup with discrete topology; we will call $(S, \cdot)$ a discrete semigroup. We can uniquely extend this operation from S to βS using the universal property of Stone–Čech compactification.

Theorem 20. [HS98, Theorem 4.1] Let $(S, \cdot)$ be a discrete semigroup.Then there is a unique extension $\odot : \beta S \times \beta S \to \beta S$ such that;

(1) For every $s, t \in S$, $s \odot t = s \cdot t$.

(2) For every $\mathcal{U} \in \beta S$, the right multiplication $\rho_{\mathcal{U}} : \beta S \to \beta S$ with $\rho_{\mathcal{U}}(x) = x \odot \mathcal{U}$ is continuous.

(3) For every $s \in S$, the function $\bar{l_s} : \beta S \to \beta S$ with $\bar{l_s}(x) = s \odot x$ is continuous.

Proof. Let $(S, \cdot)$ be a discrete semigroup.

For $s \in S$, define $l_s, r_s : S \to S$ as $l_s(x) = s \cdot x$ and $r_s(x) = x \cdot s$. By the universal property of Stone–Čech compactification, we can lift $l_s : S \to S \subseteq \beta S$ uniquely to $\bar{l_s}$. That is, the following diagram commutes.

$$\begin{array}{ccc} \beta S & & \\ \uparrow & \overset{\bar{l_s}}{\searrow} & \\ S & \xrightarrow[l_s]{} & \beta S \end{array}$$

Now, we have an operation $\bar{\cdot} : S \times \beta S \to \beta S$, defined as $s \bar{\cdot} \mathcal{U} = \bar{l_s}(\mathcal{U})$.

We will now lift the right multiplication of our new operation. Given $q \in \beta S$, define $r_q : S \to \beta S$ as $r_{\mathcal{U}}(s) = s \bar{\cdot} \mathcal{U}$. Again, by the universal property of Stone–Čech compactification, we can lift $r_{\mathcal{U}}$ uniquely to $\rho_{\mathcal{U}}$. Therefore, we can define $\odot : \beta S \times \beta S \to \beta S$ as $\mathcal{U} \odot \mathcal{V} = \rho_{\mathcal{V}}(\mathcal{U})$.

Since the universal property gives **unique** lifts, this operation is the unique extension of $\cdot$ satisfying (1), (2), and (3). □

Recall that when S and T are discrete, the unique lift $\bar{f} : \beta S \to \beta T$ of the function $f : S \to T$ can be explicitly written as

$$\bar{f}(\mathcal{U}) = f_*(\mathcal{U}) = \{f^{-1}(A) \mid A \in \mathcal{U}\}$$

Observe that $l_s^{-1}(A) = \{x \in S \mid s \cdot x \in A\} = s^{-1} \cdot A$. So, if we take l_s as f, we have

$$A \in \bar{l_s}(\mathcal{U}) \iff l_s^{-1}(A) \in \mathcal{U} \iff s^{-1} \cdot A \in \mathcal{U}$$

In particular,

$$A \in s \odot \mathcal{U} = \mathcal{U}_s \odot \mathcal{U} \iff s^{-1} A \in \mathcal{U}$$

Observe that $r_{\mathcal{V}}^{-1}(A) = \{s \in S \mid s^{-1} \cdot A \in \mathcal{V}\}$

Thus, If we take $r_{\mathcal{V}}$ as f, we have

$$A \in \rho_{\mathcal{V}}(\mathcal{U}) = \mathcal{U} \odot \mathcal{V} \iff \{s \in S \mid s^{-1} \cdot A \in \mathcal{V}\} \in \mathcal{U}$$

We have the following characterizations.

Proposition 32. Let $(S, \cdot)$ be a discrete semigroup, $\mathcal{U}, \mathcal{V} \in \beta S$ be ultrafilters and $A \subseteq S$ be a subset. Denote $s^{-1} \cdot A = \{t \in S \mid s \cdot t \in A\}$. Then,

$$\begin{aligned} A \in \mathcal{U} \odot \mathcal{V} &\iff \{s \in S \mid s^{-1} \cdot A \in \mathcal{V}\} \in \mathcal{U} \\ &\iff \mathcal{U}x\, \mathcal{V}y \ (x \cdot y \in A) \end{aligned}$$

We have two standard examples we are familiar with that are the extensions of $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$.

Example 9. Consider the semigroups $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$. We can extend these operations to $\oplus$ and $\odot$. Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on $\mathbb{N}$.

Recall that $-n + A = \{m \mid n + m \in A\}$ and $n^{-1} \cdot A = \{m \mid n \cdot m \in A\}$. Then we have

$$A \in \mathcal{U} \oplus \mathcal{V} = \{n \mid -n + A \in \mathcal{V}\} \in \mathcal{U}$$

$$A \in \mathcal{U} \odot \mathcal{V} = \{n \mid n^{-1} \cdot A \in \mathcal{V}\} \in \mathcal{U}$$

These operations $\oplus, \odot$ are not commutative, even though $+$ and $\cdot$ are.

- $\oplus$ **is not commutative:** Let $X_i = \{10^{2k} \mid k \geq i\}$ and $Y_i = \{10^{2k+1} \mid k \geq i\}$ Consider the collection of subsets

  $$\mathcal{X} = \{X_i \mid i \geq 1\}$$

  $$\mathcal{Y} = \{Y_j \mid j \geq 1\}$$

  Both collections have FIP, thus can be extended to ultrafilters, say $\mathcal{X} \subseteq \mathcal{U}$ and $\mathcal{Y} \subseteq \mathcal{V}$. Since for every $X_i \in \mathcal{X}$ and $Y_j \in \mathcal{Y}$, we have $X_i \cap Y_j = \varnothing$; $Y_j \notin \mathcal{U}$ and $X_i \notin \mathcal{V}$.

  Consider the set $A = \{10^{2i} + 10^{2j+1} \mid i < j\}$.

· $\underline{A \in \mathcal{U} \oplus \mathcal{V}}$:

Observe that, for $k \in \mathbb{N}$,

$$-10^{2k} + A = \{10^{2i} + 10^{2j+1} \mid i + k < j\}$$

In particular, $Y_j \subseteq -10^{2k} + A$ for all $j > k$. Thus $-10^{2k} + A \in \mathcal{V}$.

So, for all $x \in X_1 \in \mathcal{U}$, $-x + A \in \mathcal{V}$.

$$X_1 \subseteq \{n \mid -n + A \in \mathcal{V}\} \in \mathcal{U} \implies A \in \mathcal{U} \oplus \mathcal{V}$$

· $\underline{A \notin \mathcal{V} \oplus \mathcal{U}}$:

Let $B = \{10^{2i} + 10^{2j+1} \mid i > j\}$. Observe that $A \cap B = \varnothing$. As any number in A has even digits, while any number in B has odd digits.

If $B \in \mathcal{V} \oplus \mathcal{U}$, then $A \notin \mathcal{V} \oplus \mathcal{U}$.

We will show $B \in \mathcal{V} \oplus \mathcal{U}$.

For $k \in \mathbb{N}$,

$$-10^{2k+1} + B = \{10^{2i} + 10^{2j+1} \mid i > j + k\}$$

In particular, $X_i \subseteq -10^{2k+1} + B$ for all $i > k$. Thus $-10^{2k+1} + B \in \mathcal{U}$.

So, for all $y \in Y_1 \in \mathcal{U}$, $-y + B \in \mathcal{U}$.

$$Y_1 \subseteq \{n \mid -n + B \in \mathcal{U}\} \in \mathcal{V} \implies B \in \mathcal{V} \oplus \mathcal{U}$$

Therefore, $A \notin \mathcal{U} \oplus \mathcal{V}$.

Hence $\mathcal{U} \oplus V \neq \mathcal{V} \oplus \mathcal{U}$.

- $\underline{\textbf{$\odot$ is not commutative:}}$

Let $X_i = \{2^k \mid k \geq i\}$ and $Y_j = \{3^k \mid k \geq i\}$.

Extend the families $\{X_i \mid i \geq 1\}$ and $\{Y_j \mid j \geq 1\}$, to ultrafilters $\mathcal{U}$ and $\mathcal{V}$.

Set $A = \{2^i \cdot 3^j \mid i < j\}$. Observe that $A \in \mathcal{U} \odot V$ and $A \notin \mathcal{V} \odot \mathcal{U}$.

Proposition 33. Let $(S, \cdot)$ be a discrete semigroup. Then $(\beta S, \odot)$ is a semigroup.

Proof. Since $\cdot$ is associative, we will write $x \cdot y \cdot z$ instead of writing the parentheses.

Recall that if $f_1, f_2 : S \to T$ are functions and $B \subseteq T \times T$, then

$$\mathcal{U}x_1\, \mathcal{V}x_1\, ((f_1(x_1), f(x_2)) \in B) \iff f_{1*}(\mathcal{U})y_1\, f_{2*}(\mathcal{V})y_2\, ((y_1, y_2) \in A)$$

We will use this fact with function $f = \cdot : S \times S \to S$ paired with the following fact

$$\mathcal{U}x\, \mathcal{V}y((x, y) \in B) \iff (\mathcal{U} \otimes V)\bar{x}\, (\bar{x} \in B)$$

Recall that $f_*(\mathcal{U} \otimes V) = \mathcal{U} \odot V$ when $f(x, y) = x \cdot y$.

So we have;

$$\begin{aligned}
A \in (\mathcal{U} \odot \mathcal{V}) \odot \mathcal{W} &\iff (\mathcal{U} \odot V)\bar{y}\, \mathcal{W}z\, (\bar{y} \cdot z \in A) \\
&\iff f(\mathcal{U} \otimes \mathcal{V})\bar{y}\, \mathcal{W}z\, (\bar{y} \cdot z \in A) \\
&\iff (\mathcal{U} \otimes \mathcal{V})\bar{x}\, \mathcal{W}z\, (f(\bar{x}) \cdot z \in A) \\
&\iff \mathcal{U}x\, \mathcal{V}y\, \mathcal{W}z\, (x \cdot y \cdot z \in A)
\end{aligned}$$

Similarly, we have;

$$\begin{aligned}
A \in \mathcal{U} \odot (V \odot \mathcal{W}) &\iff \mathcal{U}x\, (\mathcal{V} \odot \mathcal{W})\bar{y}\, (x + \bar{y} \in A) \\
&\iff \mathcal{U}x\, f(\mathcal{V} \otimes \mathcal{W})\bar{y}\, (x \cdot \bar{y} \in A) \\
&\iff \mathcal{U}x\, (\mathcal{V} \otimes \mathcal{W})\bar{z}\, (x \cdot f(\bar{z}) \in A) \\
&\iff \mathcal{U}x\, \mathcal{V}y\, \mathcal{W}z\, (x \cdot f(y, z) \in A) \\
&\iff \mathcal{U}x\, \mathcal{V}y\, \mathcal{W}z\, (x \cdot y \cdot z \in A)
\end{aligned}$$

Therefore $A \in \mathcal{U} \odot (V \odot \mathcal{W}) \iff A \in (\mathcal{U} \odot \mathcal{V}) \odot \mathcal{W}$

Hence $\mathcal{U} \odot (V \odot \mathcal{W}) = (\mathcal{U} \odot \mathcal{V}) \odot \mathcal{W}$. Thus, $\odot$ is associative. □

Thus $(\beta S, \odot)$ is a compact semigroup containing the discrete semigroup $(S, \cdot)$, so it is a semigroup compactification.

The algebraic structure of βS is not so well-behaved. A small observation is that the compactification $(\beta\mathbb{Z}, \oplus)$ of the additive cyclic group $(\mathbb{Z}, +)$ is not a group at all.

Example 10. Observe that if $\beta\mathbb{Z}$ were a group, its identity has to be the principal ultrafilter generated by 0 as the identity is unique in a group. Recall that we have identified 0 with the principal ultrafilter generated by it.

Now we will show that any non-principal ultrafilter $\mathcal{V}$ is not invertible.

Assume $\mathcal{V}$ is non-principal and invertible. Then there is $\mathcal{U}$ such that $\mathcal{U} \oplus \mathcal{V} = 0$. That means the singleton $\{0\} \in \mathcal{V} \oplus \mathcal{U}$.

Since $\mathcal{V}$ is non-principal, it cannot contain any finite set.

$$\begin{aligned} \{0\} \in \mathcal{U} \oplus \mathcal{V} &\iff \{x \in \mathbb{Z} \mid \{0\} - x \in \mathcal{V}\} \in \mathcal{U} \\ &\iff \{x \in \mathbb{Z} \mid \{-x\} \in \mathcal{V}\} \in \mathcal{U} \\ &\iff \varnothing \in \mathcal{U} \end{aligned}$$

A contradiction. Therefore, $\mathcal{V}$ is principal or not invertible.

So only principal elements in $\beta\mathbb{Z}$ are the elements of $\mathbb{Z}$.

This is true in general, that is, if S is discrete, then βS is not a group. But that does not mean βS does not contain groups. In fact $(\beta\mathbb{N}, \oplus)$ contains free groups on $2^{\mathfrak{c}}$ generators, the proof can be found at [HS98, Corollary 7.36].

While the algebraic structure of βS is interesting and contains many open questions, we are particularly interested in the idempotent elements of βS, as they are especially useful for combinatorial results.

5.4 Idempotent Ultrafilters and Hindman's Theorem

βS is a compact space and $\rho_{\mathcal{U}} : \beta S \to \beta S$ is continuous. So βS is a compact right topological semigroup. Thus, we can apply Ellis' lemma.

Corollary 21. $(\beta S, \odot)$ contains an idempotent element. ($E(\beta S) \neq \varnothing$.)

In particular, idempotents of $(\beta\mathbb{N}, \oplus)$ exists.

Definition 23. Let $(S, \cdot)$ be a discrete semigroup, $A \subseteq S$ and $\mathcal{V} \in \beta S$. We generalize

the notation $A - \mathcal{U}$ we introduced in Chapter 3.

$$A \div \mathcal{V} = \{s \in S \mid s^{-1} \cdot A \in \mathcal{V}\}$$

We have $(A \div V) \div \mathcal{U} = A \div (\mathcal{U} \odot \mathcal{V})$ and

$$A \div \mathcal{V} \in \mathcal{U} \iff A \in (\mathcal{U} \odot \mathcal{V})$$

So, if $\mathcal{U}$ is idempotent, then

$$A \in \mathcal{U} = \mathcal{U} \odot \mathcal{U} \iff A \div \mathcal{U} \in \mathcal{U}$$

Define $A^* = (A \div \mathcal{U}) \cap A$. For any idempotent $\mathcal{U} \in \beta S$ we have

$$A \in \mathcal{U} \iff A^* \in \mathcal{U}$$

A^* is the set of elements in A who shifts A to a large set. We have the following lemma, saying, for a large set A in an idempotent ultrafilter $\mathcal{U}$, elements in A^* shift A^* to a large set as well.

Lemma 5. [HS98, Lemma 4.14] Let $\mathcal{U} \in \beta S$ be an idempotent ultrafilter and $A \subseteq S$. If $A \in \mathcal{U}$, then $A^* \in \mathcal{U}$ and $x^{-1} \cdot A^* \in \mathcal{U}$ for all $x \in A^*$.

Proof. Let $\mathcal{U}$ be an idempotent ultrafilter and $A \in \mathcal{U}$. Then $A^* \in \mathcal{U}$. Let $x \in A^*$, so $x^{-1} \cdot A \in \mathcal{U}$. Thus $(x^{-1} \cdot A)^* \in \mathcal{U}$. We will show that $(x^{-1} \cdot A)^* \subseteq x^{-1} \cdot A^*$.

Pick $y \in (x^{-1} \cdot A)^* = \{y \in x^{-1} \cdot A \mid y^{-1} \cdot (x^{-1} \cdot A) \in \mathcal{U}\}$. We have $y^{-1} \cdot (x^{-1} \cdot A) \in \mathcal{U}$ and $y \in x^{-1} \cdot A$. Thus $(x \cdot y)^{-1} \cdot A = y^{-1} \cdot (x^{-1} \cdot A) \in \mathcal{U}$ and $x \cdot y \in A$. So, $x \cdot y \in A^*$. Therefore $y \in x^{-1} \cdot A^*$. □

Recall that in Chapter 3, Corollary 13, we have proven that if idempotents of $(\beta\mathbb{N}, \oplus)$ exist, then for every finite coloring of $\mathbb{N}$, there is an infinite set X with monochromatic $FS(X)$. So we have finally proven IP sets are weakly partition regular. Now, we will prove the stronger result, partition regularity of IP sets. We will achieve this by showing that being an IP set is equivalent to being an element of some idempotent ultrafilter.

We will generalize the definition $FS(X)$ from Chapter 3 to semigroups.

Let $X = \{x_i\}_{i\in I} \subseteq (S, \cdot)$ is a well-ordered subset of some semigroup S. Define

$$FP(X) = \{x_1 \cdot \ldots \cdot x_n \mid x_1 < x_2 < \ldots < x_n \in X\}$$

the set of non-repeating finite products in increasing order. We defined the products in the order of increasing indices to ensure there is no ambiguity when S is not commutative.

When the set $X = \{x_i\}_{i\in I}$ is given as a sequence, we will usually abuse the notation and write $FP(x_i)_{i\in I}$ or $FP(x_i)$ instead of $FP(\{x_i\}_{i\in I})$.

Theorem 22. [HS98, Theorem 5.12] Let $(S, \cdot)$ be a semigroup and $A \subseteq S$ be any subset. Then the following are equivalent.

(1) There exists an infinite subset $X \subseteq A$ such that $FP(X) \subseteq A$.

(2) There is an idempotent ultrafilter $\mathcal{U} \in \beta S$ with $A \in \mathcal{U}$.

Proof. Let $(S, \cdot)$ be a semigroup and $A \subseteq S$.

($\Longleftarrow$) : We will do a different version of the Galvin-Glazer proof we did in Chapter 3. Observe that $A^* \in \mathcal{U}$ since $\mathcal{U} \odot \mathcal{U} = \mathcal{U}$. Take $x_1 \in A^*$. Since $A^* \in \mathcal{U}$, by the previous lemma, $x_1^{-1} \cdot A^* \in \mathcal{U}$.

Pick $x_2 \in A^* \cap (x_1^{-1} \cdot A^*)$. Similarly, we have $x_2^{-1} \cdot A^* \in \mathcal{U}$ and $(x_1 \cdot x_2)^{-1} \cdot A^* \in \mathcal{U}$ since $x_2 \in x_1^{-1} \cdot A^*$. So we pick $x_3 \in A^* \cap x_1^{-1} \cdot A^* \cap x_2^{-1} \cdot A^* \cap (x_1 \cdot x_2)^{-1} \cdot A^*$.

We will continue picking elements in this manner. Now, we shall do the induction formally.

Say $\{x_1, \ldots, x_n\}$ has been picked such that $FP(x_i)_{i\leq n} \subseteq A^*$.

Observe that for every $s \in FP(x_i)_{i\leq n}$, $s^{-1} \cdot A^* \in \mathcal{U}$ by the previous lemma. Thus,

$$x_{n+1} \in A^* \cap \left(\bigcap_{s\in FP(x_i)_{i\leq n}} s^{-1} \cdot A^* \right)$$

$FP(x_i)_{i\leq n+1} \subseteq A^*$ as for every $s \in FP(x_i)_{i\leq n}$, since $x_{n+1} \in s^{-1} \cdot A^*$, $x_{n+1} \cdot s \in A^* \subseteq A$. Therefore, $FP(x_i)_{i\leq n+1} \subseteq A^*$.

($\implies$) : Let $X = \{x_i\}_{i\in\mathbb{N}} \subseteq A$ be infinite set with $FP(X) \subseteq A$.

We claim that $T = \bigcap_{n\in\mathbb{N}} \widehat{FP(x_i)_{i\geq n}}$ is a closed subsemigroup of βS.

$\{FP(x_i)_{i\geq n} \mid n \in \mathbb{N}\}$ has FIP; thus, we can extend it into an ultrafilter $\mathcal{U}$. Then $\mathcal{U} \in T$. Thus, T is non-empty, and by compactness of βS, T is compact.

Let $\mathcal{U}, \mathcal{V} \in T$, so $FP(x_i)_{i\geq n} \in \mathcal{U}, \mathcal{V}$ for every n. Let $n \in \mathbb{N}$. So, for every $k \in FP(x_i)_{i\geq n}$, there is some ℓ with $k^{-1} \cdot FP(x_i)_{i\geq \ell} \in \mathcal{V}$. Hence we have

$$FP(x_i)_{i\geq n} \subseteq \{k \in S \mid k^{-1} \cdot FP(x_i)_{i\geq n} \in \mathcal{V}\} \in \mathcal{U}$$

Therefore, $FP(x_i)_{i\geq n} \in \mathcal{U} \odot \mathcal{V}$. T is a subsemigroup of βS.

So T is a compact semigroup; we can apply Ellis' lemma to get an idempotent ultrafilter $p \in T$. So $FP(x_i)_{i\geq n} \in p$ for every n.

Then $A \in p$ as $A \supseteq FP(X) \in p$.

□

Corollary 23 (Hindman's Theorem)**.** IP sets are partition regular.

Proof. Let $A \subseteq S$ be an IP set. Then there is an idempotent ultrafilter $\mathcal{U} \in \beta S$ with $A \in \mathcal{U}$. Let $A = C_1 \sqcup \cdots \sqcup C_r$ be any finite coloring. Then there is some i with $C_i \in \mathcal{U}$. C_i is an element of an idempotent ultrafilter; thus, by the previous theorem, C_i is an IP set. □

We also have the following characterization of being in the closure of $E(\beta\mathbb{N}, \oplus) \subseteq \beta\mathbb{N}$.

Let's denote $\Gamma = cl(E(\beta\mathbb{N})) = cl(\{\mathcal{U} \mid \mathcal{U} \oplus \mathcal{U} = \mathcal{U}\})$.

Corollary 24. [HS98, Lemma 5.19] $\mathcal{U} \in \Gamma$ if and only if for every $A \in \mathcal{U}$ is an IP set.

Proof. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$.

($\implies$) : Let $\mathcal{U} \in cl(E(\beta\mathbb{N}))$. Let $A \in \mathcal{U}$. Then $\mathcal{U} \in \widehat{A}$ and since $\mathcal{U}$ is in the closure of $E(\beta\mathbb{N})$, $\widehat{A} \cap E(\beta\mathbb{N}) \neq \varnothing$. So, there is an idempotent ultrafilter p with $A \in p$. Since A is an element of an idempotent ultrafilter, it is an IP set.

($\Longleftarrow$) : Let $\mathcal{U}$ be an ultrafilter with A is an IP set for any $A \in \mathcal{U}$. Let N be any neighborhood of $\mathcal{U}$, then there is some $A \in \mathcal{U}$ with $\widehat{A} \subseteq N$ as $\widehat{A}$'s form a basis for $\beta\mathbb{N}$. Since A is an IP set by our assumption, there is an idempotent ultrafilter p with $A \in p$. So $p \in \widehat{A} \cap E(\beta\mathbb{N}) \subseteq N \cap E(\beta\mathbb{N})$. Thus $\mathcal{U}$ is in the closure of $E(\beta\mathbb{N})$.

□

Therefore, witnesses of IP sets are in the closure of the set of all idempotents.

We will show that there are no ultrafilters that are both additive idempotent and multiplicative idempotent, but there are multiplicative idempotents "quite close" to additive idempotents.

Proposition 34. [HS98, Theorem 5.21] There is a multiplicative idempotent in Γ. That is, there is $\mathcal{U} \in cl(E(\beta\mathbb{N}, \oplus))$ with $\mathcal{U} \odot \mathcal{U} = \mathcal{U}$.

We will not give the proof, but as a guideline we will note that the proof in [HS98] uses the fact that Γ and $cl(E(K(\beta\mathbb{N}, +)))$ are left ideals of $(\mathbb{N}, \cdot)$ with a non-empty intersection.

Proposition 35. [HS98, Corollary 5.22] Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any finite coloring. Then there are infinite sets X and Y such that $FP(X) \cup FS(Y) \subseteq C_i$ for some C_i.

Proof. Let $\mathcal{U} \in \Gamma$ be a multiplicative idempotent. Then there is some i with $C_i \in \mathcal{U}$, and there is an infinite set X with $FP(X) \subseteq C_i$, as $\mathcal{U}$ is idempotent. Since $\mathcal{U} \in \Gamma$ and $C_i \in \mathcal{U}$, we have that C_i is an IP set, and so there is some infinite Y such that $FS(Y) \subseteq C_i$. Thus $FS(Y) \cup FP(X) \subseteq C_i$. □

Given a set X, define the set of all non-repeating pairwise sums and products as

$$P(X) = \{y \in \mathbb{N} \mid y = x_i + x_j \text{ or } y = x_i \cdot x_j \text{ for some } i < j\}$$

Proposition 36. [HS98, Theorem 17.16] There is a finite coloring of $\mathbb{N}$ such that there is no infinite $X \subseteq \mathbb{N}$ with monochromatic $P(X)$.

This coloring and the following corollary can be found in [HS98] Section 17.2. Observe that this result is stronger than saying there is no infinite X with monochromatic $FS(X) \cup FP(X)$ for every finite coloring of $\mathbb{N}$.

Corollary 25. [HS98, Corollary 17.17] There are no non-principal ultrafilters $\mathcal{U} \in \beta\mathbb{N}$ with $\mathcal{U} \oplus \mathcal{U} = \mathcal{U} \odot \mathcal{U}$.

Proof. Assume not, that is, assume there is an ultrafilter $\mathcal{U}$ with $\mathcal{U} \oplus \mathcal{U} = \mathcal{U} \odot \mathcal{U}$.

Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be a finite coloring such that there is no infinite X with its set $P(X)$ of all non-repeating pairwise sums and products monochromatic.

There is some color $C = C_i$ such that $C \in \mathcal{U} \odot \mathcal{U} = \mathcal{U} \oplus \mathcal{U}$. Then,

$$C \in \mathcal{U} \odot \mathcal{U} \iff C \div \mathcal{U} = \{x \in \mathbb{N} \mid C \cdot x^{-1} \in \mathcal{U}\} \in \mathcal{U}$$

$$C \in \mathcal{U} \oplus \mathcal{U} \iff C - \mathcal{U} = \{x \in \mathbb{N} \mid C - x \in \mathcal{U}\} \in \mathcal{U}$$

Thus, $C' = (C - \mathcal{U}) \cap (C \div \mathcal{U}) \in \mathcal{U}$. We will recursively construct an infinite set $\{x_n\}$ with monochromatic non-repeating pairwise sums and products.

Pick $x_1 \in C'$. Assume $x_1, \ldots, x_n$ are chosen. Pick

$$x_{n+1} \in C' \cap \left(\left(\bigcap_{i \leq n} C - x_i \right) \cap \left(\bigcap_{i \leq n} C \cdot x_i^{-1} \right) \right) \setminus \{x_i\}_{i \leq n}$$

Observe that $x_{n+1} \in C - x_i$, means $x_{n+1} + x_i \in C$ and similarly $x_{n+1} \in C \cdot x_i^{-1}$ means $x_{n+1} \cdot x_i \in C$.

Thus we have an infinite set $X = \{x_i\}_{i \in \mathbb{N}}$ with monochromatic $P(X)$, but this contradicts Proposition 36. Therefore, $\mathcal{U} \oplus \mathcal{U} \neq \mathcal{U} \odot \mathcal{U}$. □

Thus, we have finally shown that there are no ultrafilters that are both additively and multiplicatively idempotent. This result also tells us we won't get some infinite results that are both additive and multiplicative, but we can continue analysing the algebraic structure of βS and the idempotents to understand which subsets of $\mathbb{N}$ satisfy these results and if the weakenings of these results to the finite cases are true.

In particular, we will analyse the initial finite case of monochromatic $FS(X) \cup FP(X)$, which is $X = \{x_1, x_2\}$. That is *the Hindman's Conjecture*: Given finite

coloring $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$, is there monochromatic $\{x, y, x + y, x \cdot y\}$ for some $x, y \in \mathbb{N}$?

While Hindman's Conjecture is still open, given a two coloring $\mathbb{N} = C_1 \sqcup C_2$, the question is proven to be positive. We will go over the proof of this result in Chapter 6.

5.5 Minimal Idempotents and Combinatorial Largeness

Nearly everything in this section can be found in Chapters 4 and 5 of [HS98]. For other combinatorial properties of the central set, Chapters 14, 15, 16, and 17 of [HS98] can be checked.

Idempotent ultrafilters are very useful in combinatorics, as they witness nice combinatorial properties. A particularly useful and well-studied class of idempotent ultrafilters is the minimal idempotents. The minimality is in the sense of $\leq_L$ we defined in the first section of this chapter, which is $\mathcal{U} \leq_L V \iff \mathcal{U} \odot \mathcal{V} = \mathcal{U}$.

Recall that every minimal idempotent is contained in a minimal ideal. Specifically, $\mathcal{U}$ is a minimal idempotent if and only if $\beta S \odot \mathcal{U}$ is a minimal left ideal from Proposition 30

Since minimal left ideals of βS exist by Ellis' lemma, we know the smallest ideal $K(\beta S)$ exists as well, and $K(\beta S)$ contains all of the minimal idempotent ultrafilters. However, we have not said much about them except their existence. We will investigate some combinatorial properties of minimal idempotents in this section. While everything we will do can be done with arbitrary semigroups, we will specifically focus on Stone–Čech compactifications of $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$, so in both cases our main set is $S = \mathbb{N}$.

Definition 24. Let $(S, \cdot)$ be a semigroup and $A \subseteq S$ be a subset of S.

- A is **thick** if for every finite $F \subseteq S$ there is $x \in S$ with $F \cdot x \subseteq A$. That is, a finite set can be translated into A.
- A is **syndetic** if there are $t_1, \ldots, t_n \in S$ such that $\bigcup_{i=1}^{n} t_i^{-1} \cdot A = S$,

- A is **piecewise syndetic** if A is the intersection of a thick set with a syndetic set.

- A is **central** if there is a minimal idempotent ultrafilter $\mathcal{U} \in K(\beta S)$ such that $A \in \mathcal{U}$.

These notions were first defined for $\mathbb{N}$ in the topological dynamics context with different but equivalent definitions. Unfortunately, those are outside the scope of this text, but most of the notions we define in this section are actually from topological dynamics.

We call such notions notions of largeness or combinatorial largeness, as sets having these properties are large in some sense with some structure. Usually, notions of largeness and densities have characterisations with βS and ultrafilters.

Example 11. Let $(S, \cdot) = (\mathbb{N}, +)$. Then we have the following observations.

- $A \subseteq \mathbb{N}$ is thick if for any $l \in \mathbb{N}$, there is $a \in A$ with $\{a, a+1, \ldots, a+l\} \subseteq A$.

- $A \subseteq \mathbb{N}$ is syndetic if A has bounded gaps. That is there is a $k \in \mathbb{N}$ such that for every n, $\{n, n+1, \ldots, n+k\} \cap A \neq \varnothing$.

- $A \subseteq \mathbb{N}$ is piecewise syndetic if A has bounded gaps and A contains arbitrarily long consecutive numbers.

We have the following immediate observations.

Proposition 37. [HS98, Theorem 4.49] Let $A \subseteq S$ be a subset of S. Then,

1. A is syndetic if and only if $S \setminus A$ is not thick.

2. A is thick if and only if $S \setminus A$ is not syndetic.

3. A is piecewise syndetic if and only if there are $t_1, \ldots, t_n \in S$ such that $t_1^{-1} \cdot A \cup \cdots \cup t_n^{-1} \cdot A$ is thick.

Proof. Let $A \subseteq S$.

1. ($\Longrightarrow$): Let A be syndetic. That is there are $t_1, \ldots, t_n \in S$ such that $\bigcup_{i=1}^{n} t_i^{-1} \cdot A = S$.

   Assume not. That is, assume $S \setminus A$ is thick. Then for the finite set $\{t_1, \ldots, t_n\}$ there is some $x \in S$ such that $t_i \cdot x \in S \setminus A$ for every i. But this contradicts $\bigcup_{i=1}^{n} t_i^{-1} \cdot A = S$.

   ($\Longleftarrow$): Let $S \setminus A$ be not thick. That is, there is a finite set $F = \{t_1, \ldots, t_n\} \subseteq S$, and for every $x \in S$ with $F \cdot x \not\subseteq S \setminus A$.

   So for every $x \in S$, there is i with $t_i \cdot x \in A$. So $S \subseteq \bigcup_{i=1}^{n} t_i^{-1} \cdot A$. Thus, A is syndetic.

2. Same proof.

3. ($\Longrightarrow$): Let A be a piecewise syndetic set, that is $A = T \cap Y$ for some thick T and syndetic Y.

   Then there are $t_1, \ldots, t_n \in S$ such that $\bigcup_{i=1}^{n} t_i^{-1} \cdot Y = S$. We want to show that $\bigcup_{i=1}^{n} t_i^{-1} \cdot A$ is thick.

   Let $F \subseteq S$ be a finite set. We need to find $x \in S$ such that $F \cdot x \subseteq \bigcup_{i=1}^{n} t_i^{-1} \cdot A$, so for any $f \in F$, there is i with $f \cdot x \in t_i^{-1} \cdot A$ and hence $t_i \cdot f \cdot x \in A$.

   Consider $F' = \{t_i \cdot f \mid f \in F \text{ and } i \leq n\} = \{t_i \mid i \leq n\} \cdot F$ which is also a finite set. By thickness of T, there is $x \in S$ such that $F' \cdot x \in T$, that is $t_i \cdot f \cdot x \in T$ for every i.

   Given $f \in F$, by syndeticity of Y, there is i with $f \cdot x \in t_i^{-1} \cdot Y$. Thus $t_i \cdot f \cdot x \in Y$, and so, $t_i \cdot f \cdot x \in T \cap Y = A$. Therefore, $F \cdot x \subseteq \bigcup_{i=1}^{n} t_i^{-1} \cdot A$. So we are done.

   ($\Longleftarrow$): Let $A \subseteq S$ such that there is $t_1, \ldots, t_n$ with $\bigcup_{i=1}^{n} t_i^{-1} \cdot A$ is thick.

   Let $T = A \cup (\bigcup_{i=1}^{n} t_i^{-1} \cdot A)$. T is a thick set as it contains a thick set. Let $Y = A \cup (S \setminus T)$. We claim that Y is syndetic.

   Assume not. Then by part 1, $S \setminus Y$ is thick. We claim that the set $F = \{t_1, \ldots, t_n\} \cup \{t_i \cdot t_j \mid i, j \leq n\}$ is the finite set with $\bigcup_{t \in E} t^{-1} \cdot Y = S$.

   Let $s \in S$. There are two cases $s \in S$ or $s \in T = A \cup (\bigcup_{i=1}^{n} t_i^{-1} \cdot A)$. Since $s \in A$ implies $s \in Y$, we only need to consider $s \in \bigcup_{i=1}^{n} t_i^{-1} \cdot A$

   If $s \in \bigcup_{i=1}^{n} t_i^{-1} \cdot A$, then $t_i \cdot s \in A \subseteq Y$ for some i, so we are done.

By definition, $A = T \cap Y$. Therefore, A is piecewise syndetic.

□

A trivial but useful corollary is that piecewise syndetic sets are non-empty, as we expect from a notion of largeness. An implication of this is that T is thick implies $T \cap Y \neq \varnothing$ for every syndetic Y. Similarly, Y is syndetic implies $Y \cap T \neq \varnothing$ for every thick T.

Proposition 38. [HS98, Theorem 4.48] Let $(S, \cdot)$ be a semigroup and $A \subseteq S$ be a subset of S. Then

1. A is thick if and only if there is a left ideal $L \subseteq (\beta S, \odot)$ with $L \subseteq \widehat{A}$.
2. A is syndetic if and only if for every left ideal $L \subseteq (\beta S, \odot)$ $\widehat{A} \cap L \neq \varnothing$.
3. A is piecewise syndetic if and only if there is a minimal left ideal intersecting $\widehat{A}$.

Proof. Let $A \subseteq S$ be a subset.

1. ($\Longrightarrow$): Let A be thick and $L \subseteq \beta S$ be a minimal left ideal.

   First we show that $\mathcal{X} = \{t^{-1} \cdot A \mid t \in S\}$ has FIP.

   Let $t_1, \ldots, t_n \in S$. By thickness, there is $x \in S$ such that $t_i \cdot x \in A$. So, $x \in t_i^{-1} \cdot A$ for every i. Therefore $x \in \bigcap_{i \leq n} t_i^{-1} \cdot A$. Thus, $\mathcal{X}$ has FIP.

   Extend $\mathcal{X}$ into an ultrafilter $\mathcal{V}$. For every $s \in S$, $s^{-1} \cdot A \in \mathcal{V}$, and so $A \in \mathcal{U}_s \odot \mathcal{V}$. So $S \odot \mathcal{V} \subseteq \widehat{A}$.

   Since left multiplication of βS is continuous, we have $\beta S \odot V = cl(S) \odot V \subseteq cl(S \odot V)$, and hence $cl(S \odot \mathcal{V}) = \beta S \odot V$. $\widehat{A}$ is a closed set containing $S \odot \mathcal{V}$ and hence it contains $\beta S \odot \mathcal{V}$. Thus $L = \beta S \odot \mathcal{V} \subseteq \widehat{A}$. So L is the left ideal we want.

   ($\Longleftarrow$): Let $A \subseteq S$ be a subset and $L \subseteq \beta S$ be a left ideal with $L \subseteq \widehat{A}$. Let F be any finite set.

Let $\mathcal{V} \in L$, then $\mathcal{U} \odot \mathcal{V} \in L \subseteq \widehat{A}$ for every $\mathcal{U} \in \beta S$. In particular, $S \odot \mathcal{V} \in \widehat{A}$, thus given $s \in S$, $U_s \odot \mathcal{V} \in \widehat{A}$, which means $s^{-1} \cdot A \in \mathcal{V}$ for every $s \in S$.

Since $\mathcal{V}$ is an ultrafilter, the finite intersection $\bigcap_{t \in F} t^{-1} \cdot A \in \mathcal{V}$. Pick $x \in \bigcap_{t \in F} t^{-1} \cdot A$ and we have $F \cdot x \subseteq A$.

Therefore, A is thick.

2. ($\implies$): Let A be a syndetic set. Then $S \setminus A$ is not thick. So $L \not\subseteq \widehat{S \setminus A}$ for every left ideal L. Since $\widehat{S \setminus A} = \beta S \setminus \widehat{A}$, we have $L \cap \widehat{S} \neq \varnothing$.

   ($\impliedby$): Let A be a set such that for every left ideal $L \subseteq \beta S$, $L \cap \widehat{S} \neq \varnothing$.

   Assume not; that is, assume A is not syndetic, then $S \setminus A$ is thick. Therefore, there is a left ideal L such that $L \subseteq \widehat{S \setminus A}$. But this contradicts $L \cap \widehat{A} \neq \varnothing$.

3. ($\implies$): Let A be a piecewise syndetic set. Then there is a thick set T and a syndetic set Y such that $T \cap S = A$.

   By part 1, there is a left ideal $L \subseteq \beta S$ such that $L \subseteq \widehat{T}$, and by part 2, $\widehat{Y} \cap L \neq \varnothing$.

   Thus, $L \cap \widehat{A} = L \cap \widehat{T \cap S} = L \cap \widehat{T} \cap \widehat{S} \neq \varnothing$.

   Observe that every left ideal contains a minimal ideal; we could have picked L as minimal from the start to get the result we stated.

   ($\impliedby$): Let $A \subseteq S$ be a set and L be a minimal left ideal such that $\widehat{A} \cap L \neq \varnothing$. Let $\mathcal{V} \in \widehat{A} \cap L$. We claim that $A \div \mathcal{V} = \{s \in S \mid s^{-1} \cdot A \in \mathcal{V}\}$ is syndetic.

   Let $I \subseteq \beta S$ be any left ideal. Then $I \odot \mathcal{V} \subseteq L$ is a left ideal, so $I \odot \mathcal{V} = L$ by minimality of L. There is an ultrafilter $\mathcal{U} \in I$ such that $\mathcal{U} \odot \mathcal{V} = \mathcal{V}$. In particular, $A \in \mathcal{V} = \mathcal{U} \odot \mathcal{V}$, which implies $A \div \mathcal{V} \in \mathcal{U}$. So $\mathcal{U} \in \widehat{A \div \mathcal{V}}$. Therefore, $I \cap \widehat{A \div \mathcal{U}} \neq \varnothing$, and by part 2, $A \div \mathcal{U}$ is syndetic.

   $A \div \mathcal{U}$ is syndetic, so there is $t_1, \ldots, t_n \in S$ such that $\bigcup_{i \leq n} t_i^{-1} \cdot (A \div \mathcal{U}) = S$.

   Define $T' = \bigcup_{i \leq n} t_i^{-1} \cdot A$. We claim that T' is thick.

   Let $F = \{f_1, \ldots, f_k\} \subseteq S$ be a finite set. Then for any $f_j \in F$, there is some $j(i) \leq n$, $f_j \in t_{j(i)}^{-1} \cdot (A \div \mathcal{V})$, so $t_{j(i)} \cdot f_j \in A \div \mathcal{V}$. Hence, $(t_{j(i)} \cdot f_j)^{-1} \cdot A \in \mathcal{V}$. Thus the finite intersection $\bigcap_{j \leq k} (t_{j(i)} \cdot f_j)^{-1} \cdot A \neq \varnothing$. Pick an element $x \in \bigcap_{j \leq k} (t_{j(i)} \cdot f_j)^{-1} \cdot A$, so for any $f \in F$, $t_i \cdot f \cdot x \in A$ for some i.

Therefore, there is an $x \in S$ such that $F \cdot x \subseteq T'$, and so, T' is thick.

By part 3 of Proposition 38, A is piecewise syndetic.

□

Therefore, we have the algebraic characterizations of thick, syndetic, and piecewise syndetic sets.

Observe that thick sets are central, since every thick set T has a minimal left ideal L with $L \subseteq \widehat{T}$ and minimal left ideals contain minimal idempotent ultrafilters. So, there are minimal idempotent ultrafilters that contain T.

In particular, we have seen that a piecewise syndetic set gives us a characterization for being in $K(\beta S)$ by Proposition 38. A direct corollary related to understanding $K(\beta S)$ in terms of piecewise syndetic sets is the following.

Corollary 26. Let $\mathcal{U}$ be an ultrafiler on S.

$\mathcal{U} \in cl(K(\beta S))$ if and only if A is piecewise syndetic for every $A \in \mathcal{U}$.

Proof. Let $\mathcal{U} \in \beta S$ be an ultrafilter.

($\implies$) : Let $\mathcal{U} \in cl(K(\beta S))$ and $A \in \mathcal{U}$. Then $\mathcal{U} \in \widehat{A}$. By definition of closure, $\widehat{A} \cap K(\beta S) \neq \varnothing$. So A is piecewise syndetic by the last proposition.

($\impliedby$) : Let $\mathcal{U}$ be an ultrafilter with A is piecewise syndetic for every $A \in \mathcal{U}$. Let N be a neighbourhood of $\mathcal{U}$, then N contains a set of the form $\widehat{A}$ for some A, since $\widehat{A}$'s form a basis. By the assumption, $K(\beta S) \cap \widehat{A} \neq \varnothing$, every neighborhood of $\mathcal{U}$ intersects $K(\beta S)$, thus $\mathcal{U}$ is in the closure of $K(\beta S)$.

□

We have several corollaries.

Corollary 27. Central sets are piecewise syndetic.

Every minimal idempotent ultrafilter is in $cl(K(\beta S))$, and hence their elements, the central sets, are piecewise syndetic by the previous corollary.

Regarding the partition regularity for these notions, we have the necessary tools to see which notions are and which are not.

Corollary 28. Piecewise syndetic sets and central sets are partition regular. Thick and syndetic sets are not (weakly) partition regular.

Proof. Let $A \subseteq S$.

- Let A be a piecewise syndetic set and $A = C_1 \sqcup \cdots \sqcup C_r$ be a finite coloring.

  Since A is piecewise syndetic, there is a minimal left ideal $L \subseteq \beta S$ such that $L \cap \widehat{A} \neq \varnothing$. Let $\mathcal{U} \in L \cap \widehat{A}$ be an ultrafilter. Since $A \in \mathcal{U}$, there is some $C_i \in \mathcal{U}$. Then $\mathcal{U} \in L \cap \widehat{C_i}$. Therefore, C_i is piecewise syndetic.

- A be a central set, then there is a minimal idempotent ultrafilter $\mathcal{U}$ with $A \in \mathcal{U}$, so any finite coloring of A has a color in $\mathcal{U}$.

- A simple example for thick sets is the coloring $\mathbb{N} = 2\mathbb{N} \sqcup (2\mathbb{N}+1)$, which gives a coloring of $\mathbb{N}$ with no thick colors.

  An example for the syndetic sets is the following: let $A = \bigcup_{k=1}^{\infty} \{n \in \mathbb{N} \mid k^2 - k + 1 \leq n \leq k^2\}$. The coloring $\mathbb{N} = A \sqcup (\mathbb{N} \setminus A)$ has both colors thick, and no colors are syndetic.

□

When there are several semigroup structures on the same set, we will denote with respect to which structure this notion is defined. For example, a syndetic set in $(\mathbb{N}, +)$ will be called additively syndetic if there is another semigroup on $\mathbb{N}$ at play.

We have two interesting theorems about combinatorial structures of additively central and multiplicatively central sets from [BH94].

Lemma 6. [BH94, Lemma 3.3] Let $X = \{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ be any infinite subset. Then

$$\bigcup_{n\in\mathbb{N}} \{x_n + 1, \ldots, x_n + n\}$$

is additively central and

$$\bigcup_{n\in\mathbb{N}}\{x_n, x_n\cdot 2, \ldots, x_n\cdot n\}$$

is multiplicatively central.

Proof. Let $\mathcal{V}$ be a non-principal ultrafilter on $\mathbb{N}$ with $X \in \mathcal{V}$. Let $A = \bigcup_{n\in\mathbb{N}}\{x_n + 1, \ldots, x_n + n\}$.

We claim that any ultrafilter $\mathcal{U}$ containing $X - \mathcal{V}$ contains A as well.

Given $n \in \mathbb{N}$, $X - n = \{x_n + n \mid n \in \mathbb{N}\}$. If $X - n = \{x_i + n\}_{i<n} \sqcup \{x_i + n\}_{i\leq n}\mathcal{U}$, then $\{x_i + n\}_{i\leq n}\mathcal{U}$ as $\mathcal{U}$ cannot contain a finite set. Since $\{x_i + n\}_{i\leq n} \subseteq A$, $A \in \mathcal{U}$.

Therefore, $\beta\mathbb{N} \oplus \mathcal{V} \subseteq \widehat{A}$. So A is additively thick and hence additively central.

The proof for the multiplicative version is the same.

□

Theorem 29. [BH94, Theorem 3.5] Let $A \subseteq \mathbb{N}$ be a multiplicatively central set, then for every $l \in \mathbb{N}$ there is a set $\{x_1, \ldots, x_l\}$ with $FS(x_i)_{i\leq l} \subseteq A$.

Proof. We look at the set of all ultrafilters that witness the property we want. That is

$$T = \{\mathcal{U} \in \beta\mathbb{N} \mid \forall A \in \mathcal{U}\ \forall l \in \mathbb{N} \text{ there is } \{x_n\}_{n\leq l} \text{ such that } FS(x_n)_{n\leq l} \subseteq A\}$$

We will show that T is two-sided ideal of $(\beta\mathbb{N}, \cdot)$. Then T contains the smallest ideal $K(\beta\mathbb{N}, \cdot)$. Thus, every minimal idempotent ultrafilter is in T, and hence every central set is in T.

<u>T **is a two sided ideal:**</u> Let $\mathcal{V} \in T$ and $\mathcal{U} \in \beta\mathbb{N}$. Let $B \in \mathcal{U} \odot \mathcal{V}$, then $B \in \mathcal{U} \odot \mathcal{V} \iff B \div \mathcal{V} \in \mathcal{U}$.

Since $B \div \mathcal{V} \in \mathcal{U}$, we can pick $n \in \mathbb{N}$ with $(n)^{-1} \cdot B \in \mathcal{V}$, then for any l, there is $\{x_n\}_{n\leq l}$ with $FS(x_n) \subseteq (n)^{-1} \cdot B$. Thus $\{n \cdot x_n\}_{n\leq l}$ has $FS(n \cdot x_n) \subseteq B$.

Therefore $\mathcal{U} \odot \mathcal{V} \in T$.

Let $U \in T$ and $\mathcal{V} \in \beta\mathbb{N}$. Let $B \in \mathcal{U} \odot \mathcal{V}$, then $B \div \mathcal{V} \in \mathcal{U}$. So, for every l, there is $\{x_n\}_{n\leq l}$ with $FS(x_n) \subseteq B \div \mathcal{V}$. Let $l \in \mathbb{N}$ and $\{x_n\}_{n\leq l}$ have $FS(x_n) \subseteq B \div \mathcal{U}$.

Then for any $n \leq l$, $(x_n)^{-1} \cdot B \in \mathcal{V}$. Since $\mathcal{V}$ is an ultrafilter, $\bigcap (x_n)^{-1} \cdot B \in \mathcal{V}$. Pick $c \in \bigcap (x_n)^{-1} \cdot B$. Then $\{c \cdot x_n\}_{n \leq l}$ is a set with $FS(c\dot{x}_n) \subseteq B$. Hence $\mathcal{U} \odot \mathcal{V} \in T$.

Thus, we are done. □

While we have finite sets B of arbitrary lengths with $FS(B)$ in multiplicatively central sets, we might not even have $\{x_1, x_2\}$ with $\{x_1 \cdot x_2\}$ in additively central sets.

Theorem 30. [BH94, Theorem 3.4] There is $A \subseteq \mathbb{N}$ an additively central set with no $\{x, x^2\} \subseteq A$.

Proof. We will use the previous lemma. The main idea is to pick such a rapidly growing set as $X = \{x_n\}$, so the central set A cannot contain both $x = x_n + i$ for some $i \leq n$ and x^2 in A.

Set $X = \{2, 8, 512, ...\} = \{2^{(3^n)}\}$. Then

$$A = \{3, 4\} \cup (\bigcup_{n=1} \{2^{(3^n)} + 1, \ldots, 2^{(3^n)} + n\}$$

A is additively central by the previous lemma. Let $x \in A$. Then $x = 2^{(3^n)} + k$ for some $k \leq n$. $x^2 = \{(2^{(3^n)})^2 + k \cdot 2^{(3^n)+1} + k^2\} < 2^{(3^{n+1})}$. So $x^2 \notin A$. □

We can strengthen the previous theorem to multiplicatively piecewise syndetic sets.

Proposition 39. [BG20]. Let $A \subseteq \mathbb{N}$ be a multiplicatively piecewise syndetic set, then for every l, there is a set $\{x_n\}_{n \leq l}$ with $FS(x_n) \subseteq A$.

Proof. Let A be multiplicatively piecewise syndetic. Then there are $t_1, \ldots t_n \in \mathbb{N}$ such that $A' = t_1^{-1} \cdot A \cup \ldots \cup t_n^{-1} \cdot A$ is thick.

We claim that A' satisfies for every l, there is a set $\{x_n\}_{n \leq l}$ with $FS(x_n) \subseteq A'$.

Let $l \in \mathbb{N}$ and $\{y_n\}_{n \leq l} \subseteq \mathbb{N}$ such that $FS(y_n) \subseteq \mathbb{N}$. Since A' is thick and $FS(y_n)$ is finite, there is $a \in A'$ such that $FS(y_n) \cdot a \in A'$. Thus $\{x_n\} = \{y_n \cdot a\}$ satisfies $FS(x_n) \subseteq A'$.

Observe that same proof of ($\Longleftarrow$) Theorem 22 gives us that, the property, for every l there is $\{x_n\}_{n \leq l}$ with $FS(x_n) \subseteq A$, is partition regular.

So $A' = t_1^{-1} \cdot A \cup \ldots \cup t_n^{-1} \cdot A$ gives a t_j with for every l, there is $\{x_i\}_{i \leq l}$ such that $FS(x_i) \subseteq t_j^{-1} \cdot A$. Then $\{t_j \cdot x_i\}_{i \leq l}$ satisfies $FS(t_j \cdot x_i) \subseteq A$. □

Before concluding the chapter, we would like to note that there is a survey [Hin20a] on the history of central sets by Hindman, which illustrates that central sets are combinatorially rich sets from topological dynamics with a nice Stone–Čech compactification characterization.

CHAPTER 6

MONOCHROMATIC $\{x, y, x+y, xy\}$ IN 2 COLORS

6.1 Overview

In the last chapter, we have seen that we cannot have an infinite X with monochromatic $FS(X) \cup FP(X)$ for any finite colouring of $\mathbb{N}$. Consider the following finite version: *Given $n \geq 2$, does there exist $X = \{x_1, \ldots, x_n\}$ with monochromatic $FS(X) \cup FP(X)$ for any finite coloring of $\mathbb{N}$?* This question is still open for the general case. That said, for two colorings of $\mathbb{N}$, Matt Bowen proved that the answer is affirmative for $n = 2$ in [Bow25]. In this chapter, we will present this proof. The paper proves a stronger result than we will present here. All the results we present in this chapter are from the paper [Bow25].

This question can also be asked for other semirings instead of $\mathbb{N}$. For the case of $\mathbb{Q}$, the following results for $n = 2$ are proven by Bowen and Sabok in [BS24].

Theorem 31. [BS24] For any finite coloring of $\mathbb{Q}$, there is a monochromatic $\{x, y, x+y, x \cdot y\}$ for some non-zero $x, y \in \mathbb{Q}$.

The proof of $\mathbb{Q}$ in [BS24] also utilises a version of thick/syndetic dichotomy in their argument and a density statement for IP* sets.

Corollary 32. [BS24] For any finite coloring of $\mathbb{R}$, there is a monochromatic $\{x, y, x+y, x \cdot y\}$ for some non-zero $x, y \in \mathbb{Q}$.

Corollary 33. [BS24, Corollary 1.2] For any $r \in \mathbb{N}$, there is a prime p such that for any r-coloring of a field with characteristics p, there is a monochromatic $\{x, y, x+y, x \cdot y\}$ for some non-zero x, y.

We now return to the case of $\mathbb{N}$ in [Bow25]. We categorize colourings into two categories: thick and locally balanced. In a way, these categories correspond to structured and random colourings. If a color is thick, it has a nice structure we can utilize. If a coloring is locally balanced, we will use IP* van der Waerden's theorem and a Galvin-Glazer type induction to get what we want. First, we should define what a locally balanced colouring is.

Definition 25. A coloring $\mathbb{N} = C_0 \sqcup C_1$ (with two colors) is **locally balanced** if there is a multiplicatively thick set $T \subseteq \mathbb{N}$ and multiplicatively syndetic sets $A_1, A_2 \subseteq \mathbb{N}$ such that $T \cap A_i \subseteq C_i$.

We will sometimes omit the locally part and just say a coloring is balanced.

Observe that colors C_0, C_1 are multiplicatively piecewise syndetic in a balanced coloring. That is because, since there is a thick T with piecewise syndetic A_1, A_2 such that $T \cap A_i \subseteq C_i$, then $T_i = T \cup C_i$ is a thick set with syndetic set A_i such that $T_i \cap A_i = C_i$.

Proposition 40. Let $\mathbb{N} = C_0 \sqcup C_1$ be a locally balanced coloring, then there is a minimal left ideal $L \subseteq (\mathbb{N}, \cdot)$ such that $\widehat{C_0} \cap L \neq \varnothing \neq \widehat{C_1} \cap L$.

In particular, there is a minimal left ideal $L \subseteq (\mathbb{N}, \cdot)$, a minimal ultrafilter $\mathcal{U} \in L$ and an idempotent ultrafilter $e \in L$ such that $C_i \in \mathcal{U}$ and $C_{1-i} \in e$ for some $i \in \{0, 1\}$.

Proof. Let $\mathbb{N} = C_0 \sqcup C_1$ be a balanced coloring. Let T be a multiplicatively thick set and A_1, A_2 be multiplicatively syndetic sets such that $T \cap A_i \subseteq C_i$ for all $i \in \{0, 1\}$. Since T is thick, there exists a minimal ideal $L \subseteq (\mathbb{N}, \cdot)$ such that $L \subseteq \widehat{T}$. Since A_i is syndetic, for every left ideal $L \subseteq (\mathbb{N}, \cdot)$, $A_i \cap L \neq \varnothing$.

We claim that $\widehat{C_i} \cap L \neq \varnothing$ for all i.

Assume not, that is assume $\widehat{C_i} \cap L = \varnothing$ for some i. Then $L \subseteq \widehat{C_{1-i}}$ as $\beta\mathbb{N} = \widehat{C_0} \sqcup \widehat{C_1}$. Then $L \subseteq \widehat{T} \cap \widehat{C_{1-i}}$. Since A_i is syndetic, $\widehat{A_i} \cap L \neq \varnothing$. Thus, $\varnothing \neq \widehat{T} \cap \widehat{C_{1-i}} \cap \widehat{A_i} = \widehat{T} \cap \widehat{A_i} \subseteq \widehat{C_i}$. Which contradicts C_i and C_{1-i} being disjoint.

Therefore, $L \cap \widehat{C_i} \neq \varnothing$ for all i.

Now let's get on with the in particular part. Recall that any minimal left ideal contains an idempotent ultrafilter, say $e \in L$ such that for every ultrafilter $\mathcal{U} \in L$, $e \odot \mathcal{U} = \mathcal{U}$.

Let $e \in L$ be the idempotent ultrafilter. Then $C_i \in e$ for some i. Since $\widehat{C_{1-i}} \cap L = \varnothing$, pick $\mathcal{U} \in \widehat{C_{1-i}} \cap L$. $C_{1-i} \in \mathcal{U}$ by definition. So we are done. □

Proposition 41. If a coloring $\mathbb{N} = C_0 \sqcup C_1$ is not balanced, then there is a minimal left ideal $L \subseteq (\mathbb{N}, \cdot)$ with $L \subseteq \widehat{C_i}$ for some $i \in \{0, 1\}$, that is C_i is multiplicatively thick.

Proof. Let $\mathbb{N} = C_1 \sqcup C_2$ be a non-balanced coloring, that is, for every thick set T and for every syndetic sets A_1, A_2, $T \cap A_i \not\subseteq C_i$ for some i. We want to show C_i is thick for some i. Assume not; that is, assume both C_1 and C_2 are not thick. So there is some minimal ideal $K \subseteq (\beta\mathbb{N}, \odot)$ with $K \cap \widehat{C_i} \neq \varnothing$ for all i.

Observe that C_1 and C_2 cannot both be syndetic; otherwise, the coloring would be balanced. Let C_i be the non-syndetic color. Then there is some minimal left ideal $L \subseteq (\beta\mathbb{N}, \odot)$ such that $\widehat{C_i} \cap L = \varnothing$. So C_i is thick. □

Thus, we have only two kinds of 2- colorings. Balanced colorings and colorings with a thick color.

The proof for the thick case follows from the structure of being thick and combinatorics.

We have the following important lemma regarding piecewise syndetic sets

Lemma 7. Let $A \subseteq \mathbb{N}$.

1. Piecewise syndetic sets are partition regular.
2. A is additively piecewise syndetic if and only if $k \cdot A$ is additively piecewise syndetic.
3. If A is multiplicatively piecewise syndetic, then for any $l \in \mathbb{N}$, there are $x_i \in \mathbb{N}$ such that $FS(\{x_1, \ldots, x_l\}) \subseteq A$. That is, multiplicatively piecewise syndetic sets contain the non-repeating sums of sets of arbitrary sizes.

We have proven 1. in Corollary 28, 2. follows from Lemma 5.19.2 of [HS98], and we have proven 3. in Proposition 39.

6.1.1 Thick Color

Given a two coloring $\mathbb{N} = R \sqcup B$, we will show that when one of the colors is multiplicatively thick, there is a monochromatic $\{x, y, x + y, x \cdot y\}$.

Theorem 34. Let $\mathbb{N} = R \sqcup B$ be a two coloring of $\mathbb{N}$. If R or B is multiplicatively thick, then there exists $\{x, y, x + y, x \cdot y\}$.

To prove this, we will use thickness to construct $\{r_0, r_1, r_2, r_3, r_4\} \subseteq R$ with any nice property we want, and then do case-by-case combinatorics and use those properties to show we get $\{x, y, x + y, x \cdot y\} \subseteq R$.

Proof. Let $R \subseteq \mathbb{N}$ be a multiplicatively thick set without loss of generality.

That is for any finite set $F \subseteq \mathbb{N}$, there is $r \in R$ such that $F \cdot r \in R$.

We will construct a set $\{r_0, r_1, r_2, r_3, r_4\} \subseteq R$ using the thickness of R repeatedly.

Let $r_0 \in R$. Set

$$F_0 = \{r_0, r_0^2\}$$

Then by thickness, there is $r_1 \in R$ such that $F_0 \cdot r_1 \subseteq R$.

Here, we could have added more (finitely many) things to F_0 to have an r_1 that ensures the new things multiplied by r_1 are in R as well. We will use this trick at every step, but we will only add things we will use in the case-by-case proof.

Let

$$F_1 = \{r_0, r_1, r_0 + r_1\}$$

By thickness, there is $r_2 \in R$ such that $F_1 \cdot r_2 \subseteq R$.

Again, here we could have added any polynomial $P(r_0, r_1) \subseteq \mathbb{N}(r_0, r_1)$ to F_1 so that $P(r_0, r_1) \cdot r_2 \in R$.

Let

$$F_2 = F_1 \cup \{r_2, (r_0 + r_1 + r_2), (r_0 + r_1)^2 \cdot r_2\}$$

By thickness, there is $r_3 \in R$ such that $F_2 \cdot r_3 \subseteq R$.

Let

$$F_3 = F_2 \cup (F_2 + r_3) \cup \{r_3, (r_0 + r_1 + r_2)^2 \cdot r_3\}$$

By thickness, there is $r_4 \in R$, such that $F_3 \cdot r_4 \subseteq R$.

Thus, we constructed $\{r_0, r_1, r_2, r_3, r_4\} \subseteq R$.

Observe that $F_i \subseteq F_j$ for $i \leq j$, and, $F_i \cdot r_j \subseteq R$ for $i < j$.

Now, we shall begin the case-by-case combinatorics.

1. For all $i < j$, $r_i \cdot r_j \in R$ and $r_i, r_j \in R$. If $r_i + r_j \in R$ as well, we are done. So assume $r_i + r_j \in B$.

2. $r_0 + r_1 \in F_1 \subseteq F_2$. Thus, $(r_0 + r_1) \cdot r_2, (r_0 + r_1) \cdot r_3 \in R$.

   Their multiplication $(r_0 + r_1)^2 \cdot r_2 \cdot r_3 \in R$ as well, since $(r_0 + r_1)^2 \cdot r_2 \in F_2$.

   Therefore, if their sum is in R, we are done. Thus, assume their sum $(r_0 + r_1)(r_2 + r_3) \in B$.

3. By 1, $(r_0 + r_1)$ and $(r_2 + r_3)$ are in B. By 2, their product is in B. So their sum $(r_0 + r_1 + r_2 + r_3)$ must be in R, otherwise we are done.

4. $(r_0+r_1+r_2+r_3) \in R$ by 3, Since $(r_0+r_1+r_2+r_3) \in F_3$, $(r_0+r_1+r_2+r_3)\cdot r_4 \in R$. But then $(r_0 + r_1 + r_2 + r_3)$, r_4 and their product is in R. So their sum $(r_0 + r_1 + r_2 + r_3) + r_4$ must be in B.

5. $(r_0 + r_1 + r_2) \in F_2$, so $(r_0 + r_1 + r_2) \cdot r_3 \in R$. Also, By 3, we have $(r_0 + r_1 + r_2 + r_3) \in R$. Thus, r_3 is in R, and its sum and product with $(r_0 + r_1 + r_2)$ is in R, as well. So $(r_0 + r_1 + r_2)$ must be in B.

6. By 5, we have $(r_0+r_1+r_2) \in B$, By 1) $(r_3+r_4) \in B$, and, by 4, we have their sum $(r_0+r_1+r_2)+(r_3+r_4) \in B$. So their product $(r_0+r_1+r_2)\cdot(r_3+r_4) \in R$.

   But now, set $x = (r_0 + r_1 + r_2) \cdot r_3$ and $y = (r_0 + r_1 + r_2) \cdot r_4$. We have just shown $x + y = (r_0 + r_1 + r_2) \cdot (r_3 + r_4) \in R$.

$(r_0 + r_1 + r_2) \in F_2 \subseteq F_3$, so $x, y \in R$.

Since $(r_0+r_1+r_2)^2 \cdot r_3 \in F_3$, thus their product $x \cdot y = (r_0+r_1+r_2)^2 \cdot r_3 \cdot r_4 \in R$. Therefore, there is $\{x, y, x + y, x \cdot y\} \subseteq R$. We are done.

□

6.1.2 Locally Balanced Colors

The proof for the locally balanced coloring uses a variant of van der Waerden's theorem. The proof can be found in the paper [Bow25] as Theorem 2.5.

Theorem 35 (IP* van der Waerden)**.** Let $A \subseteq \mathbb{N}$ be an additively piecewise syndetic set, let $Q \subseteq \mathbb{Q}$ be a finite subset of rationals. Then there is $n \in \mathbb{N}$ such that for every subset $S \subseteq \mathbb{N}$ with $|S| \geq n$ there is some $d \in FS(S)$ satisfying

$$\bigcap_{q \in Q} (A - qd) \text{ is additively piecewise syndetic.}$$

Lemma 8. Let $A \subseteq \mathbb{N}$ be a multiplicatively central set. Then there exists sequence of sets $S_1, S_2, \ldots \subseteq \mathbb{N}$ with $|S_n| = n$ such that for every finite $F \subseteq \bigcup_n S_n$,

$$\prod_{n : S_n \cap F \neq \varnothing} \sum_{s \in S_n \cap F} s \in A$$

We will do a Galvin-Glazer type of proof.

Proof. Let A be a multiplicatively central set. Then there is a minimal idempotent $\mathcal{U} \in (\beta\mathbb{N}, \odot)$ such that $A \in \mathcal{U}$.

Define $A^* = (A \div \mathcal{U}) \cap A$. Then $A^* \in \mathcal{U}$, since $\mathcal{U} \odot \mathcal{U} = \mathcal{U}$.

Pick $s_1 \in A^*$ and set $S_1 = \{s_1\}$.

Let $A_1 = A^* \cap s_1^{-1} \cdot A \in \mathcal{U}$.

Assume we have defined S_n, A_n and A_n^*.

Since A_n^* is central, it is piecewise syndetic; and since A_n^* is piecewise syndetic, it contains a set S_{n+1} with $FS(S_{n+1}) \subseteq A_n^*$ and $|S_{n+1}| = n + 1$.

Set

$$A_{n+1} = (\bigcap_{s \in FS(S_{n+1})} s^{-1} \cdot A_n^*) \cap A_n^*$$

$A_{n+1} \in \mathcal{U}$, since it is finite intersection of things in $\mathcal{U}$.

We claim

$$\prod_{n:S_n \cap F \neq \varnothing} \sum_{s \in S_n \cap F} s \in A$$

□

We will modify this theorem to be able to control the colors in the product as well, and we will also control the set sizes.

Theorem 36. [Bow25, Theorem 3.2] Let $L \subseteq (\beta\mathbb{N}, \odot)$ be a minimal ideal, $e \in L$ be an idempotent and $\mathcal{U} \in L$ be an ultrafilter. Then for every $A \in \mathcal{U}$, $B \in e$ and function $f : \mathbb{N} \to \mathbb{N}$,there exists sequence of sets $S_1, S_2, \ldots \subseteq \mathbb{N}$ with $|S_n| = f(n)$ such that for every finite $F \subseteq \bigcup_n S_n$,

$$\prod_{n:S_n \cap F \neq \varnothing} \sum_{s \in S_n \cap F} s \in \begin{cases} A & \text{if } S_0 \cap F \neq \varnothing \\ B & \text{if } S_0 \cap F = \varnothing \end{cases}$$

Proof. We will modify the last proof. Recall that $e \odot \mathcal{U} = \mathcal{U}$ and $A \in e \odot \mathcal{U} \iff A \div e \in \mathcal{U}$.

Set $A^* = (A \div e) \cap A \in \mathcal{U}$. Since A^* is multiplicatively central, it is multiplicatively piecewise syndetic. Thus, there is S_0 with $FS(S_0) \subseteq A^*$ and $|S_0| = f(0)$.

Observe that for every $s \in FS(S_0)$, $s \in A^*$, so, $s^{-1} \cdot A \in e$. Therefore, $C = B \cap s^{-1} \cdot A \in e$.

Now, we will do the same proof we did in the last theorem with C. Let $C^* = C \div e \cap C \in e$. Then C^* is central, and hence it contains some S_1 such that $FS(S_1) \subseteq C^*$ and $|S_1| = f(1)$. Set $C_1 = C^* \cap (\bigcap_{s \in FS(S_1)} s^{-1} \cdot C^*)$

Say S_n, C_n and C_n^* are chosen. C_n^* is central, so there is S_{n+1} with $FS(S_{n+1}) \subseteq C_n^*$ and $|S_{n+1}| = f(n+1)$. Set $C_{n+1} = C^* \cap (\bigcap_{s \in FS(S_{n+1})} s^{-1} \cdot C^*)$ and our induction is completed.

□

Theorem 37. Let $\mathbb{N} = C_0 \sqcup C_1$ be a balanced coloring, then there is monochromatic $\{x, y, x + y, x \cdot y\}$.

We aim to use the last theorem paired with IP* van der Waerden to control the color of y. We need to show that our colors are additively piecewise syndetic to be able to apply IP* van der Waerden to them.

Lemma 9. Let $\mathbb{N} = C_1 \sqcup C_2$ be a balanced coloring. Then both colors are additively piecewise syndetic, or there is a monochromatic $\{x, y, x + y, x \cdot y\}$.

Proof. Recall that being additively piecewise syndetic is partition regular. Therefore, $\mathbb{N} = C_1 \sqcup C_2$ tells that at least one color is additively piecewise syndetic. Say C_i is. We want to show C_{1-i} is additively piecewise syndetic or there is a monochromatic $\{x, y, x + y, x \cdot y\}$.

By the definition of being balanced, both colors are multiplicatively piecewise syndetic. We will apply IP* van der Waerden with C_i and $Q = \{0, 1\} \subseteq \mathbb{Q}$ for some q such that, which gives us that $C_i \cap (C_i - y)$ is additively piecewise syndetic.

Set $X' = C_i \cap (C_i - y)$ and $X = \{x \in X' \mid x \cdot y \in C_i\} = (C_i/y) \cap X'$.

If X is non-empty, then any $x \in X$ gives us $\{x, y, x + y, x \cdot y\} \subseteq C_i$.

If X is empty, set $X_{1-i} = \{x \in X' \mid x \cdot y \in C_{i-1}\} = X' \setminus X$. Since X' is additively piecewise syndetic and piecewise syndeticity is partition regular, we have $X_{1-i} = X' \setminus X$ is additively piecewise syndetic. Then $y \cdot X_{1-i} \subseteq C_{1-i}$ is additively piecewise syndetic as well. So C_{1-i} is additively piecewise syndetic. □

Proposition 42. Let $\mathbb{N} = C_1 \sqcup C_2$ be a locally balanced coloring, then there is monochromatic $\{x, y, x + y, x \cdot y\}$.

Proof. Let $L \subseteq (\beta\mathbb{N}, \odot)$ be a minimal left ideal and let $e \in L$ be an idempotent ultrafilter and $\mathcal{U} \in L$ be an ultrafilter such that $C_1 \in \mathcal{U}$ and $C_0 \in e$ without loss of generality.

We have shown that both colors are additively piecewise syndetic.

By the Theorem 36, there exists sequence of sets $S_1, S_2, \ldots \subseteq \mathbb{N}$ with finite sizes such that, for any finite $F \cap \bigcup S_i$,

$$\prod_{n:S_n\cap F\neq\varnothing} \sum_{s\in S_n\cap F} s \in \begin{cases} C_0 & \text{if } S_0 \cap F \neq \varnothing \\ C_1 & \text{if } S_0 \cap F = \varnothing \end{cases}$$

Thus we have $FS(S_0) \subseteq C_0$, $FS(S_i) \subseteq C_1$ for $i > 0$, and $FS(S_0) \cdot FS(S_i) \subseteq C_0$.

C_0 is additively piecewise syndetic and S_0 is a finite set with $FS(S_0) \subseteq C_0$, we can apply IP* van der Waerden to get $y_0 \in FS(S_0)$ with $C_0 \cap (C_0 - y_0)$ is additively piecewise syndetic.

Set $A_0 = C_0 \cap (C_0 - y_0)$. Then $y_0 \cdot A_0$ is additively piecewise syndetic as well. Since piecewise syndeticity is partition regular, $(y_0 \cdot A_0 \cap C_0)$ or $y_0 \cdot A_0 \cap C_1$ is additively piecewise syndetic.

- **If $y_0 \cdot A_0 \cap C_0$ is additively piecewise syndetic:**

  Then for every $x' \in y_0 \cdot A_0 \cap C_0$, we have $\frac{x'}{y_0} \in A_0$, which implies $\frac{x'}{y_0} + y_0 \in C_0$.

  Therefore, $\{\frac{x'}{y_0}, y_0, \frac{x'}{y_0} + y_0, x'\}$ is a monochromatic set we want. So, we are done.

- **If $y_0 \cdot A_0 \cap C_1$ is additively piecewise syndetic:**

  We will apply IP* van der Waerden to C_1 and S_i with $Q_0 = \{0, 1, y_0^2\}$ to get $y_1 \in FS(S_i)$ such that $A_1 = C_1 \cap (C_1 - y_1) \cap (C_1 - y_0^2 \cdot y_1)$ is additively piecewise syndetic.

  We will do the same thing we did with A_1, that is, $y_1 \cdot A_1$ is additively piecewise syndetic, since A_1 is. By partition regularity of piecewise syndetic sets, $y_1 \cdot A_1 \cap C_0$ or $y_1 \cdot A_1 \cap C_1$ is additively piecewise syndetic.

  We will again look at both cases.

  - **If $y_1 \cdot A_1 \cap C_1$ is additively piecewise syndetic:**

    Then for every $x' \in y_1 \cdot A_1 \cap C_1$, we have $\frac{x'}{y_1} \in A_1$, which implies $\frac{x'}{y_1} + y_1 \in C_1$.

    Therefore, $\{\frac{x'}{y_1}, y_1, \frac{x'}{y_1} + y_1, x'\}$ is a monochromatic set we want. So, we are done.

– **If $y_1 \cdot A_1 \cap C_0$ is additively piecewise syndetic:**

Let $y = y_0 \cdot y_1$. For every $x' \in y_1 \cdot A_1 \cap C_0$, $x = \frac{x'}{y} \in C_0$.

$$x + y = \frac{x'}{y_0 \cdot y_1} + y_0 \cdot y_1 = \frac{\frac{x'}{y_1} + y_0^2 \cdot y_1}{y_0} \in A_1/y_1 \subseteq C_0$$

Since $x \cdot y = x' \in C_0$, we have a monochromatic $\{x, y, x+y, x \cdot y\} \subseteq C_0$.

Thus, we are done.

□

We also give the following result for balanced colorings in [Bow25] as a fact.

Theorem 38. Let $\mathbb{N} = C_1 \sqcup C_2$ be a balanced coloring and $n, N \in \mathbb{N}$. Then there is a monochromatic set of the form

$$\{x_i\,,\, (x_1 \cdot \ldots \cdot x_i) + (c_{i+1} \cdot x_{i+1} + \ldots + c_n \cdot x_n) \mid i \leq n \text{ and } c_i \leq N\}$$

The proof idea we did is improved and used in [ABS25] by Alweiss, Bowen, and Sabok. They ensure the arithmetic progression is also monochromatic and prove a multiplicative/exponential version of the statement. So, we have the following stronger result.

Theorem 39. Let $\mathbb{N} = C_1 \sqcup C_2$ be a 2-coloring and $l \in \mathbb{N}$, then there are monochromatic sets $\{x, y, xy, x + iy \mid i \leq l\}$ and $\{a, b, a^b, a \cdot b^j \mid j \leq l\}$

The proofs also use thick/syndetic duality and use the arguments presented in this chapter, such as IP* van der Waerden and multiplicatively syndetic colorings. There is also an alternative proof in the paper using the distributions among polynomials.

We will not cover the proof of this result and instead present a proof of the existence of a monochromatic $\{x, y, x^y\}$ from [DNR24] in the next chapter. We will utilize a Galvin-Glazer type induction and present the proof of an infinitary version as well.

CHAPTER 7

EXPONENTIAL MONOCHROMATIC PATTERNS

7.1 Exponential Monochromatic Triples

Exponentiation is not an associative operation; thus, to study exponential patterns, we cannot take exponentiation as our operation and extend it into a βS as a semigroup as easily, and more importantly, we cannot apply Ellis' Lemma to get idempotents. The initial question, the exponential version of Schur's theorem, whether there is a monochromatic $\{x, y, x^y\}$ in any finite coloring of $\mathbb{N}$, was answered positively in [Sah18] by J. Sahasrabudhe. We will present the proof of this result by Mauro Di Nasso and Mariaclara Ragosta. Their proof uses a Galvin-Glazer type induction with ultrafilters. All the results in this chapter can be found in [DNR24].

Thus, this chapter presents two main theorems. A finite simple result and an infinitary result.

Theorem 40 (Exponential Schur's Theorem)**.** Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any finite coloring. Then there is monochromatic $\{a, b, b^a\}$.

Theorem 41. Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any finite coloring. Then there is i and an infinite increasing sequence $\{b_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with $\{b_n, b_n^{b_{n+1}}\} \subseteq C_i$ for every n.

For both theorems, we will use ultrafilters $\mathcal{U}$ such that for every $A \in \mathcal{U}$ and for every $\ell \in \mathbb{N}$, there is $b, c \in \mathbb{N}$ such that $\{b, c, b + c, \ldots, b + \ell c\} \subseteq A$. Existence of these ultrafilters is equivalent to a stronger version of van der Waerden, named Brauer's theorem.

Theorem 42 (Brauer's Theorem)**.** [Bra28] For any coloring $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ and for any $\ell \in \mathbb{N}$, there is $b, c \in \mathbb{N}$ with monochromatic $\{b, c, b + c, \ldots, b + \ell c\}$.

We should note that given $\ell \in \mathbb{N}$, Rado's theorem implies that the system

$$x_2 - x_1 = x_0$$

$$x_3 - x_2 = x_0$$

$$\dots$$

$$x_\ell - x_{\ell-1} = x_0$$

which corresponds to the linear system with matrix

$$A = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is weakly partition regular.

We will take Brauer's theorem as a fact, and note that minimal ultrafilters of $(\beta\mathbb{N}, \oplus)$ already witness Brauer's theorem.

Theorem 43. The following are equivalent.

(1) There is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that for every $A \in \mathcal{U}$ and for all $\ell \in \mathbb{N}$ there is $b, c \in \mathbb{N}$ with $\{b, c, b + c, \dots, b + \ell c\} \subseteq A$.

(2) For any coloring $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ and for any $\ell \in \mathbb{N}$,there is $b, c \in \mathbb{N}$ with monochromatic $\{b, c, b + c, \dots, b + \ell c\}$.

Proof. ($\implies$) : Let $\mathcal{U}$ be an ultrafilter with for every $\ell \in \mathbb{N}$ and $A \in \mathcal{U}$, there is $\{b, c, b + c, \dots, b + \ell c\} \subseteq A$. Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any finite coloring. Then some $C_i \in \mathcal{U}$ and we are done.

($\impliedby$) : We are going to use an almost identical argument to Corollary 13 in Chapter 3. Given a set A, we will consider the collection of pairs (b, c) such that $b, c, b + c, \dots, b + \ell c$ does not split up by A, and show these collections have FIP and extend them into an ultrafilter. Then we will push this ultrafilter with the same T_i functions.

That is given $A \subseteq \mathbb{N}$ and $\ell \in \mathbb{N}$, define

$$\Gamma_\ell(A) = \{(b,c) \mid \{b, c, b+\ell c\} \subseteq A \text{ or } \{b, c, b+\ell c\} \subseteq \mathbb{N} \setminus A\}$$

We have to show collection $\Gamma = \{\Gamma_\ell(A) \mid A \subseteq \mathbb{N}, \ell \in \mathbb{N}\} \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ has FIP. Let $\Gamma_{\ell_1}(A_1), \ldots \Gamma_{\ell_n}(A_n) \in \Gamma$. Then let $(s_1, \ldots, s_n) \in \{0,1\}^n$ be a string of length n. Define $B_s = A_1^{s_1} \cap \ldots \cap A_n^{s_n}$ where $A_i^0 = A_i^c$, the complement of A_i and $A_i^1 = A_i$. So $\mathbb{N} = B_1 \sqcup \cdots \sqcup B_s$ is a finite coloring. Then by Brauer's theorem there is $b, c \in \mathbb{N}$ with monochromatic $\{b, c, b+c, \ldots, b+\ell c\}$. Thus, $(b,c) \in \bigcap_{i \leq n} \Gamma_\ell(A_i)$ and hence Γ has FIP. So extend Γ into an ultrafilter $\mathcal{W}$ on $\mathbb{N} \times \mathbb{N}$.

Define $T_i : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $T_i(n,m) = n + im$ for $\ell \in \mathbb{N}$.

We claim that $T_{i*}(\mathcal{W}) = T_{j*}(\mathcal{W})$ for every $i \leq j$. Let $B \in T_i(\mathcal{W})$. Then $T_i^{-1}(B) \in \mathcal{W}$. That is $\{(n,m) \mid n + im \in B\} \in \mathcal{W}$. Since $\Gamma_j(B) \in \mathcal{W}$, we have $\Gamma_j(B) \cap T_i^{-1}(B) \in W$.

Observe that for $\Gamma_j(B) \cap T_i^{-1}(B) = \Gamma_j(B) \cap T_j^{-1}(B)$, as if a pair (b,c) is in $\Gamma_j(B) \cap T_i^{-1}(B)$, then $\{b, c, b+c, \ldots, b+ic, \ldots, b+jc\} \subseteq B$. Thus $B \in T_j(\mathcal{W})$ and hence $T_{i*}(\mathcal{W}) \subseteq T_{j*}(\mathcal{W})$. By maximality of ultrafilters $T_{i*}(\mathcal{W}) = T_{j*}(\mathcal{W})$.

Set $\mathcal{U} = T_{i*}(\mathcal{W})$. We claim that for all $A \in \mathcal{U}$, there is $\{b, c, b+c, \ldots, b+\ell c\} \subseteq A$.

Let $A \in \mathcal{U}$ and $\ell \in \mathbb{N}$. Then for every $t \leq \ell$, $\Gamma_t(A) \in \mathcal{W}$. In particular $A' = T_1^{-1}(A) \cap (\bigcap_{t \leq \ell} \Gamma_t(A)) \in \mathcal{W}$. Pick an element $(b,c) \in A'$, so there is some c with $\{b, c, b+c, \ldots, b+\ell c\} \subseteq A$.

□

Let $f : \mathbb{N}^2 \to \mathbb{N}$ such that $f(n,m) = 2^n \cdot m$, we will use this function to push-forward $\mathcal{U} \otimes \mathcal{U}$ and move from an arithmetic pattern to an exponential pattern. Then we will pull back the coloring.

Given ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and $A \subseteq \mathbb{N}$,

$$\begin{aligned} A \in f_*(\mathcal{U} \otimes \mathcal{U}) &\iff f^{-1}(A) \in \mathcal{U} \otimes \mathcal{U} \\ &\iff \{(n,m) \mid 2^n \cdot m \in A\} \in \mathcal{U} \otimes \mathcal{U} \\ &\iff \{n \mid (2^n)^{-1} \cdot A \in \mathcal{U}\} \in \mathcal{U} \end{aligned}$$

Even though f is not associative, we can treat f as an operation; then its idempotents would look like

$$A \in \mathcal{U} \iff \{n \mid (2^n)^{-1} \cdot A \in \mathcal{U}\} \in \mathcal{U}$$

In fact, we can denote $f_*(\mathcal{U} \otimes \mathcal{U})$ with $\mathcal{U} * \mathcal{U}$, but we have to be careful with any parentheses involved, as this is not an associative operation.

So we have

$$A \in \mathcal{U} * \mathcal{V} \iff \{n \mid (2^n)^{-1} A \in \mathcal{V}\} \in \mathcal{U}$$

We will denote $A * (\mathcal{V})^{-1} = \{n \mid (2^n)^{-1} A \in \mathcal{V}\}$.

Now, we prove the first theorem.

Theorem 44. Let $\mathcal{U}$ be an ultrafilter such that for every $A \in \mathcal{U}$ and for all $\ell \in \mathbb{N}$ there is $b, c \in \mathbb{N}$ with $\{b, c, b + c, \ldots, b + \ell c\} \subseteq A$. Then for every $A \in \mathcal{U}$ there is $\{x, y, 2^x \cdot y\} \subseteq A$.

Proof. Let $\mathcal{U}$ be an ultrafilter such that for every $A \in \mathcal{U}$ and for all $\ell \in \mathbb{N}$ there is $b, c \in \mathbb{N}$ with $\{b, c, b+c, \ldots, b+\ell c\} \subseteq A$ for every $i \leq \ell$. Then for every $A \in \mathcal{U} * \mathcal{U}$, $A * (\mathcal{U})^{-1} \in \mathcal{U}$. Pick $a \in A^* = A \cap A * (\mathcal{U})^{-1}$.

We set $\ell = 2^a$, and by definition of $\mathcal{U}$, there is b, c such that $\{b, c, b + c, \ldots, b + lc\} \in A^* \cap (2^a)^{-1} \cdot A$.

So $(2^b)^{-1} \cdot A$, $(2^{b+2^a c})^{-1} \cdot A$ and $(2^c)^{-1} \cdot A$ are in $\mathcal{U}$. Pick $d \in ((2^b)^{-1} \cdot A) \cap ((2^{b+2^a \cdot c})^{-1} \cdot A) \in \mathcal{U}$.

Therefore,

$$2^{2^a c} \cdot 2^b \cdot d = 2^{2^a c + b} \cdot d \in A$$

Set $x = 2^{2^a c}$ and $y = 2^b d$, to get $\{x, y, 2^x \cdot y\} \subseteq A$. □

Corollary 45. Given any coloring $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$, there is monochromatic $\{x, y, 2^x \cdot y\}$.

Proof. Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any coloring. Brauer's Theorem holds, so there is an ultrafilter $\mathcal{U}$ such that for every $A \in \mathcal{U}$ and for all $\ell \in \mathbb{N}$ there is $b, c \in \mathbb{N}$ with

$\{b, c, b+c, \ldots, b+\ell c\} \subseteq A$. There is i with $C_i \in \mathcal{U}$. So there is $\{x, y, 2^x \cdot y\} \subseteq C_i$, and we are done. □

Corollary 46. Given any coloring $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$, there is monochromatic $\{a, b, b^a\}$.

Proof. Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any coloring. We will color $\mathbb{N} = \bar{C_1} \sqcup \cdots \sqcup \bar{C_r}$ such that

$$n \in \bar{C_i} \iff 2^n \in C_i$$

By the previous corollary, there is some i with $\{x, y, 2^x y\} \subseteq \bar{C_i}$. Then $a = 2^x$, $b = 2^y$ has $\{a, b, b^a\} \subseteq C_i$. □

Now, we will do the infinite case.

Theorem 47. Let $\mathcal{U}$ be an ultrafilter such that for every $A \in \mathcal{U}$ and for all $\ell \in \mathbb{N}$ there is $b, c \in \mathbb{N}$ with $\{b, c, b+c, \ldots, b+\ell c\} \subseteq A$. Then for every $A \in \mathcal{U}$ there is an infinite increasing sequence $\{a_n\}_{n\in\mathbb{N}} \in \mathbb{N}$ such that, for every i, j, k with $i < 2j$, $2j+1 < k$, there is $x = 2^{a_i} \cdot a_{2j}$, $y = 2^{a_{2j+1}} a_k$ with $\{x, y, 2^x \cdot y\} \subseteq A$.

Proof. We will do a Galvin-Glazer type induction, picking triples a, b, c as we did in the finite case at every step, then using the property of U, we get $a, b, a + \ell b \in A^*$ with

$$(2^a)^{-1} \cdot A \,, (2^b)^{-1} \cdot A \,, (2^{a+\ell b})^{-1} \cdot A \in \mathcal{U}$$

We will pick several ℓ's as the 2^{a_n} where n are previous steps in the induction.

Let's do it formally.

Base Step: Let $\mathcal{U}$ be an ultrafilter such that for every $A \in \mathcal{U}$ and for all $\ell \in \mathbb{N}$ there is $b, c \in \mathbb{N}$ with $\{b, c, b+c, \ldots, b+\ell c\} \subseteq A$ for every $i \leq \ell$. Then for every $A \in \mathcal{U} * \mathcal{U}$, $A * (\mathcal{U})^{-1} \in \mathcal{U}$. Pick $a_1 \in A^* = A \cap A * (\mathcal{U})^{-1}$.

We set

$$A_1 = A^* \cap ((2^{a_1})^{-1} \cdot A) \in \mathcal{U}$$

We set $\ell = 2^{a_1}$, and by definition of $\mathcal{U}$, there are a_3, a_2 such that $\{a_3, a_2, a_3 + a_2, \ldots, a_3 + 2^{a_1} a_2\} \in A_1$.

So $2^{a_1}a_3, 2^{a_1}a_2, 2^{a_1}(a_3 + t \cdot a_2) \in A$ for $t \leq 2_1^a$. We also have $(2^{a_2})^{-1} \cdot A, (2^{a_3})^{-1} \cdot A, (2^{a_3+2^{a_1}a_2})^{-1} \cdot A \in \mathcal{U}$.

We set

$$A_2 = A_1 \cap ((2^{a_2})^{-1} \cdot A) \cap ((2^{a_3})^{-1} \cdot A) \cap ((2^{a_3+2^{a_1}a_2})^{-1} \cdot A) \in \mathcal{U}$$

We set $\ell = max(2^{a_1}, 2^{a_2}, 2^{a_3})$, and by definition of $\mathcal{U}$, there are a_5, a_4 with $a_5 + t \cdot a_4 \in A_2$ for every $t \leq \ell$.

So, we have $a_5 + 2^{a_1} \cdot a_4, a_5 + 2^{a_2} \cdot a_4, a_5 + 2^{a_3} \cdot a_4 \in A_2$. Thus, by definition of A_2,

$$\left((2^{a_5+2^{a_1}\cdot a_4})^{-1} \cdot A\right), \left((2^{a_5+2^{a_2}\cdot a_4})^{-1} \cdot A\right), \left((2^{a_5+2^{a_3}\cdot a_4})^{-1} \cdot A\right) \in \mathcal{U}$$

We also have $(2^{a_4})^{-1} \cdot A, (2^{a_5})^{-1} \cdot A \in \mathcal{U}$. Thus, we will intersect all those sets in the last two lines with A^* to get $A_3 \in \mathcal{U}$.

We will continue one more step before going to the inductive step.

By assumption, there are a_7, a_6 such that $a_7 + t \cdot a_6 \in A_3$ for all $t \leq max\{2^{a_i} \mid i \leq 5\}$.

Like the last step, we have $(2^{a_7+2^{a_i}a_6})^{-1} \cdot A \in \mathcal{U}$ for all $i \leq 5$.

Since $a_7 + i \cdot a_6 \in A_3$, we have $2^{a_7+2^{a_i}\cdot a_6} \in (2^{a_j})^{-1} \cdot A$ for every $j \leq 5$. Thus, we have $2^{a_7+2^{a_i}\cdot a_6}2^{a_j} \in A$.

Inductive Step: Say n steps has been done, that is $a_1 \ldots, a_{2n-1}$ and $A_n \in \mathcal{U}$ has been picked such that

$$A_n = A^* \cap \left(\bigcap_{i \leq 2n-1} (2^{a_i})^{-1} \cdot A\right) \cap \left(\bigcap_{i<2j<2n-1} (2^{a_{2j+1}+2^{a_i}a_{2j}})^{-1} \cdot A\right)$$

with $a_i \in A_n$ for all $i \leq 2n-1$.

Since $A_n \in \mathcal{U}$, there is $a_{2n+1}, a_{2n} \in A_n$ such that $a_{2n+1} + t \cdot a_{2n} \in A_n$ for all $t \leq max\{2^{a_i} \mid i \leq 2n-1\}$.

Let $A_{n+1} = A^* \cap \left(\bigcap_{i \leq 2n+1} (2^{a_i})^{-1} \cdot A\right) \cap \left(\bigcap_{i<2j<2n+1} (2^{a_{2j+1}+2^{a_i}a_{2j}})^{-1} \cdot A\right)$. We have $A_{n+1} \in \mathcal{U}$ and $a_i \in A_{n+1}$ for all $i \leq 2n+1$. Thus, we are done with the induction, but we will check if our sequence works as intended.

Let i, j, k with $i < 2j$ and $2j + 1 < k$. Then $a_i, a_{2j+1}, a_k \in A_k$, where

$$A_k = A^* \cap \left(\bigcap_{i \leq 2k-1} (2^{a_i})^{-1} \cdot A \right) \cap \left(\bigcap_{i<2j<2k-1} (2^{a_{2j+1}+2^{a_i}a_{2j}})^{-1} \cdot A \right)$$

Let $x = 2^{a_i} \cdot a_{2j}$, since $a_{2j} \in A_k \subseteq (2^{a_i})^{-1} \cdot A$, we have $x \in A$.

Let $y = 2^{a_{2j+1}} \cdot a_k$, since $a_k \in A_k \subseteq (2^{a_{2j+1}})^{-1} \cdot A$, we have $y \in A$

$$2^x \cdot y = 2^{2^{a_i} \cdot a_{2j}} \cdot 2^{a_{2j+1}} \cdot a_k = 2^{a_{2j+1}+2^{a_i} \cdot a_{2j}} \cdot a_k$$

Since $a_k \in A_k \subseteq (2^{a_{2j+1}+2^{a_i}a_{2j}})^{-1} \cdot A$, $2^x \cdot y \in A$. We are done.

□

Corollary 48. Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any coloring, then there is an infinite increasing sequence $\{a_n\}_n \in \mathbb{N}$ such that, for every i, j, k with $i < 2j$, $2j + 1 < k$, there is $x = 2^{a_i} \cdot a_{2j}$, $y = 2^{a_{2j+1}} a_k$ with monochromatic $\{x, y, 2^x \cdot y\}$.

Again, we will push back the coloring to obtain an infinite monochromatic exponential pattern.

Proof. Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any coloring. Brauer's Theorem holds, so there is an ultrafilter $\mathcal{U}$ such that for every $A \in \mathcal{U}$ and for all $\ell \in \mathbb{N}$ there is $b, c \in \mathbb{N}$ with $\{b, c, b+c, \ldots, b+\ell c\} \subseteq A$. There is i with $C_i \in \mathcal{U}$. So there is an infinite increasing sequence $\{a_n\}_n \in \mathbb{N}$ such that, for every i, j, k with $i < 2j$, $2j + 1 < k$, there is $x = 2^{a_i} \cdot a_{2j}$, $y = 2^{a_{2j+1}} a_k$ with monochromatic $\{x, y, 2^x \cdot y\}$. □

Corollary 49. Let $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$ be any coloring; then there is an infinite increasing sequence $\{b_n\}_n \in \mathbb{N}$ with monochromatic $\{b_n, b_{n+1}^{b_n}\}$ for every n.

Proof. Given a coloring $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_r$, we pull back the coloring to $n \in \bar{C}_i \iff 2^n \in C_i$, in the same way we did before. There is a sequence $\{a_n\}$ as stated by the previous corollary.

Set $b_n = 2^{2^{a_{2n-1}} a_{2n}}$. Then

$$\begin{aligned} b_{n+1}^{b_n} &= (2^{2^{a_{2n+1}} a_{2n+2}})^{2^{2^{a_{2n-1}} a_{2n}}} \\ &= 2^{(2^{a_{2n+1}} a_{2n+2}) 2^{2^{a_{2n-1}} a_{2n}}} \\ &= 2^{(2^{a_{2n+1}} + 2^{a_{2n-1}} a_{2n}) a_{2n+2}} \end{aligned}$$

Since for $k = 2n+1$, $j = n$ and $i = 2n-1$, we have some color $\bar{C}_t$ with $(2^{a_{2n+1}} + 2^{a_{2n-1}} a_{2n}) a_{2n+2} \in \bar{C}_t$, $b_{n+1}^{b_n} \in C_t$. □

CHAPTER 8

CONCLUSION

This thesis was intended to serve as an exposition of the study of partition regularity via ultrafilters and the algebra of βS. We have surveyed some partition regularity results and presented proofs of them via ultrafilters and the algebra of the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. We have shown that idempotent ultrafilters, which exist by Ellis' lemma, witness Ramsey theoretic results and, in particular, minimal idempotent ultrafilters witness abundant combinatorial properties. We have illustrated that every partition regularity result is equivalent to the existence of some ultrafilter and conversely, ultrafilters that satisfy nice algebraic properties also witness nice partition regularity results.

Most of the results we have presented were for $\mathbb{N}$. While some of them can be proven for arbitrary semigroups with several adaptations in the definitions, that is not the case for all partition regularity results. For example, Schur's theorem in non-commutative semigroups can be stated in the following ways:

Proposition 43. Let $(S, \cdot)$ be a semigroup. Let $S = C_1 \sqcup \cdots \sqcup C_r$ be a finite coloring, then there is some $x, y \in S$ with monochromatic $\{x, y, x \cdot y\}$.

Question 2. Let F_2 be the free group on 2 generators. Is it the case that for every finite coloring $F_2 = C_1 \sqcup \cdots \sqcup C_r$, there is some $x, y \in F_2$ with monochromatic $\{x, y, x \cdot y, y \cdot x\}$ and $x \cdot y \neq y \cdot x$?

The first proposition is true, while the second remains open. The same question for amenable groups was answered positively in [Bow24]. There has been active research on this problem and many other open problems in the study of partition regularity for arbitrary semigroups.

Another important tool for the study of partition regularity is the notions of largeness and density statements. We have covered some notions of largeness and notions of density, but there are other useful notions we did not have the chance to introduce in this thesis. Notions of largeness and density statements generally have ultrafilter characterizations, and they are frequently used to construct ultrafilters witnessing a specific property. For example, in the [DNR24] paper, asymptotic density and Δ sets are used to construct an ultrafilter $\mathcal{U}$ such that for all $A \in \mathcal{U}$ and for all $\ell \in \mathbb{N}$, there is $\{b, c, b + \ell c\} \subseteq A$. A survey on notions of largeness on semigroups and their hierarchy can be found in [Hin20b].

For a selection of open problems related to partition regularity, notions of largeness, and the Stone–Čech compactification of discrete semigroups, we refer the reader to [BR09].

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